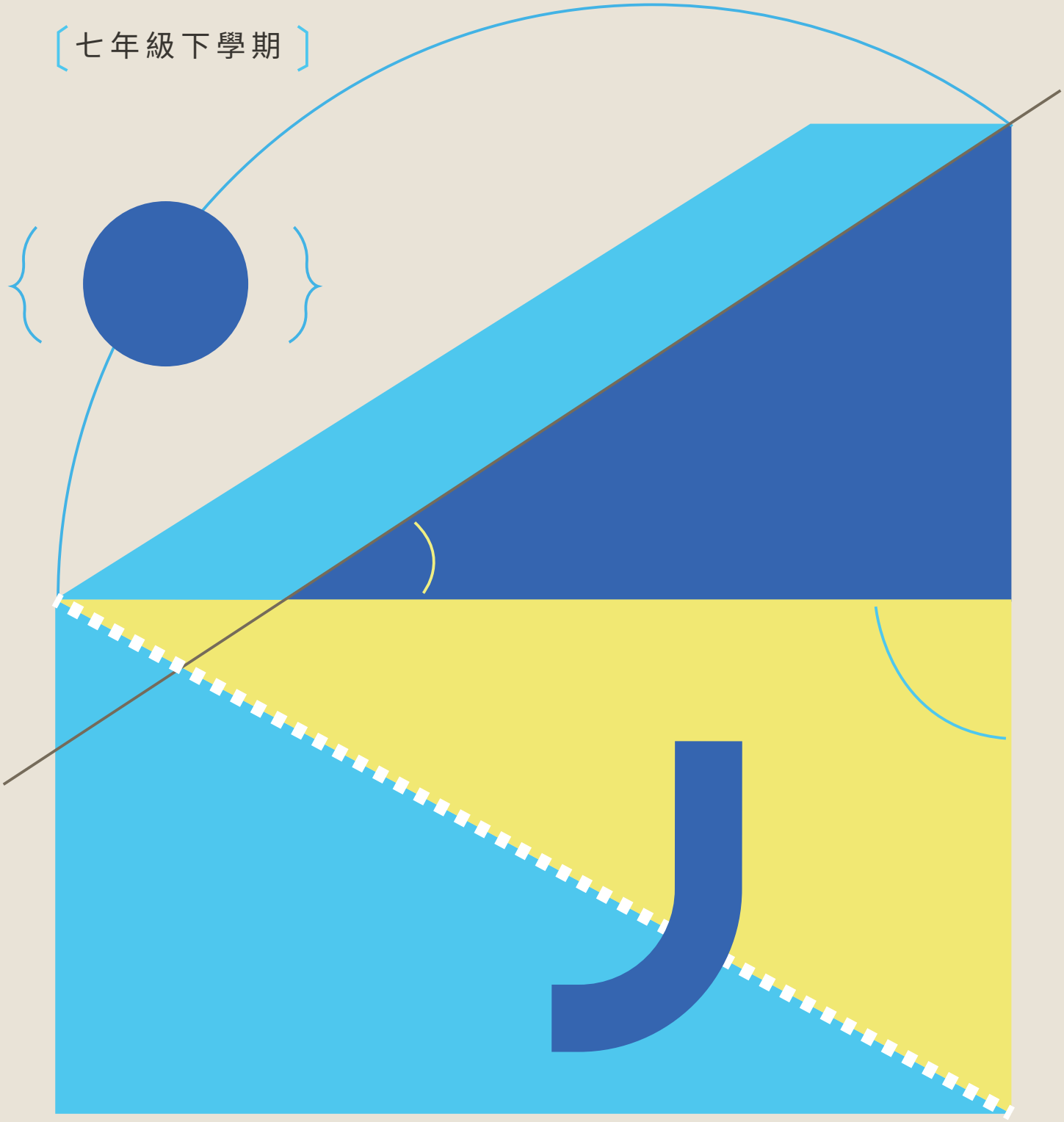
國中數學領域

雙語教學資源手冊 英語授課用語

A Reference Handbook for Junior High School Bilingual Teachers in the Domain of Mathematics: Instructional Language in English

〔七年級下學期〕



臺師大雙語教學研究中心 發行

目次 Table of Contents

單元一 二元一次方程式
Linear Equations in Two Variables

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■ 前言 Introduction

學生認識二元一次方程式的定義，能依題列出二元一次方程式，並且求出可能的解。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中文	單字	中文
polynomial	多項式	linear expression in/with two variables	二元一次式
simplify	**化簡**	**linear equation in/with two variables**	**二元一次方程式**
coefficient	係數	real number	實數
ordered pair	有序數對	commutative property of addition	加法交換律
like terms	**同類項**	distributive property	分配律
solution	解		

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ Simplify _____.

例句：**Simplify** the expression $3x + 5y - 4x + 2y$.

化簡多項式 $3x + 5y - 4x + 2y$。

❷ Decide whether _____ is _____.

例句：**Decide whether** the given ordered pair **is** a solution of the equation.
判斷已知數對是否為該方程式的解。

❸ Substitute ___ for *x* and ___ for *y* (_____).

例句：**Substitute** 1 **for *x* and** −3 **for *y*** in the equation, and see whether the equation is satisfied.
將 1 代入 x，−3 代入 y，看看方程式的值是否成立。

■ 問題講解 Explanation of Problems

ଓ 說明 ଓ

Linear expressions and linear equations 以例子介紹二元一次式與二元一次方程式的不同

We have learned linear equations in one variable before, and now we will learn linear equations in two variables. This means that we can use two variables to represent two unknown things and describe their relationships.

For example, your mom goes shopping and buys two Cokes. You ask how much a Coke is, and your mom says, “Well, two Cokes are NT$58.” Your brain works quickly, and you may have $\frac{58}{2} = 29$ for each Coke. Suppose I ask you to write an equation to describe the relationships between quantity and price. In that case, you will use a variable, such as x, to represent the unit price for a Coke. Two Cokes cost “2 times x.” Therefore, “two Cokes are NT$58” gives you the equation “$2x = 58$.” This is a linear equation in one variable, as we learned before.

Your mom goes shopping again and buys three cups of coffee and two cupcakes. How much are they? Now, we don’t have the price of one cup of coffee and one cupcake. We can use two variables to represent their unit price, respectively, x for coffee and y for the cupcake. Three cups of coffee will cost you “3 times x,” and two cupcakes are “2 times y.” Therefore, the total price is $3x + 2y$. If your mom paid NT$215 for coffees and cupcakes, then you can have the equation $3x + 2y = 215$.

A polynomial is an algebraic expression made up of variables and coefficients, such as $3x + 2y$, $-4m + 2n$. In this expression, x, y, m, and n are variables while the number in front of the variable is its coefficient. Also, the degree of each variable is 1, so the expression is called a "linear expression." $3x + 2y = 215$ is a linear equation. The clear difference between an expression and an equation is that an equation has an equal sign.

When an equation has two variables of degree one, written in the form of "$ax + by = c$," it is called a linear equation in two variables. In this equation, x and y are the variables, and a and b are the coefficients of x and y, respectively. Also, a, b and c are all real numbers.

Operations, values of linear expression 二元一次式的運算

When adding expressions, $(a_1x + b_1y + c_1) + (a_2x + b_2y + c_2)$, we group the like terms together. This means that we put all the x-terms together, all the y-terms together, and all the constants together, like this: $(a_1x + a_2x) + (b_1y + b_2y) + (c_1 + c_2)$. We do simple math to combine the like terms. So, $(a_1 + a_2)$ is the coefficient of x, $(b_1 + b_2)$ is the coefficient of y, and $(c_1 + c_2)$ is the constant.

Let's try an example: $(3x + 2y - 5) + (x + 3y + 8)$. First, we put all the x-terms together in parentheses $(3x + x)$, put all the y-terms together in parentheses $(2y + 3y)$, and all constants together in parentheses $(-5 + 8)$. We connect all parentheses with addition, and we get $(3x + x) + (2y + 3y) + (-5 + 8)$. Then, we do simple math to combine the like terms. $3x + x$ equals $4x$, $2y + 3y$ equals $5y$, $-5 + 8$ equals 3. Finally, we get the answer $4x + 5y + 3$.

The distributive property states $a(m + n) = am + an$, and $a(m - n) = am - an$. We multiply each term in parentheses by the factor "a." When subtracting an expression, like in $a_1x + b_1y + c_1 - (a_2x + b_2y + c_2)$, we can use the distributive property: multiplying each term in parentheses by (-1). We get: $a_1x + b_1y + c_1 - a_2x - b_2y - c_2$, then combine the like terms to get: $(a_1 - a_2)x + (b_1 - b_2)y + (c_1 - c_2)$.

Let's practice with $3x + 4y - (-7x + y - 3)$. We multiply $-7x + y - 3$ by (-1) and take away the parentheses. We get: $3x + 4y + 7x - y + 3$. Remember that a negative number times a

negative number equals a positive number; a negative number times a positive number equals a negative number. This is one of the most common mistakes that students make, so remember it and do not make the same mistake!

When simplifying expressions, such as $m(a_1x + b_1y + c_1) - n(a_2x + b_2y + c_2)$, you need to apply the distributive property two times. Multiply each term in the first parentheses by "m," multiply each term in the second parentheses by "$-n$," and then combine the like terms.
Take: $2(x + 5y - 4) - 3(3x - y + 1)$ as an example. Apply the distributive property, multiply each term in the first parentheses by 2, and we have $2x + 10y - 8$. Multiply each term in the second parentheses by -3, and we get: $-9x + 3y - 3$. Then combine the like terms of $2x + 10y - 8$ and $-9x + 3y - 3$.
We get: $(2x - 9x) + (10y + 3y) + (-8 - 3) = -7x + 13y - 11$.

Solution to linear equations 列式與解，判斷數對是否為二元一次方程式的解

Take the coffee and cupcake above as an example. If the equation is $3x + 2y = 215$, decide whether $x = 45$, $y = 40$ is the solution to this equation. We plug in the x value and y value into the equation: so for 3 times 45 and 2 times 40, the sum is 215. The solution satisfies the equation. We can also write the answer as (45, 40).

The solution of linear equations in two variables, $ax + by = c$, can be written in the form of an ordered pair (x, y). When the x value is multiplied by a, and the y value is multiplied by b, the sum of these two values will be c. If the equation isn't true, which means the sum of these values is not c, the ordered pair is not the solution.

∞ 運算問題的講解 ∞

例題一

說明：讓學生練習二元一次式的化簡

（英文）Simplify the expression.
（中文）化簡二元一次式。
(1) $4x + 10y + 3 - (-5x - 7y)$　　(2) $6(2m-n) + 3(-4n + 2m - 1)$

(1)

Teacher: We use the properties of operations to group like terms, and then we simplify. First, we apply the distributive property. Multiply each term in $(-5x - 7y)$ by (-1). Remember that a negative number times a negative number equals a positive number. You will have $4x + 10y + 3 + 5x + 7y$.

Here are the steps:

$4x + 10y + 3 - (-5x - 7y)$

$= 4x + 10y + 3 + 5x + 7y$ ····· **Distributive property**

$= 4x + 5x + 10y + 7y + 3$ ····· **Commutative property of addition**

$= (4x + 5x) + (10y + 7y) + 3$ ····· **Group like terms.**

$= 9x + 17y + 3$ ····· **Combine like terms**

When applying the distributive property, you have to multiply "each term" by the factor, not just the first term. This is a common mistake made by students, so don't make the same mistake! Now, work on the next question.

(2)

Teacher: What is your first step?

Student: Apply the distributive property.

Teacher: Please write each step on the board.

Student: $6(2m - n) + 3(-4n + 2m - 1)$

$= 6(2m) - 6n + 3(-4n) + 3(2m) + 3(-1)$ ····· **Distributive property**

$= 12m - 6n - 12n + 6m - 3$ ····· **Multiply.**

$= (12m + 6m) + (-6n - 12n) - 3$ ····· **Group like terms.**

$= 18m - 18n - 3$ ····· **Combine like terms.**

Teacher: Great!

老師： 我們使用運算法則合併同類項，並進行化簡。首先，我們使用分配律。
將$(-5x-7y)$中的每一項乘以(-1)。請記住，負負得正。
因此，你們會得到$4x+10y+3+5x+7y$。
過程如下：
$4x+10y+3-(-5x-7y)$
$=4x+10y+3+5x+7y$……分配律
$=4x+5x+10y+7y+3$……加法交換律
$=(4x+5x)+(10y+7y)+3$……同類項分類
$=9x+17y+3$……同類項合併
在使用分配律時，括號內每一項都要乘到，而不僅僅是第一項。這是學生經常犯的錯誤，所以不要犯同樣的錯！
現在，看第(2)小題。

老師： 你的第一步是什麼？

學生： 使用分配律。

老師： 來黑板上寫下步驟。

學生： $6(2m-n)+3(-4n+2m-1)$
$=6(2m)-6n+3(-4n)+3(2m)+3(-1)$ …… 分配律
$=12m-6n-12n+6m-3$ …… 乘法運算
$=(12m+6m)+(-6n-12n)-3$ …… 同類項分類
$=18m-18n-3$ …… 同類項合併

老師： 很棒！

例題二

說明：判斷某數對是否為方程式的解，並引導學生思考二元一次方程式是否只有唯一解。

（英文）Tell if the ordered pair (2, 6) is a solution to the linear equations below.

Equation 1: $2x-3y=-11$

Equation 2: $4x-y-2=0$

（中文）判斷數對 (2, 6) 是否為方程式的解。

方程式 1：$2x-3y=-11$

方程式 2：$4x-y-2=0$

Teacher: Substitute 2 for x and 6 for y in each equation. If the equation is satisfied, then the ordered pair is a solution for the equation. Try this question, and share your answers.

Student 1: In equation 1, I got $4-18=-14$, which is not -11. So, the equation is not satisfied. (2, 6) is not a solution for the first equation.

Student 2: In equation 2, I got $8-6-2=0$. This is the same as the question. So, (2, 6) is a solution for the second equation.

Teacher: Correct! I want to follow up. Is (2, 6) the ONLY solution to equation 2?

Student 1: I guess so. When you plug in the numbers, the equation gives you 0 as the answer.

Teacher: I understand. But I was asking for other possible answers, except (2, 6). Can you substitute other numbers for x and y and still get 0 as the answer? Try to substitute 1 for x. What equation do you have now?

Student: $4-y-2=0$.

Teacher: Find the value for y.

Student 2: $4-y-2=0$, then y is 2.

Teacher: Correct. The equation is satisfied when x is 1 and y is 2. We will say that (1, 2) is a solution to this equation. Are there any other solutions?

Student 3: That could be a lot. If I plug in a different number for x, I will get a different number for y as a result. This action will form another solution. I may have not only integer solutions but also fraction and decimal solutions. If I repeat this process again and again, then I could have infinitely many solutions for this equation!

Teacher: I like your reasoning. That's correct. (2, 6) is not the only solution to equation 2.

老師： 將 2 代入x，6 代入y。如果等式成立，那麼這個數對就是方程式的解。試著解這個題目，並說說看你的答案。

學生： 在方程式(1)中，我得到 $4-18=-14$，不等於-11。所以這個方程式不成立。因此，(2, 6)不是第一個方程式的解。

學生： 在方程式(2)中，我得到 $8-6-2=0$。這跟題目敘述的答案一樣。所以，(2, 6)是第二個方程式的解。

老師： 正確！讓我更進一步地問，(2, 6) 是方程式(2)的唯一解嗎？

學生： 我想是。當你代入這些數字時，方程式的值是 0，就跟答案一樣！

老師： 我知道。但我想問除了(2, 6) 以外其他可能的答案。你還能將其他數字代入x和y，並仍然得到0作為答案嗎？試著將 1 代入x。你的方程式是什麼？

學生： $4-y-2=0$。

老師： 解出y的值。

學生： $4-y-2=0$，那麼y就是2。

老師： 正確。當x是1，y是2時，方程式成立。我們會說$(1, 2)$是這個方程式的解。還有其他解嗎？

學生： 應該有很多。如果我將不同的數字代入x，我會得到不同的y。不僅可能有整數解，還可能有分數解和小數解。如果我一再重複這個過程，那麼我可能有無限多個解！

老師： 我喜歡你的解釋，沒錯。$(2, 6)$不是方程式(2)的唯一解。

∞ 應用問題 / 會考素養題 ∞

例題一

說明：這題讓學生依照情境列出二元一次方程式。

（英文）The cost of a chocolate cake is \$20 more than twice the cost of a cheesecake. The cost of a cream-filled donut is \$3 less than the cost of a cinnamon donut. The cost of a cheesecake is m, and the cost of a cinnamon donut is n. If Mary bought two chocolate cakes, one cheesecake, three cream-filled donuts, and five cinnamon donuts. How much did she pay? Please show your answer in terms of m and n.

（中文）巧克力蛋糕的價錢比乳酪蛋糕價錢的 2 倍還貴 20 元，奶油內餡甜甜圈的價錢比肉桂甜甜圈價錢便宜 3 元。一個乳酪蛋糕的價格是 m 元，一個肉桂甜甜圈的價格是 n 元。瑪麗買了 2 個巧克力蛋糕，1 個乳酪蛋糕，3 個奶油內餡甜甜圈和五個肉桂甜甜圈，請問總共付多少錢？以 m 和 n 的形式寫出答案。

Teacher: Mary bought a lot of cakes and donuts, and we are helping her to figure out the total price. Let's start with the chocolate cake and cheesecake. The cost of a cheesecake is defined as m, and the cost of a chocolate cake is \$20 more than twice the cost of a cheesecake. Which one is more expensive?

Student: A chocolate cake is much more expensive than a cheesecake. A chocolate cake is twice the cost of a cheesecake plus 20.

Teacher: Correct, "twice the cost of a cheesecake" means 2 times m. Can you describe chocolate cake in terms of m?

Student: $2m + 20$.

Teacher: Yes. Now we move on to the donuts. The cost of a cinnamon donut is defined as n. The cost of a cream-filled donut is \$3 less than the cost of a cinnamon donut. Which flavor is more expensive?

Student: A cinnamon donut.

Teacher: Correct. Can you describe the cost of a cream-filled donut in terms of n?

Student: Cream is cheaper. The cost is $n - 3$.

Teacher: Correct. Now we are going to list an expression to show the total items that Mary bought.

Student: $2(2m + 20) + m + 3(n - 3) + 5n$.

Teacher: Indeed. Please combine like terms to simplify your expression.

Student: $(4m + m) + (3n + 5n) + (40 - 9) = 5m + 8n + 31$.

Teacher: Good job. This is the answer in terms of m and n.

老師： 瑪麗買了很多蛋糕和甜甜圈，我們來幫她算算看總價。讓我們從巧克力蛋糕和起司蛋糕開始。一個起司蛋糕的價格定為 m 元，一個巧克力蛋糕的價格比起司蛋糕的兩倍貴 20 元。哪一個更貴呢？

學生： 巧克力蛋糕比起司蛋糕貴得多。巧克力蛋糕是起司蛋糕價格的兩倍再加上 20 元。

老師： 正確，“起司蛋糕的兩倍”可以寫成 2 乘 n。你能用 m 來描述巧克力蛋糕的價格嗎？

學生： $2m + 20$。

老師： 沒錯。現在我們來看甜甜圈的部分。一個肉桂甜甜圈的價格定為 n 元。一個奶油內餡甜甜圈的價格比一個肉桂甜甜圈便宜 3 元。哪一個更貴呢？

學生： 肉桂甜甜圈。

老師： 正確。你能用 n 來描述奶油內餡甜甜圈的價格嗎？

學生： 奶油比較便宜。價格是$n - 3$。

老師： 正確。現在我們要列式來表示瑪麗所買的總物品數。

學生： $2(2m + 20) + m + 3(n - 3) + 5n$。

老師： 沒錯。請合併同類項並化簡。

學生： $(4m + m) + (3n + 5n) + (40 - 9) = 5m + 8n + 31$。

老師： 很好。這就是用 m 和 n 所表示出來的答案。

例題二

說明：列出二元一次方程式，並找出多組解。

（英文）A fraction becomes $\frac{1}{3}$ when 1 is subtracted from the numerator. Find 3 possible solutions for the original fraction.

（中文）有一分數分子減 1 後，分數為 $\frac{1}{3}$。找出三組可能的解。

Teacher: There are a numerator and a denominator in a fraction. The numerator is the number on the top. The denominator is the number on the bottom. We have two unknown numbers. Use two variables for the equation.

Student: The fraction is x over y ($\frac{x}{y}$).

Teacher: $\frac{x}{y}$ is the original fraction. 1 is subtracted from the numerator. Write the new fraction with variables.

Student: $\frac{x-1}{y}$

Teacher: Okay. We have an equation: $\frac{x-1}{y} = \frac{1}{3}$

Student: I can see the answer. Just compare these two fractions.
The numerator part: $x - 1 = 1$, so $x = 2$. The denominator part: $y = 3$. So, the original fraction is $\frac{2}{3}$.

Teacher: Good job. What about two other possible solutions?

Student: There's no other solution. $x - 1 = 1$, so x can only be 2! This is the only answer.

Teacher: $\frac{1}{3}$ is the simplest form. Did you forget other fractions whose simplest form is $\frac{1}{3}$, such as $\frac{2}{6}$, $\frac{3}{9}$, or $\frac{10}{30}$?

Student: Oh, yes. But in this case, there are infinitely many solutions. Can I pick three random fractions?

Teacher: Yes, you can. But before you pick three random fractions that are equivalent to $\frac{1}{3}$, I would like you to simplify your equation first. It would be helpful when you look for other solutions. Use cross multiplication to simplify it.

Student: 3 times $(x-1)=y$ times 1

$3(x-1)=y$

$3x-3=y$

Teacher: This equation talks about the relationships between x and y. Now you can plug in numbers for x and solve for y.

Student: When $x=1, y=3-3=0$.

Teacher: But, is this a correct solution?

Student: The fraction is $\frac{1}{0}$. Well, it is undefined. So, it is not a solution.

Teacher: Good. Try another number.

Student: When x is 3, y is 6. When x is 10, y is 27.

Teacher: We usually use a table to organize the numbers. You can fill in the number for x first, and then solve for y. We can show all the possible answers and cross out those that don't satisfy the equation. The table is like the one below.

x	1	2	3	4	
y	0	3	6	9	

↑ undefined

Don't forget to write your answers in the form of a fraction— $\frac{2}{3}$, $\frac{3}{6}$, $\frac{4}{9}$, or $\frac{10}{27}$.

老師： 分數分為分子和分母兩個部分。分子在分數的上方，分母在下方。我們有兩個未知數。試著用兩個變數來表示方程式。

學生： 這個分數是$\frac{x}{y}$。

老師： $\frac{x}{y}$是原始分數。從分子減去 1。用變數寫出新的分數。

學生： $\frac{x-1}{y}$。

老師： 好的。現在我們有方程式：$\frac{x-1}{y}=\frac{1}{3}$。

學生： 我知道答案了。只需要比較這兩個分數就好。分子部分是$x-1=1$，所以$x=2$。分母是$y=3$。因此，原來的分數是$\frac{2}{3}$。

老師： 做得好。其他兩個可能的解呢？

學生： 沒有其他解。$x-1=1$，所以x只能是 2！這是唯一的解。

老師： $\frac{1}{3}$ 是最簡分數形式。你忘了還有其他最簡形式也是 $\frac{1}{3}$ 的分數嗎？

例如 $\frac{2}{6}$、$\frac{3}{9}$ 或 $\frac{10}{30}$。

學生： 哦，是的。但在這種情況下，有無限多個解。我可以隨便選三個嗎？

老師： 可以，但在你隨機選擇三個等於 $\frac{1}{3}$ 的分數之前，我想讓你先簡化你的方程式。這會讓你在尋找其他解時很有幫助。使用交叉相乘法簡化它。

學生： 3 乘以 $(x-1)$ 等於 y 乘以 1

$3(x-1)=y$

$3x-3=y$

老師： 這個方程式描述了x和y之間的關係。現在你可以將數字代入x來找出y。

學生： 當$x=1$，$y=3-3=0$。

老師： 但是，這是正確的解嗎？

學生： 分數會變成 $\frac{1}{0}$。好吧，這沒有定義。因此，它不是解。

老師： 很好。試試其他的數字。

學生： 當$x=3$，$y=6$。當$x=10$，$y=27$。

老師： 我們通常會畫表格列出解。你可以先填入x的數字，然後再求解y。我們可以列出所有可能的答案，並刪除不符合方程式的答案。表格如下所示。

x	1	2	3	4	
y	0	3	6	9	

不要忘記以分數的形式寫出答案—— $\frac{2}{3}$、$\frac{3}{6}$、$\frac{4}{9}$ 或 $\frac{10}{27}$。

例題三

說明：依情境列出二元一次方程式，並探討可能的解。

（英文）Given a two-digit-number, A is the tens digit, and B is the units digit. The sum of this number and its reverse is 143. Write a linear equation and solve for all solutions.

（中文）已知某二位數，十位數字是 A，個位數字是 B，若位置對調，所得到的新數字與舊數字之和為 143。列出二元一次方程式，並找出所有可能解。

Teacher: Let me use a two-digit number to explain the relationship between the number and its reverse. 46 is a two-digit number. 4 is the tens digit, and 6 is the units digit. The value of this number is 10× the digit in the tens place + 1× the digit in the units place = $10 \times 4 + 1 \times 6 = 46$. Its reverse is 64. The two digits are interchanged. So, the value is $10 \times 6 + 1 \times 4 = 64$. This question gives you AB as the two-digit number. Which one is the tens digit? Which one is the units digit?

Student: A is the tens digit, and B is the units digit.

Teacher: Can you write the value of this number?

Student: Is that $10 \times A + 1 \times B = 10A + B$?

Teacher: Yes. What about its reverse number?

Student: BA. $10 \times B + 1 \times A = 10B + A$?

Teacher: Yes. The sum of these two numbers is 143. Can you show the expression?

Student: $BA + AB = 143$, $2A + 2B = 143$, $A + B = 71.5$? No way!

Teacher: The sum of these two numbers is 143. A and B are "one-digit" numbers. We will write $(10A+1B) + (10B+1A)$ as the sum of these two numbers.
Can anyone follow up?

Student: $(10A+1B) + (10B+1A) = 143$, so $11A+11B=143$.
$11(A+B) = 143$, so $A+B=13$.

Teacher: How many A and B fit this equation, $A + B=13$? Please note that A and B are integers.

Student: $A=1$, $B=12$; $A=2$, $B=11$…

Teacher: Hold on. If A is 1, B is 12, what will be your two-digit number?

Student: 112?

Teacher: Isn't that a three-digit number? Please note that A and B are "one-digit" numbers.

Student: Oh. $A=4$, $B=9$; $A=5$, $B=8$; $A=6$, $B=7$; $A=7$, $B=6$; $A=8$, $B=5$; $A=9$, $B=4$.
That's all.

Teacher: You can use this table to organize the answers.

A	4	5	6	7	8	9
B	9	8	7	6	5	4

Can you list the possible two–digit numbers?

Student: The numbers can be 49, 58, 67, 76, 85, or 94.

Teacher: Let's double-check the answer. 49 plus its reverse, 94, equals 143. Can you check other numbers?

Student: $58 + 85 = 143$, $67 + 76 = 143$. They are all good.

Teacher: Great!

老師： 讓我用一個二位數來解釋一個數字和它在位數顛倒後之間的關係。46 是一個二位數，其中 4 是十位數，6 是個位數。這個數字的值為10 ×十位數+1 ×個位數等於$10 \times 4 + 1 \times 6 = 46$。46的個位數和十位數對調後變成64，因此值為10 ×個位數+1 ×十位數等於$10 \times 6 + 1 \times 4 = 64$。
題目說AB是一個二位數，哪一個是十位數？哪一個是個位數？

學生： A是十位數，B是個位數。

老師： 你能寫出這個數字的值嗎？

學生： 是不是$10 \times A + 1 \times B = 10A + B$？

老師： 是的。它在位數顛倒後呢？

學生： BA。$10 \times B + 1 \times A = 10B + A$？

老師： 是的。這兩個數字的和是143。你能列出算式嗎？

學生： $BA + AB = 143$，$2A + 2B = 143$，$A + B = 71.5$？不可能！

老師： 這兩個數字的和是143。A和B是一個「位數」。所以我們將$(10A + 1B) + (10B + 1A)$表示成這兩個數字的和。有人可以接下去嗎？

學生： $(10A + 1B) + (10B + 1A) = 143$，$11A + 11B = 143$。
$11(A + B) = 143$，所以$A + B = 13$。

老師： 有多少個 A 和 B 符合這個方程式，$A + B = 13$？請注意A和B是整數。

學生： $A = 1$，$B = 12$；$A = 2$，$B = 11$..

老師： 等等。如果A是1，B是12，那由它們所組成的二位數是多少？

學生： 112？

老師： 那變成是一個三位數了！請注意 A 和 B 是一位數。

學生： 哦。$A = 4$，$B = 9$；$A = 5$，$B = 8$；$A = 6$，$B = 7$；$A = 7$，$B = 6$；$A = 8$，$B = 5$；$A = 9$，$B = 4$。就這些了。

老師： 你可以畫表格來列出答案。

A	4	5	6	7	8	9
B	9	8	7	6	5	4

能列出所有可能的兩位數嗎？

學生： 49、58、67、76、85、或 94。

老師： 我們來驗算。49 加上它在位數顛倒後的數字 94 等於 143。
你能驗算其他數字嗎？

學生： $58 + 85 = 143, 67 + 76 = 143$，都是正確的。

老師： 很好！

單元二 解二元一次聯立方程式
Solving System of Linear Equations

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■ 前言 Introduction

此單元為解二元一次聯方程式，包括代入消去法、加減消去法的教學。教學生解方程式的英文和第一冊解一元一次方程式大同小異，所以老師應該會較得心應手。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中文	單字	中文
system of linear equations 或 linear system	二元一次聯立方程式	**solution of a system of linear equations**	**二元一次聯立方程式的解**
substitution	**代入消去法**	**elimination**	**加減消去法**
original equation	原始的方程式	revised equation	改寫後的方程式
coefficient	係數	like terms	同類項
ordered pair	有序數對	curly bracket	大括弧
least common multiple	最小公倍數	distributive property	分配律

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ Solve for _____ in _____.

例句：**Solve for** *y* **in** equation 1.

解第一條方程式的 *y*。

❷ Substitute _____ for _____.

例句：**Substitute** −1 **for** *x*.

x 用 −1 取代。

❸ Substitute ____ for _____ in _____.

例句：**Substitute** $x + 2$ **for** y **in** equation 2.

第二條方程式的 *y* 用 $x + 2$ 取代。

❹ Subtract _____ from _____.

例句：**Subtract** 5 **from** each side.

兩邊同減 5。

❺ Check the solution _____ by substituting_____.

例句：**Check that** (0, 2) is a solution **by** substituting 0 for *x* and 2 for *y* in each of the original equations.

檢驗(0, 2)是一組解，$x = 0$，$y = 2$ 代入兩個原始的方程式。

❻ Use substitution _____.

例句：**Use the substitution (method)** to solve the linear system.

用代入消去法解二元一次聯立方程式。

❼ the coefficients of _____

例句：**The coefficients of** *y* are opposites.

y 的係數互為相反數。

❽ arrange like terms

例句：**Arrange like terms** in columns.
把直行的同類項排列整齊。

❾ Use elimination _____

例句：**Use the elimination (method)** to solve the system of linear equations.
用加減消去法解二元一次聯立方程式。

❿ Choose a method _____

例句：**Choose a method** to solve the system of linear equations. Explain your choice.
選一個方法解聯立方程式，並且解釋原因。

■ 問題講解 Explanation of Problems

說明

We have learned one equation with two variables, such as $x + 2y = 5$ (read as x plus $2y$ equals 5). In this chapter, we will see two equations at a time, which have two variables. We usually use a curly bracket to indicate that equations form a system. They are called the system of linear equations. There can be more than two equations or more than two variables, but don't worry, we only deal with the easy linear systems in this chapter.

We will learn two different methods to solve the linear systems, one is substitution, and the other is elimination. We will learn to determine whether the ordered pair is a solution of the system of linear equations. Then we will apply the linear system in real-life questions.

運算問題的講解

例題一

說明：老師使用代入消去法解聯立方程式，教學生檢驗答案。再引導學生從不同的方程式開始解題，讓學生自行比較便利性。

（英文）Solve the system of linear equations by substitution.

$$\begin{cases} x+4y=36 \\ x-2y=0 \end{cases}$$

（中文）以代入消去法解聯立方程組。

Teacher: There are two equations and two variables in the linear system. We first solve one of the equations for one of the variables. You can choose either one equation to start. In order to communicate easily, we can label the linear system: the first equation as Equation1, and the second equation as Equation 2.

Let's start with Equation 1: x plus $4y$ equals 36. We solve for x, and x equals 36 minus $4y$. We write $x = 36 - 4y$, and this is revised Equation 1.

Then we substitute $36 - 4y$ for x in Equation 2. We have $36 - 4y - 2y = 0$, read as 36 minus $4y$ minus $2y$ equals 0. Then simplify, 36 minus $6y$ equals 0.

So, y is 6.

To find the value for x, substitute 6 for y in Equation 2. x minus 12 equals 0, so x is 12. We can write the solution in an ordered pair (12, 6). The first number is the solution for x, and the second number is the solution for y.

$x = 36 - 4y$ ····· Revised equation 1
$x - 2y = 0$ ····· Equation 2
$(36 - 4y) - 2y = 0$ ····· Substitute $36 - 4y$ for x.
$36 - 6y = 0$ ····· Simplify.
$y = 6$ ····· Solve for y.
$x - 2(6) = 0$ ····· Substitute 6 for y.
$x - 12 = 0$ ····· Simplify.
$x = 12$ ····· Solve for x.

We have to check that (12, 6) is the solution in each of the original equations. Substitute 12 for x and 6 for y in each equation. In Equation 1, 12 plus 24 is 36; in

equation 2, 12 minus 12 is 0. Now, we solve the case.

We solve for x in Equation 1, and then substitute x in Equation 2. This method is called substitution.

Teacher: We isolate the x variable in Equation 1 as the first step. Now I want you to think about a question, can we isolate the x variable in Equation 2 as the first step?

Student: Let me try. I will have $x = 2y$ if I start with Equation 2. Should I substitute $2y$ for x in the other equation?

Teacher: Yes. So what will you have?

Student: $2y + 4y = 36$. Simplify the equation, and y will be 6. Same answer!

Teacher: Correct. If I ask you to choose which equation you want to start with when isolating the x variable?

Student: I am fine with either one. But I would start with Equation 2 in this question because "$x = 2y$" is easier than "$x = 36 - 4y$" for the simplification later.

Teacher: Smart choice.

老師：有兩條方程式和兩個未知數，我們先解一條方程式中的一個未知數開始，任一條方程式開始都可以。為了便利溝通，我們把第一條方程式標記為(1)，第二條方程式標記為(2)。

我們從方程式(1)開始：$x + 4y = 36$。我們解出x，得到$x = 36 - 4y$，這是重新改寫的方程式(1)。

接著我們把$36 - 4y$代入方程式(2)的 x。

得到$36 - 4y - 2y = 0$，化簡後變成$36 - 6y = 0$。所以，y=6。

為了找到 x 的值，我們把 y 代入方程式(2)。x減12等於0，所以x是12。我們可以用有序對(12,6)表示解答。第一個數字是x的解，第二個數字是y的解。

$x = 36 - 4y$ ……重寫後的方程式(1)

$x - 2y = 0$ ……方程式(2)

$(36 - 4y) - 2y = 0$ ……將$36 - 4y$ 代入 x

$36 - 6y = 0$ ……化簡

$y = 6$ ……得到 y 值

$x - 2(6) = 0$ ……將 6 代入 y

$x - 12 = 0$ ……化簡

$x = 12$ ……得到 x 值

我們必須檢查 $(12, 6)$ 是否是每個原始方程式的解。在每個方程式中，將 12 代

入 x，6代入y。在方程式(1)中，$12 + 24 = 36$；在方程式(2)中，$12 - 12 = 0$。於是，我們找到答案了。

我們從方程式(1)解出 x，然後把x代入方程式(2)中。這種方法被稱為代入法。

老師： 我們首先把方程式(1)中的x變數孤立出來。現在我想讓你思考一個問題，我們可以在第一步就孤立方程式(2)中的 x 變數嗎？

學生： 讓我試試看。如果我從方程式(2)開始，我會得到$x = 2y$。我應該在另一個方程式中將 x 替換為 $2y$ 嗎？

老師： 是的。那你會得到什麼？

學生： $2y + 4y = 36$。簡化方程式，得到$y = 6$。同樣的答案！

老師： 正確。如果我問你，你會選擇從哪個方程式開始孤立x變數呢？

學生： 我兩個都可以。但在這個問題中，我會從方程式(2)開始，因為“$x = 2y$” 比“$x = 36 - 4y$”更容易化簡。

老師： 明智的選擇。

例題二

說明：老師引導學生選擇適合的方程式開始，引導學生做代入消去法，最後再教學生快速的檢驗答案。

（英文）Solve the system of linear equations by substitution.

$$\begin{cases} 3x - 2y = 12 \\ y - x = 60 \end{cases}$$

（中文）以代入消去法解聯立方程組。

Teacher: Please solve the linear system with the substitution method. Which equation do you want to start with? Or which variable do you want to solve first?

Student: Does it matter which to start with?

Teacher: Frankly speaking, the solution should be the same, if your works are all correct. However, it would be easier to isolate the variable in the second equation. If I were you, I would start with Equation 2.

Student: I can see that. There will be a fraction if I solve for x or y in Equation 1, right?

Teacher: Correct. It will be more complicated if we have a fraction in the equation. Now you have chosen Equation 2, which variable do you want to solve first? x or y?

Student: I want to isolate y first. I will have “y equals 60 plus x.”

Does it matter which variable to solve first?

Teacher: Well, in this equation, there's no difference. We can solve either *x* or *y* first. The coefficients for both variables are 1, so there's no fraction when we isolate either variable. Then what's your next step?

Student: Should I solve for the other variable?

Teacher: Yes, that's your next step. Substitute *y* in Equation 1. What will you have?

Student: 3*x* minus 2 times 60 plus *x* equals 12. The equation is 3*x* minus 120 minus 2*x* is 12. (Calculating.) I have the answer for *x*, and it is 132.

Teacher: Good, then what's next?

Student: Because *y* equals 60 plus *x*, *y* equals 192.

Teacher: Ok. Can you double-check the solution?

Student: Do I have to check two equations? Too much work...

Teacher: Well, if you are in a hurry, you don't have time to check two equations.
Let me tell you how to do the one-shot check.
You have used the revised Equation 2 "*y* equals 60 plus *x*" to solve for *y*.

Teacher: Since Equation 1 has not been used to confirm the value of *y* yet, you can check the solution in Equation 1. Substitute 132 for *x* and 192 for *y* in Equation 1.3 times 132 minus 2 times 192 equals 12. So your solution is correct.

老師： 請用代入法解這個聯立方程式。你們會從哪個方程式下手呢？或者說，你想先求哪個未知數呢？

學生： 從哪個開始有影響嗎？

老師： 說實話，只要你的計算都正確，答案應該會是一樣的。但是，在第二個方程式中，將未知數孤立會比較容易。如果我是你，我會從方程式(2)開始。

學生： 我看得出來。如果我在方程式(1)中解x或y，會出現一個分數對吧？

老師： 沒錯。如果方程式中有分數，計算會變得更加複雜。
現在你選擇了方程式(2)，你想先孤立哪個未知數呢？x還是y？

學生： 我想先孤立y。我會得到"$y = 60 + x$"。 先解決哪個未知數會有影響嗎？

老師： 在這個方程式中，沒有區別。我們可以先解決x或y。兩個未知數的係數都是 1，因此在孤立任何一個未知數時都不會有分數。接下來該怎麼做？

學生： 我應該解另一個未知數嗎？

老師： 對，這是下一步。在方程式(1)中取代y。你會得到什麼？

學生： $3x - 2 \times (60 + x) = 12$。方程式是$3x - 120 - 2x = 12$。
（計算中）。我求出x的答案了，是132。

老師： 好的，那下一步呢？
學生： 因為$y = 60 + x$，所以$y = 192$。
老師： 好的。你能驗算嗎？
學生： 我要驗算兩個方程式嗎？這樣太麻煩了……
老師： 好吧，如果你趕時間，你可能會沒空檢查兩個方程式。
讓我告訴你如何快速進行驗算。
你已經使用重寫後的方程式(2)"$y = 60 + x$"來解出 y 了。
因為方程式(1)尚未用於確認 y 的值，所以可以在方程式(1)中進行驗算。
將$x = 132, y = 192$代入方程式(1)。得到$3 \times 132 - 2 \times 192 = 12$。因此，你們算的答案是正確的。

例題三

說明：老師說明使用加減消去法的時機，教學生解聯立方程式，再教學生檢驗答案。

（英文）Solve the system of linear equations by addition.

$$\begin{cases} 4a - 5b = 8 \\ 2a + 5b = 10 \end{cases}$$

（中文）以加減消去法解聯立方程組。

Teacher: Let's start with the substitution first. No matter which equation you start with, no matter which variable you want to solve first, you can see that a fraction will represent each variable. The coefficients of *a* and *b* are not 1.

When it is not easy to isolate one variable, we can try a new method: The elimination method. Add one of the equations to the other one, or add a multiple of one of the equations to the other one. When we combine like terms, one variable will be eliminated. We solve for the remaining variable, substitute the value into one of the other equations, and then solve for the other variable.

Teacher: In the first step, we arrange the equations with like terms in columns. The equations are already arranged, so we don't need to revise the equations.

Next, we add Equation 1 to Equation 2 and combine like terms. Because the coefficients for *b* are opposites, *b* will be eliminated. Negative 5*b* plus positive 5*b* equals 0. You have 6*a* left on the left-hand side, and 18 on the right-hand side. The

equation will be 6a equals 18. The value of a is 3. Then substitute 3 for a in Equation 1 and solve for b.

5b equals 4, and b is 0.8.

$4a - 5b = 8$ ····· Equation 1
$2a + 5b = 10$ ····· Equation 2
$6a = 18$ ····· Add the equations.
$a = 3$ ····· Solve for a.
$4(3) - 5b = 8$ ····· Substitute 3 for a.
$b = 0.8$ ····· Solve for b.

Don't forget to check the solution in the linear system. Substitute 3 for a and 0.8 for b in the original equations. Therefore, (3, 0.8) is the solution.

老師： 讓我們先從代入法開始。無論從哪個方程式開始，或是想先解哪個未知數，你會發現每個變數都會出現帶有分數的表示方式。也就是說，a和b的係數不是 1。當每一個未知數都不容易被孤立時，我們可以嘗試另一種方法：消去法。將其中一個方程式加到另一個方程式中，或是將其中一個方程式的倍數加到另一個方程式中。當合併同類項時，其中一個未知數會被消除。我們解出剩下的未知數，再將其值代入另一個方程式中，然後解出另一個未知數。

老師： 第一步，我們將方程式以對齊同類項的方式上下排好。方程式已經排好了，所以不需要修改方程式。

接著，我們將方程式(1)加到方程式(2)中，合併同類項。由於在兩式中，b 的係數的正負性是相反的，所以 b 將被消去。$-5b + 5b = 0$。左側剩下$6a$，右側剩下18。方程式變成$6a = 18$。得到$a = 3$。然後將3代入方程式(1)中的 a，並解出 b。$5b = 4$，得到 $b = 0.8$。

$4a - 5b = 8$ ····· 方程式(1)
$2a + 5b = 10$ ····· 方程式(2)
$6a = 18$ ····· 方程式(1) +方程式(2)
$a = 3$ ····· 得到a值
$4(3) - 5b = 8$ ····· 將$a =$ 3代入方程式(2)
$b = 0.8$ ····· 得到b值.

不要忘記驗算。將 $a = 3$、$b = 0.8$ 代入原始方程式中。得到解為$(3, 0.8)$。

例題四

說明：老師說明係數不同的解法，教學生使用加法或減法解聯立方程式的時機。

（英文）Solve the system of linear equations by addition or subtraction.

$$\begin{cases} 9a-3b=20 \\ 3a+6b=23 \end{cases}$$

（中文）以加減消去法解聯立方程組。

Teacher: This question is different from the last one. Do you see the difference?

Student: Coefficients?

Teacher: Coefficient is the key point! Sometimes you can add the original equations directly because the coefficients of a variable are already opposites. In this question, we need to multiply one or both equations by appropriate numbers to get opposite coefficients.

Let's target b. We can get the coefficients of b to be opposites by multiplying Equation 1 by 2. The coefficient of b becomes negative 6. Then you add these two equations, and $-6b$ and $+6b$ will be eliminated. Here are the works.

$18a - 6b = 40$ ····· Revised Equation 1.

$3a + 6b = 23$ ····· Equation 2

$21a = 63$ ····· Add the equations

$a = 3$ ····· Solve for a.

$3(3) + 6b = 23$ ····· Substitute 3 for a.

$6b = 14$ ····· Simplify.

$b = \frac{7}{3}$ ····· Solve for b.

Don't forget to check the solution. Substitute 3 for a and $\frac{7}{3}$ for b in Equation 1.

27 minus 7 equals 20. The order paired $(3, \frac{7}{3})$ is the solution.

Teacher: Now, let's try something different. If I multiply Equation 1 by 4 and multiply Equation 2 by 2, what will I get?

Student: $36a - 12b = 80$ and $6a + 12b = 46$

Teacher: Can I add these two equations to cross out b?

Student: Yes, you can. b will be eliminated.

Teacher: $42a = 126$, so a is 3. The answer is the same.

Which method do you think is easier? This one or the previous one.

Student: The previous one. The number is smaller, and I need to operate multiplication once.

Teacher: Because the coefficient of b, 6, is the least common multiple of 3 and 6; while the coefficient 12 is the common multiple of 3 and 6. In the previous method, we multiply a smaller number to get the least common multiple of 3 and 6. So you think it's easier.

Student: Can we first eliminate a instead?

Teacher: This is a good question. Let's target a this time. How to make the coefficients of a be opposites?

Student: 3 times 3 equals 9. But they are not opposites.

Teacher: 9 is the least common multiple of 3 and 9. We multiply Equation 2 by 3, and we will get $9a + 18b = 69$. When we have the same coefficients, we can use subtraction to eliminate the variable. On the left-hand side of the equation, $9a$ minus $9a$ equals 0. a is eliminated while -21b is left. On the right-hand side, you have 20 minus 69.

$9a - 3b = 20$ ……Equation 1

$9a + 18b = 69$ ……Revised Equation 2

$-3b - (18b) = 20 - 69$ ……Subtract the equations. (Equation 1 − Equation 2)

$-21b = -49$ ……Simplify.

$b = \frac{7}{3}$ ……Solve for b.

Substitute $\frac{7}{3}$ for b in Equation 1. a equals 3. Same answers!

When the coefficients of a variable are opposites, we can add these equations to eliminate the variable. When the coefficients of a variable are identical, we can use subtraction.

We have learned substitution and elimination to solve the linear system. No matter which method we apply, the main focus is to eliminate one variable and create a linear equation with the other variable.

We solve for one variable first, and then plug the value in one of the other equations to solve for the remaining variable.

老師： 這道問題和上一題不同，你看出差別了嗎？

學生： 係數？

老師： 係數是關鍵！有時候因為未知數的係數已經是相反數，所以可以直接將原始方

程式相加。在這道問題中，我們需要將一個或兩個方程式乘上適當的數字，以獲得相反的係數。

我們以未知數 b 為目標。通過將方程式(1)乘以 2，我們可以將 b 的係數變成-6。然後，透過將這兩個方程式相加，$-6b$ 和 $+6b$ 便可以被消去。以下是運算過程：

$18a - 6b = 40$ ……改寫後的方程式(1).
$3a + 6b = 23$ ……方程式(2)
$21a = 63$ ……方程式(1) +方程式(2)
$a = 3$ ……得到 a 值
$3(3) + 6b = 23$ ……$a = 3$代入
$6b = 14$ ……化簡
$b = \frac{7}{3}$ ……得到 b 值

別忘了驗算答案是否正確。將 $a = 3$ 和 $b = \frac{7}{3}$ 代入方程(1)。$27 - 7 = 20$。

於是得到 $(3, \frac{7}{3})$ 為解。

老師： 現在，讓我們試試不同的方法。如果我將方程(1)乘以 4，方程式(2)乘以 2，會得到什麼？

學生： $36a - 12b = 80$和$6a + 12b = 46$。

老師： 我可以藉由相加這兩個方程式來消去b嗎？

學生： 可以。b會被消去。

老師： $42a = 126$，因此$a = 3$。答案相同。
哪一種方法你覺得比較容易？這一個還是上一個？

學生： 上一個方法。數字較小，而且我只需要乘一次。

老師： 因為未知數b的係數6，是3和6的最小公倍數；而係數12是3和6的公倍數。在上一個方法中，我們乘以一個較小的數字來得到3和6的最小公倍數。所以你認為這種方法比較容易。

學生： 那我們可以先消除未知數 a 嗎？

老師： 這是個好問題。這次我們以未知數a為目標。如何讓a的係數成為相反數？

學生： $3 \times 3 = 9$。但它們並不互為相反數。

老師： 9是3和9的最小公倍數。我們將方程式(2)乘以3，得到$9a + 18b = 69$。當係數相同時，我們可以使用減法以消除未知數。在方程式的左側，$9a$減去$9a$等於0。a被消除，剩下$-21b$。而在右側，我們得到$20 - 69$

$9a - 3b = 20$…… 方程式(1)

$9a + 18b = 69$…… 改寫後的方程式(2)

$-3b - (18b) = 20 - 69$ 兩式相減

$-21b = -49$…… 化簡

$b = \frac{7}{3}$…… 得到b值

將$b = \frac{7}{3}$代入方程式(1)得到$a = 3$，答案相同！

老師： 當未知數的係數互為相反數時，我們可以將這些方程式相加以消除該未知數；當未知數的係數相同時，我們可以使用減法。

現在我們學會了使用代換法和消元法來解聯立方程式。無論應用哪種方法，主要都是先消除一個未知數，然後得到只有另一個未知數的線性方程式。

我們首先解出一個未知數，再將這個值代入另一個方程式以求剩餘的未知數。

∞ 應用問題 / 會考素養題 ∞

例題一

說明：老師引導學生列式，及解題。

（英文）Hudson is going to the National Science Museum with his family. The price of each adult admission is NT$70. Children with the ages of 4 to 12 pay the child admission, NT$30. Children under the age of 4 will not be charged. There are no children under 4 in his family. If everyone goes to the museum, they will spend NT$390 in total. It's known that there are twice as many children as adults. How many adults and children are going together?

（中文）哈德森要和家人一同逛國家自然博物館。博物館的成人票價是 70 元，4～12 歲的孩童票是 30 元，4 歲以下的孩童免門票。他家沒有 4 歲以下的孩童。如果每個成員都要逛博物館，門票花費總共是 390 元。已知小孩人數是大人的兩倍，請問大人與小孩各有多少人？

Teacher: The first step is to assume the variables. Let's use a for adults, and c for children. How much do they spend on the adult admission tickets?

Student: They will spend 70a.

Teacher: Correct. How much do they spend on the child admission tickets?

Student: 30c.

Teacher: Now we can write the first equation with the information: the total admission fee is

390. Does anyone want to share your equation with us?

Student: $70a + 30c = 390$.

Teacher: Yes. Now we move on to the next information: There are twice as many children as adults. The English idiom "as many as" means "the same quantity as". We can tell the relationship between adults and children. Children's number equals 2 times adults.

We can write $c = 2a$. The linear system is $\begin{cases} 70a + 30c = 390 \\ c = 2a \end{cases}$

Teacher: The next step is to solve the system of linear equations. Which method do you want to apply? Substitution or elimination?

Student: Because $c = 2a$, I want to try substitution first. I can replace c with $2a$.

Teacher: It's a good start. Can anyone give me the equation?

Student: $70a + 30(2a) = 390$. Then simplify, I will have $130a = 390$.

Teacher: Then, what is the solution for a?

Student: a is 3.

Teacher: What's the next step?

Student: Plug in $c = 2a$. So c is 6.

Teacher: Can you verify your answer?

Student: We can substitute a and c in the equation $70a + 30c = 390$. 70 times 3 plus 30 times 6 is 390! Correct answer.

Teacher: Good job. Does anyone have questions?

Student: Can we simplify the equation $70a + 30c = 390$ first? They all have "0." Will they affect the final answer?

Teacher: Thanks for your question. What do other people think about it? Can we simplify $70a + 30c = 390$ first?

Student: Not sure.

Teacher: Let's try it together. We divide every coefficient by 10, and it means we cross out a zero for all coefficients. We will have $7a + 3c = 39$.

Let's plug in a as 3 and b as 6 in this equation. What will you see?

Student: 7 times 3 plus 3 times 6 equals 39.

Teacher: The values of a and c also work in this simplified equation. If we simplify the equation in the first, the answers are still the same. Some people prefer to simplify first, because they want to deal with the smaller numbers.

老師： 首先要假設變數。我們使用 a 代表成人，c 代表兒童。

他們在成人票上花了多少錢？

學生： 會花費 $70a$ 元。

老師： 正確。他們在兒童票上花了多少錢？

學生： $30c$ 元。

老師： 現在我們可以用這些資訊寫出第一個方程式：總門票費用為 390 元。有人想分享一下你列的算式嗎？

學生： $70a + 30c = 390$。

老師： 沒錯。現在我們來看下一個資訊：兒童人數是成人的兩倍。英語中的"as many as"意思是「與…相同的數量」。我們可以知道成人人數和兒童人數之間的關係。兒童人數等於成人人數的兩倍。

我們可以寫出$c = 2a$。所以我們得到聯立方程式 $\begin{cases} 70a + 30c = 390 \\ c = 2a \end{cases}$

老師： 下一步是解這個聯立方程組。你想採用哪種方法？代入法還是消去法？

學生： 因為$c = 2a$，我想先嘗試代入法。我可以用$2a$代替c。

老師： 這是一個不錯的開始。有人能告訴我方程式會變什麼樣子嗎？

學生： $70a + 30(2a) = 390$，然後化簡成$130a = 390$

老師： 那麼 a 的解是多少？

學生： $a = 3$。

老師： 接下來呢？

學生： 將$c = 2a$代入。得到c是6。

老師： 你能驗算你的答案嗎？

學生： 我們可以在方程式$70a + 30c = 390$ 中代入a和c的值。$(70 \times 3) + (30 \times 6) = 390$！答案正確。

老師： 做得好。有人有問題嗎？

學生： 我們可以先簡化方程式$70a + 30c = 390$ 嗎？他們都有 0。這會影響最後的答案嗎？

老師： 謝謝同學的提問。其他人怎麼想呢？我們可以先簡化$70a + 30c = 390$嗎？

學生： 不太確定。

老師： 我們一起試看看，將每個係數都除以10，這表示所有係數的 0 都可以被消去。我們將得到$7a + 3c = 39$。

現在，我們把a代入3，把c代入6，你會得到什麼？

學生： 7 乘 3 加上 3 乘 6 等於 39。

老師： a 和 c 的值也符合這個簡化的方程式。如果我們先簡化方程式，答案仍然相同。有些人喜歡先簡化方程式，因為他們想處理較小的數字。

單元三 直角坐標平面
The Coordinate Plane

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■ 前言 Introduction

學生在這一章節中，從之前學到在數線(number line)表示一個數的位置，進展到用數對(ordered pair)來表示在坐標平面(coordinate plane)上一個點的位置，也發現到坐標平面上的點與數對有一對一的關係(one-to-one correspondence)。接下來會討論平面坐標上各個象限(quadrant)上每個點的有序數列的正負關係，並在坐標平面上畫出(graph)數對代表的點。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中文	單字	中文
rectangular coordinate plane	直角坐標平面	translation	平移
***x*-axis**	***x* 軸**	**quadrant**	**象限**
***y*-axis**	***y* 軸**	Quadrant I	第一象限
origin	**原點**	Quadrant II	第二象限
perpendicular	垂直	Quadrant III	第三象限
***x*-coordinate**	***x* 坐標**	Quadrant IV	第四象限
***y*-coordinate**	***y* 坐標**	graph	畫出坐標平面上的點的“畫”
ordered pair	數對	distance	距離

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ Give the coordinates of _____.

例句：**Give the coordinates of** point *O*.

請找出 *O* 點的坐標。

❷ Graph _____ in the coordinate plane.

例句：**Graph** these ordered pairs **in the coordinate plane**.

請在坐標平面上找到數對的位置。

❸ Name the _____ containing the _____.

例句：**Name the** quadrant **containing the** point.

請問該點在第幾象限。

❹ The _____ is called the _____.

例句：**The** horizontal line **is called the** *x*-axis. The vertical line is called the *y*-axis.

坐標平面上的橫軸稱為 *x* 軸，縱軸為 *y* 軸。

❺ The _____ direction is to the _____ of the origin on the *x*-axis and _____ on the *y*-axis.

例句：**The** positive **direction is to the** right **of the origin on the *x*-axis** and upward **on the *y*-axis.**

x 軸方向往右及 *y* 軸方向往上均表示為正向。

❻ The _____-coordinate of _____ is _____.

例句：**The *x*-coordinate of** a point *P* **is** 5. **The *y*-coordinate of** a point *P* **is** 4.

P 的 *x* 坐標為 5。*P* 的 *y* 坐標為 4。

❼ **The coordinate of _____ is _____.**
_____ has coordinates _____.

例句：**The coordinate of** a point *Q* **is** (6, 5). Or, a point *Q* **has coordinates** (6, 5)
Q 的坐標為(6, 5)

❽ **If the *x*-coordinate is _____ then move the point *x* units to the _____.**

例句：First, locate the *x*-coordinate on the *x*-axis. **If the *x*-coordinate is** positive, **then move the point *x* units to the** right. Then move up or down to locate the *y*-coordinate.
在坐標平面上，如果要標出數對的位置，先看 *x* 坐標的值，如果是正的，則向右移動 *x* 單位，接下來再看 *y* 坐標的值，如果是正的則向上移動 *y* 單位，負的則向下移動 *y* 單位。

❾ **A point _____ is in Quadrant __.**

例句：**A point** *D* **is in Quadrant** IV.
D 點在第四象限。

■ 問題講解 Explanation of Problems

說明

We use an ordered pair of numbers, separated by a coma and expressed as (x, y) to locate a point in the rectangular coordinate plane. The positive direction is to the right of the origin on the *x*-axis and upward on the *y*-axis, and negative direction and vice versa.
我們用數對來表示一個點在坐標平面上的位置，原點向上及向右定義為正向，反之則為負向。

We divide the coordinate plane into four quadrants by *x*-axis and *y*-axis.
我們將坐標平面用 *x* 軸及 *y* 軸分成四個象限。

(5, −4) and (−4, 5) are coordinates that represent two different points.
數對的順序改變也會改變該點的位置，就像(5, −4)及(−4, 5) 是分別代表兩個不同位置的點。

∞ 運算問題的講解 ∞

例題一

說明：本題是跟學生介紹去如何利用畫垂直線跟水平線去找到該點的坐標。

（英文）Give the coordinates of the point *P*, *Q*, *R*, and *S*.
（中文）坐標上有 *P*, *Q*, *R*, *S* 四點，寫出它們的坐標。

Teacher: We draw a vertical line and a horizon line through the point *P*. What is the intersection of the perpendicular line and the *x*-axis?

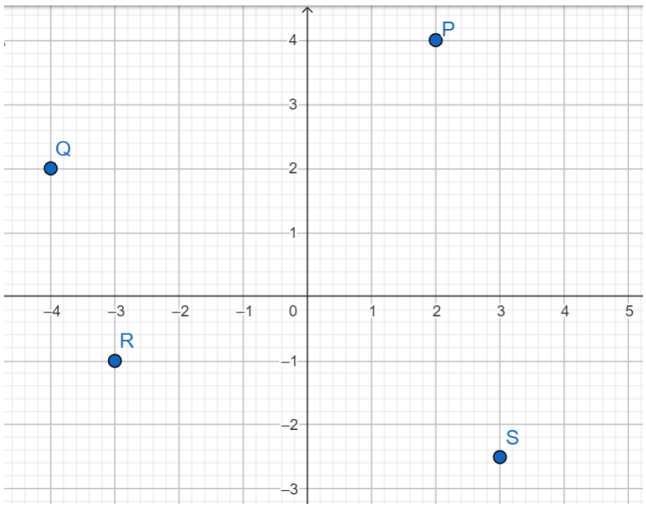
Student: The perpendicular line intersects the *x*-axis at 2.

Teacher: Then we find that the *x*-coordinate of point *P* is 2. What is the intersection of the horizontal line and the *y*-axis?

Student: The horizontal line intersects the y-axis at 4. So, the y-coordinate of the point *P* is 4. Thus, the coordinates of the point *P* is (2, 4).

Teacher: Good job. Then you could use the same method of drawing one horizontal and perpendicular line through a given point to find its coordinates.

Student: I find the coordinates of the point *Q* is (−4, 2), the point *R* is (−3, −1), and the point S is (3, −2.5).



Teacher: Excellent, all the answers are correct. We can move to the next question now.

老師： 我們在 *P* 點畫一條鉛直線和一條水平線。鉛直線和 *x* 軸的交點是什麼？

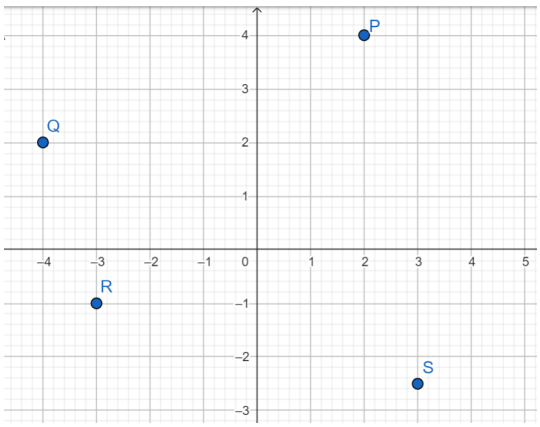
學生： 鉛直線與 *x* 軸的交點是 2。

老師： 所以我們知道點 *P* 的 *x* 坐標為 2。那水平線和 *y* 軸的交點是什麼？

學生： 水平線與 *y* 軸的交點是 4。因此，*P* 點的坐標是(2, 4)。

老師： 做得好。接著你可以使用同樣的方法，從給定的點畫一條水平線和一條鉛直線，以找到它的坐標。

學生： 我知道了 Q 點的坐標是 $(-4, 2)$，R 點的坐標是$(-3, -1)$，S 點的坐標是$(3, -2.5)$。



老師： 太棒了，都答對了。我們現在可以繼續下一個例題。

例題二

說明：學生能從在坐標平面上標出點的位置中了解到 x 坐標及 y 坐標的正負及他們的數字所代表的意思，並發現到坐標(m, n)和坐標(n, m)可能會代表不同的點。

（英文）Graph these ordered pairs: $(-1, 2)$, $(2, -1)$, and $(-1.5, -3)$.
（中文）在坐標平面上標出各點的位置：$(-1, 2)$、$(2, -1)$、$(-1.5, -3)$

Teacher: Let's graph the ordered pair $(-1, 2)$ first. Please identify the x-coordinate and the y-coordinate.

Student: The x-coordinate is -1, and the y-coordinate is 2.

Teacher: Correct. Since the x-coordinate is -1, to find the ordered pair $(-1, 2)$, we start from the origin and move to the left by one unit. Then, do you know which direction and how many units you should move to locate the y-coordinate?

Student: Yes, I can do that. The y-coordinate is positive 2.
I will move up two units to find the y-coordinate.

Teacher: Please graph the following two ordered pairs: $(2, -1)$ and $(-1.5, -3)$.

Student: For the ordered pair $(2, -1)$, I will move two units to the right from the origin to locate the x-coordinate and move one unit downward to locate the y-coordinate.

Teacher: Excellent. Please compare the location of ordered pairs $(-1, 2)$ and $(2, -1)$. Do those two points overlap, or are they in a different location?

Student: No, they are in different locations.

The point $(-1, 2)$ is in Quadrant II, but the point $(2, -1)$ is in Quadrant IV.

Teacher: Yes, your observation is exact. We can conclude that the ordered pair (m, n) and (n, m) represent two different points if $m \neq n$. Now please graph the last ordered pair.

Student: For the ordered pair $(-1.5, -3)$, I will move one and a half units to the left from the origin to locate the x-coordinate and move 3 units downward to locate the y-coordinate.

Teacher: Good job.

老師： 讓我們先標出 $(-1, 2)$，請確認 x 坐標和 y 坐標。

學生： xx坐標是-1，y 坐標是 2。

老師： 正確。因為 x 坐標是 -1，要找到 $(-1, 2)$，我們從原點向左移動一個單位。然後，你知道要移動多少個單位和方向才能找到 y 坐標嗎？

學生： 知道。y 坐標是正 2，所以向上移動兩個單位就會找到 y 坐標。

老師： 請標出以下兩個座標的位置：$(2, -1)$ 和 $(-1.5, -3)$。

學生： $(2, -1)$的話，我將從原點向右移動兩個單位來找到 x 坐標，向下移動一個單位來找到y坐標。

老師： 很棒。請比較$(-1, 2)$和$(2, -1)$的位置。這兩個點是重疊還是位於不同位置？

學生： 它們位於不同的位置。點$(-1, 2)$位於第二象限，但點$(2, -1)$位於第四象限。

老師： 是的，你的觀察是正確的。我們可以得出結論，如果(m, n)和(n, m)表示兩個不同的點，則$m \neq n$。現在請標出最後一點。

學生： $(-1.5, -3)$，我將向左移動 1.5 單位來找到 x 坐標，向下移動 3 個單位來找到 y 坐標。

老師： 做得好。

∞ 應用問題 / 會考素養題 ∞

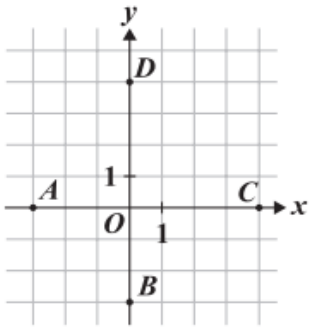
例題一

說明：學生如果了解坐標平面上點的坐標，和水平線及垂直線的概念，就可以順利找到答案。

（英文）There are four points A, B, C, and D in the coordinate plane. The origin is O. If there is a line L passing through the point $(-3, 4)$ and perpendicular to the y-axis, which one of the following four points would L pass through?

（中文）圖（四）的坐標平面上有原點 O 與 A、B、C、D 四點。若有一直線 L 通過點 $(-3, 4)$ 且與 y 軸垂直，則 L 也會通過下列哪一點？

(A) A　(B) B　(C) C　(D) D



圖（四）

（108 年國中會考 7）

Teacher: Plot the point $(-3, 4)$ in the plane first.

Student: I would start from the origin, moving 3 units to the left and moving 4 units upward.

Teacher: According to the question, L is perpendicular to the y-axis. What kind of line is L? Vertical, or horizontal?

Student: L is a horizontal line through the point $(-3, 4)$. I would draw a horizontal line through the point and check which point the line would intersect at. After I draw L, I find that L passes through $(0, 4)$, which is the location of point D.

Teacher: Well Done. The answer is point D.

老師：首先在平面上畫出點$(-3, 4)$。

學生：我會從原點開始，向左移動3個單位，再向上移動4個單位。

老師：題目說，直線L與y軸垂直。那麼直線L會是什麼樣的線段？鉛直的還是水平的？

學生：L會是通過點$(-3, 4)$的水平直線。我會畫一條水平線段穿過這個點，然後檢查線段會穿過哪個點。畫完L之後，我發現L穿過點$(0, 4)$，也就是D點所在的位置。

老師：做得好。答案是D點

例題二

說明：讓學生練習判斷該點是在哪一個象限或是哪一個坐標軸上。

（英文）Name the quadrant or axis containing the point.

(1) (5, 2)　(2) (4, −3)　(3) (−1, −1)　(4) (−2, 0)

（中文）判別下列各點分別在哪一象限內或哪一坐標軸上？

(1) (5, 2)　(2) (4, −3)　(3) (−1, −1)　(4) (−2, 0)

Teacher: The *x*- and *y*-axis divide the coordinate plane into Quadrants I, II, III, and IV. In Quadrant I, both x-coordinate and y-coordinate are positive. In Quadrant II, the *x*-coordinate is negative, and the y-coordinate is negative. In Quadrant III, both *x*-coordinate and y-coordinate are negative. In Quadrant IV, the x-coordinate is positive, and the y-coordinate is negative.

For the first point (5, 2), both *x*-coordinate and *y*-coordinate are positive. Therefore, it is in Quadrant I. Now name the quadrant containing the following points.

Student: For the second point (4, −3), the *x*-coordinate is positive, and the *y*-coordinate is negative; therefore, it is in Quadrant IV. For the third point, the *x*-coordinate is negative, and the *y*-coordinate is negative, it is in Quadrant III. For the last point, the *y*-coordinate is 0; therefore, it is on the *x*-axis.

Teacher: You did a good job. All the answers are correct.

老師：x 軸和 y 軸將坐標平面分成第一、第二、第三和第四象限。在第一象限中，x坐標和y坐標都是正的。在第二象限中，x 坐標為負，y 坐標為正。在第三象限中，x 坐標和 y 坐標都是負的。在第四象限中，x 坐標為正，y 坐標為負。

第一個點 $(5,2)$，x 坐標和 y 坐標都是正的，因此它在第一象限。

現在判別其他點分別在哪一象限。

學生：第二個點 $(4,-3)$，x坐標為正，y坐標為負，因此在第四象限。第三個點，x坐標為負，y坐標為負，它在第三象限。而最後一個點，y坐標為0；因此，它在x軸上。

老師：做得很好，都答對了。

例題三

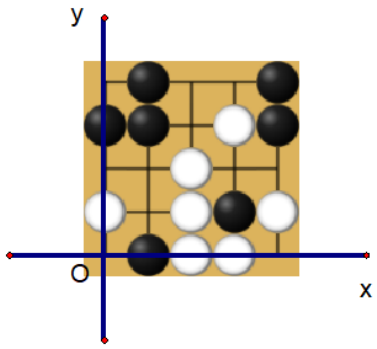
說明：利用圍棋下棋的位置，及氣的概念，讓學生練習用坐標平面去描述黑棋及白棋的位置。

（英文）Go(圍棋) is a game between two players, called Black and White. We played Go on a plane grid of 19 horizontal and 19 vertical lines, called a *board*. The token used to play in Go is known as stones. At any time in the game, each intersection on the board is in one and only one of the following three states: (1) empty, (2) occupied by a black stone, or (3) occupied by a white stone.

(A) If you draw the x-axis and y-axis in the graph below, and the coordinates of point *C* is (2, 0) and the coordinates of point *A* is (0, 3), please follow the subsequent instructions.
(1) Give the coordinates of one intersection which is empty.
(2) Give the coordinates of one intersection which is occupied by a black stone.
(3) Give the coordinates of one intersection which is occupied by a white stone.

（中文）圍棋是一個由黑白二色棋子進行對弈的遊戲。我們在一個由 19 條水平線和 19 條鉛直線組成的平面網格上下圍棋，這個網格被稱為棋盤。使用的棋子英文稱為 stones。在遊戲進行中，棋盤上的每個交叉點只有以下三種狀態之一：(1)空的，(2)被黑子佔據，或(3)被白子佔據。

(A)如果你在下圖中畫出 x 軸和 y 軸，並且 C 點的坐標為(2, 0)， A 點的坐標為(0, 3)，請按照以下指示進行。
(1)給出一個空的交叉點的坐標。
(2)給出一個被黑子佔據的交叉點的坐標。
(3)給出一個被白子佔據的交叉點的坐標。



Teacher: The empty intersections are (0, 0), (1, 2), (4, 0), etc. Find the coordinates of a black stone that is on the axes.
Student: (1, 0).
Teacher: Correct. Find the coordinates of a white stone that is not on the axes.
Student: (2, 1).
Teacher: Great. Now we move to the next step to see when a stone will be captured.

老師： 空白交叉點是(0,0), (1,2), (4,0).....等。找出一個位於坐標軸上的黑子坐標。
學生： (1,0)。
老師： 正確。找出一個不在坐標軸上的白子坐標。
學生： (2,1)。
老師： 很好。現在我們進入下一個步驟，看看什麼情況下棋子會被吃掉。

（英文）**(B)** A *position* consists of an indication of the state of each intersection. In a given position, the *liberty* of a stone is an empty intersection adjacent to that stone or adjacent to a stone which is connected to that stone. A player can remove from the board any stones of their opponent's color that have no liberties. We say that the stones removed from the board have been *captured* by the player.

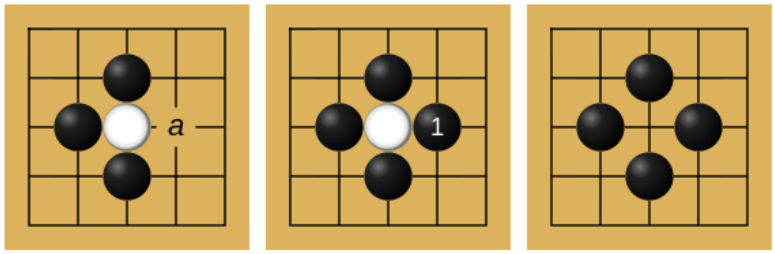
Here are two examples in which the capture rule is applied.

（中文）(B) 佈局是指每個交叉點狀態的指示。在特定的棋局中，與棋子直線相連的空白交叉點叫做氣。玩家可以從棋盤上移除任何沒有氣的對手顏色棋子。當沒有氣的棋子被拿離棋盤，我們「吃子」或「提子」。

以下舉兩個「提子」的例子。

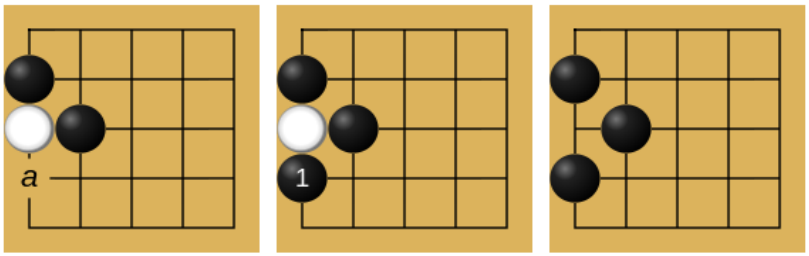
（英文）The diagrams below show the capture of a white stone by Black. Since the white stone has a single liberty at a, if the black stone is placed at a, then the white stone has no liberty left and will be captured and removed.

（中文）下圖顯示了黑方捕獲白方棋子。由於白子只有一個氣在 a 位置，如果黑子下在 a，則白子就沒有氣了並且會被吃（提）。



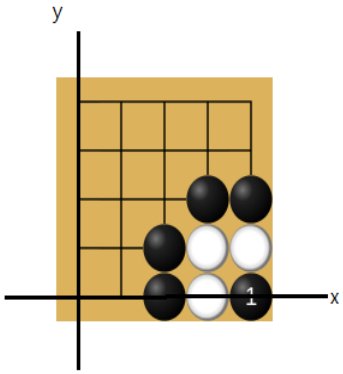
（英文）At the edge of the board and especially in the corners, stones have fewer liberties than the one in the interior and are more easily captured. The white stone at the edge has only one liberty left, and after the black stone is placed at 1, the white stone has no liberty left and is removed. Please find the diagrams below.

（中文）在棋盤邊緣以及特別是在角落位置，棋子的氣比內部的少，因此更容易被吃（提）。在邊緣的白棋只剩下一個氣，當黑棋放在 1 處後，白棋就沒有氣了而被吃。請參見下面的圖示。



（英文）If the black stone “1” is at (4, 0), give the coordinates of the white stone that will be captured after the black stone “1” is placed.

若黑子 1 下在坐標平面(4, 0)的位置，則會被吃（提）的白子坐標為何？



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（圖片及規則取自 https://en.wikipedia.org/wiki/Rules_of_Go）

Teacher: From the instruction above, we can tell all three white stones have no liberty left and will all be captured after the black stone “1” is placed. Give the coordinates of the white stones that are captured.

Student: (3, 0), (3, 1), and (4, 1).

Teacher: Excellent. There are more rules to learn about “Go.” If you are interested in this game, you can use the following website as a reference or play with your friends interested in this game.

老師： 從上面的指示可以看出，當黑子 1 被下在棋盤上，三顆白子都沒有「氣」，它們就都會被吃掉。請給出被吃掉的白子坐標。

學生： (3, 0), (3, 1)以及(4, 1)。

老師： 非常好。還有更多關於“圍棋”的規則需要學習。如果你對這個遊戲感興趣，你可以使用以下網站作為參考或者和你對這個遊戲感興趣的朋友一起玩。

單元四 二元一次方程式的圖形
Graphs of Linear Equations in Two Variables

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■ 前言 Introduction

二元一次方程式的圖形是一直線。老師在介紹本單元時可以先由例題一的引導讓學生發現找出的數組解在坐標平面上都是共線的，進而了解二元一次方程式的圖形是一直線。同時以圖形配合英語敘述如何在坐標平面上標示坐標及繪出直線。之後介紹畫二元一次方程式的圖形時，只要找出任意兩組解，再畫一條直線通過這兩點，該直線就是這個二元一次方程式的圖形。接下來介紹水平線與鉛垂線的方程式，及二元一次方程組的圖形。最後由加入二元一次方程式（組）圖形的應用及二元一次方程式解的幾何意義。因為本章節的範例及概念內容較多，建議在例題時老師能反覆提問，讓學生能熟悉本章節的英文名詞及用語後，再以應用問題來加深學生們對二元一次方程式圖形的了解。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中文	單字	中文
coordinate plane	**坐標平面**	**vertical line**	**鉛直線**
coordinate axis	**坐標軸**	**horizontal line**	**水平線**
quadrant	**象限**	linear equation in/with two variables	二元一次方程式
intersection	相交；交集	system of linear equations in two variables	二元一次方程組
備註 1. 二元一次方程式(linear equation in two unknowns) 在美國中小學課程內容常用 linear equation in two variables。			

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ plot _____ in _____.

例句：**Plot** the five ordered pairs **in** a coordinate plane.
將這五組解（數對）標示在坐標平面上。

❷ There are infinitely many _____.

例句：**There are infinitely many** solutions in a linear equation with two variables.
二元一次方程式有無限多組解。

❸ It is a good idea to_____.

例句：**It is a good idea to** find a third solution just to check.
用第三組解來檢查答案是否正確是不錯的主意。

❹ _____ pass through _____.

例句：The graph of a linear equation in two variables **passes through** the points, A and B.
這直線方程式的圖形通過 A、B 兩點。

■ 問題講解 Explanation of Problems

ᘓ 說明 ᘒ

In the former section, we learned that there are infinitely many solutions in a linear equation with two variables. In this section, we will discuss how to draw a graph of a linear equation in two variables. A graph of a linear equation in two variables consists of all the points that are the graphs of the solutions of the equation. Therefore, by plotting all the ordered pairs of the solutions to the equation in two variables, we can recognize a pattern that these points are all on the same line.

A linear equation is an equation with two variables whose graph is a line. To graph a linear equation, you need to have at least two points. First, draw a linear equation in two variables by plotting more points on the coordinate plane. Then read the pattern of these points carefully.

To draw the graph of any linear equation in two variables with the standard form $Ax + By = C$, first, graph it by finding any two solutions (x_1, y_1) and (x_2, y_2). Then plot these two points. Last, draw a line connecting them. It is convenient to use the two points where the graph crosses the axes to determine the graph of the line.

∞ 運算問題的講解 ∞

例題一

說明：觀察找出的解標示在坐標平面上都是共線的。學生已經於之前的單元學過二元一次方程式有無限多組解，老師引導學生將二元一次方程式的五組解的數對當成坐標標示在坐標平面上，且觀察其形成的圖形都在一直線上。

（英文）Find the solutions to the linear equation $x + 2y = 6$ by choosing a couple of values for x: -2, 0, 2, 4 and 6. Then plot these solutions on the coordinate plane.

（中文）找出方程式 $x + 2y = 6$ 的五組解，並將它們標示在坐標平面上。

x	-2	0	2	4	6
y		3			
(x, y)		(0, 3)			

Teacher: We have learned how to find the solutions to a linear equation before. Please calculate the corresponding y values and find the ordered pairs now. You have two minutes.
Student: OK.
Teacher: Time's up. Please check your answers and see if you got all of the answers correct.

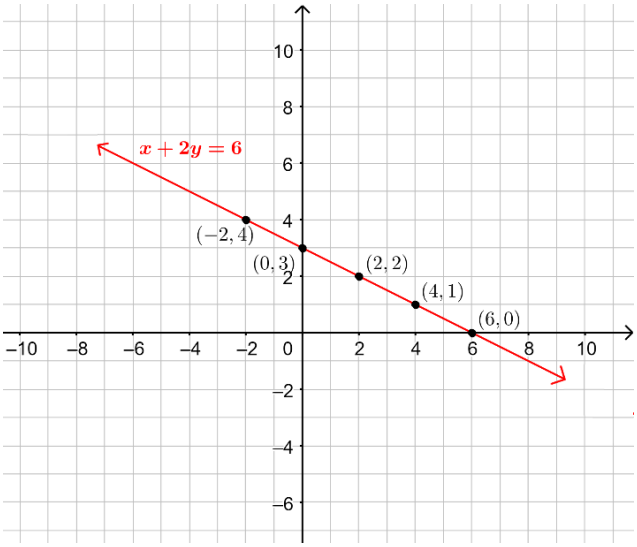
x	-2	0	2	4	6
y	4	3	2	1	0
(x, y)	$(-2, 4)$	(0, 3)	(2, 2)	(4, 1)	(6, 0)

Teacher: Now, let's plot the five ordered pairs on the coordinate plane. Tell me something about the graphs of these points.

Student: They are on the same line.

Teacher: Yes, you can find that the graphs of these solutions lie on the same line. There are many other solutions to the equation $x + 2y = 6$, such as $(1,\ 2\frac{1}{2})$, $(3,\ 1\frac{1}{2})$, and $(-1,\ 3\frac{1}{2})$. If you mark these graphs of the solutions on the same coordinate plane, you will find that these points also lie on the same line. You can also find that the y value is one unit down when x is two units right.

So, the equation $x + 2y = 6$ is called a **linear equation** because the graph is a line.



老師： 我們以前已經學過如何解一元一次方程式。請在 2 分鐘找出五組對應的y值。

學生： 好。

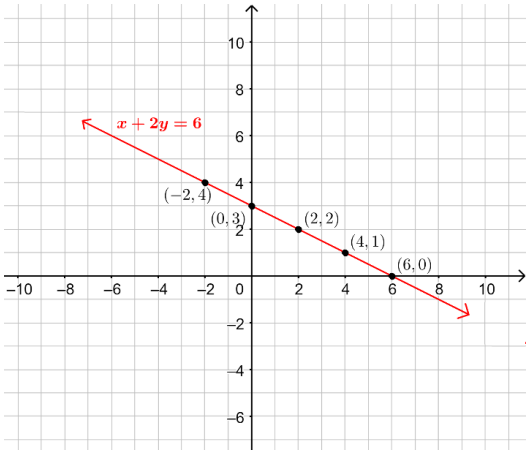
老師： 時間到囉，看看你們的答案是不是都正確。

接下來讓我們在坐標平面上標出這五個點。這些點的圖形有什麼特點呢？

學生： 它們都在同一條直線上。

老師： 是的，你可以發現這些解的圖形都在同一條直線上。方程式$x + 2y = 6$還有其他很多組解，例如 $(1, 2\frac{1}{2})$、$(3, 1\frac{1}{2})$ 和 $(-1, 3\frac{1}{2})$ 等。

如果你在同一個坐標平面上標出這些解的圖形，你會發現這些點也在同一條直線上。因此，方程式$x + 2y = 6$被稱為線性方程式，因為它的圖形是一條直線。



例題二

說明：找出二元一次方程式的兩組解，將對應的點描在座標平面上，再畫出該方程式的直線圖形。因通過不同的兩點的直線只有一條，所以要畫二元一次方程式的圖形時，只要找出任意兩組解，把它們對應的點標在座標平面上，然後畫出一條直線通過這兩點，該直線即為這個二元一次方程式的圖形。一般最便捷的求解方法是找出方程式與 x 軸($y = 0$)及 y 軸($x = 0$)的交點。同時，我們可以找出第三組解來檢查所畫圖形是否正確。

（英文）Graph $2x - 3y = 6$ on a coordinate plane.
（中文）在座標平面上，畫出方程式 $2x - 3y = 6$ 的圖形。

Teacher: From Example 1, we know that the graph of a linear equation in two variables is a line. Since two different points determine a line, we can graph the line $2x - 3y = 6$ by finding two solutions first. Then we can plot them on the coordinate plane. Last, let's draw a line connecting these two points. Now, give me two solutions to this linear equation.

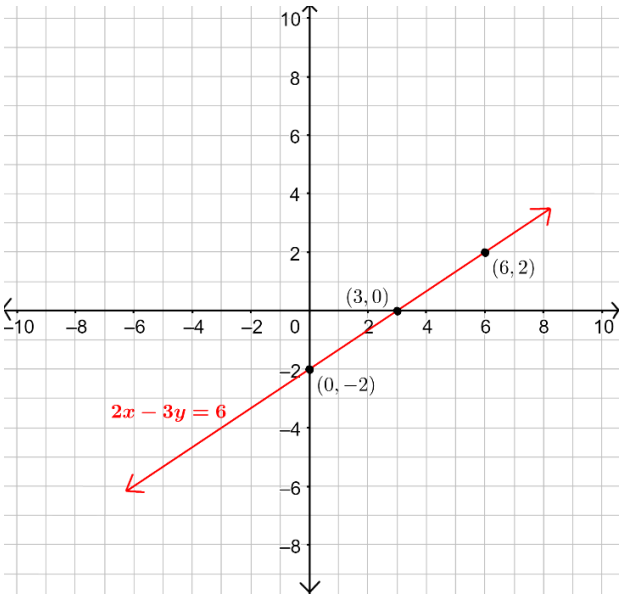
Student: (3, 0) and (6, 2)

Teacher: Very good. But the most straightforward solutions are those where the line crosses the x-axis ($y = 0$) and the y-axis ($x = 0$).
Here you can find the point (3, 0) where the line crosses the x-axis. What about the point where the line crosses the y-axis?

Student: (0, −2)

Teacher: Excellent! Let's plot these points on a coordinate plane and draw a line through them.

This line is the linear equation graph $2x - 3y = 6$., It is a good idea to find a third solution just to check.



老師： 從例題一，我們知道二元一次方程式的圖形是一條直線。由於兩個不同的點可以決定出一條直線，所以要畫出方程式$2x - 3y = 6$的圖形，我們可以先找到它的不同兩組解，然後標記在坐標平面上。接著就可以畫出一條連接這兩個點的直線。現在，試著找出這題一次方程式的兩個解。

學生： $(3, 0)$和$(6, 2)$。

老師： 很好。但最方便的解是找出線與x軸$(y = 0)$和y軸$(x = 0)$的交點。
你已經找到直線與x軸相交的點$(3,0)$。那麼該直線與y軸相交的點呢？

學生： $(0, -2)$。

老師： 很好！將這兩點標記在坐標平面上，並畫出一條通過它們的直線。這條直線就是線性方程式$2x - 3y = 6$的圖形。我們可以再找一組解來驗證圖形是否正確。

例題三

說明：找出直線與 x 軸、y 軸的交點來畫出該二元一次方程式的圖形，並判斷此圖形不通過哪一象限。老師引導學生用 $y = 0$ 或 $x = 0$ 來找出直線與 x 軸、y 軸的交點。畫出該二元一次方程式的圖形後，即可判斷該方程式的圖形的直線不通過哪一象限。

（英文）(a) Find the intersections where the line $3x + 4y = 12$ crosses the x-axis and the y-axis.

(b) Graph the linear equation $3x + 4y = 12$. Then determine which quadrant the linear equation graph doesn't pass through.

（中文）(a) 找出二元一次方程式 $3x + 4y = 12$ 與 x 軸、y 軸的交點坐標。

(b) 畫出方程式 $3x + 4y = 12$ 的圖形，並判斷此圖形不通過哪一象限。

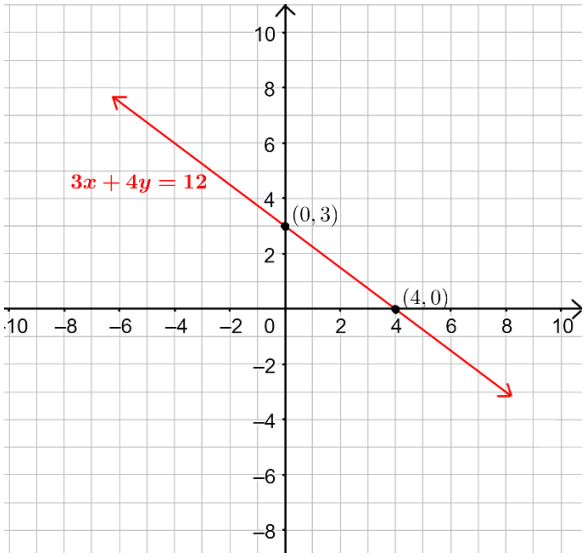
Teacher: Let's find out the points where the graph of the equation crosses both of the axes. First, let's find the point where the line intersects with the y-axis by setting $x = 0$. What is the value of y?

Student: The y-value is 3.

Teacher: Good. So what is the x-value when $y = 0$ (the intersection with x-axis)?

Student: Four. ($x = 4$)

Teacher: Great. By setting $x = 0$ or $y = 0$, we get the points (4, 0) and (0, 3) where the line $3x + 4y = 12$ crosses the x-axis and y-axis. And this is the answer to Part (a). We can plot the points (4, 0) and (0, 3) on the coordinate plane. Then we can draw the line that connects both points. Tell me which quadrant this line doesn't pass through.



Student: It does not pass through the third quadrant. (Quadrant III).

Teacher: Yes, you are right.

老師： 我們一起找出這條直線與兩個坐標軸相交的點。先讓$x = 0$，找出該直線與y軸的交點。y 值是多少？

學生： $y = 3$。
老師： 很好。那麼當$y = 0$時，x值是多少？
學生： $x = 4$
老師： 很好。設 $x = 0$或 $y = 0$，可以得到直線 $3x + 4y = 12$ 與 x 軸和 y 軸分別相交於$(4, 0)$ 和 $(0, 3)$。這是問題 (a) 的答案。
接下來，我們可以在坐標平面上標出 $(4, 0)$ 和 $(0, 3)$ 兩點，然後畫一條連接兩點的直線。這條線不通過哪個象限？
學生： 它不通過第三象限。
老師： 答對了！

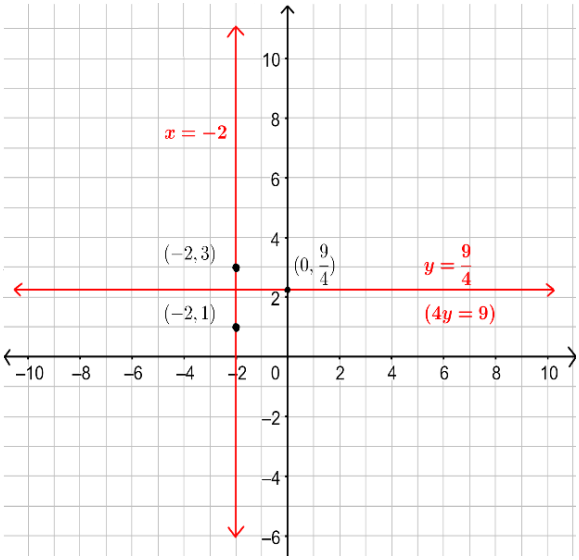
例題四

說明：討論鉛垂線$(ax = c)$及水平線$(by = c)$的圖形。老師引導學生找出方程式 $x = -2$ 的兩組解$(-2, 1)$和$(-2, 3)$，然後在座標平面上畫出這兩點及通過這兩點的直線。再由學生自行練習畫出方程式 $4y = 9$，讓學生觀察其圖形為水平線。
由例題四方程式的圖形討論鉛垂線$(ax = c)$及水平線$(by = c)$的圖形。

（英文）(a) Graph $x = -2$ on a coordinate plane.
(b) Graph $4y = 9$ on a coordinate plane.
（中文）在座標平面上畫出下列方程式的圖形。
(a) $x = -2$ (b) $4y = 9$

Teacher: The equation $x = -2$ places no restriction on y. All points with x-coordinate -2 are graphs of solutions. So, please find any two solutions to $x = -2$.
Student: $(-2, 1)$ and $(-2, 3)$
Teacher: Good. Let's plot these two points and draw a line connecting them. You can tell this line is a vertical line. Now it's your turn to draw the equation of the second part $4y = 9$. You have three minutes.
(After three minutes)
Teacher: OK. Time is up. Are there any special parts of the graph?
Student: A horizontal line!

Teacher: Yes, you are right! It's a horizontal line perpendicular to the *y*-axis at $(0, \frac{9}{4})$. From Example 4, we can tell that the graph of the equation $ax = c$ $(a \neq 0)$ is a vertical line that is perpendicular to the *x*-axis at the point $(\frac{c}{a}, 0)$. And the graph of the equation $by = c$ $(a \neq 0)$ is a horizontal line that is perpendicular to the *y*-axis at the point $(0, \frac{c}{b})$.



老師： 第一個方程式$x = -2$並未限制 y 的值，因此所有 x坐標為 -2 的點都是方程式的解。所以，請找出$x = -2$的任意兩個解。

學生： $(-2, 1)$和$(-2, 3)$

老師： 很好。讓我們將這兩個點繪製在坐標平面上，然後畫一條連接它們的直線。你可以看出這條線是一條鉛直線。現在換你來繪製第二個方程式 $4y = 9$ 的圖形。你有三分鐘時間。

（三分鐘後）好的，時間到了。圖形有特別的地方嗎？

學生： 有，是一條水平線！

老師： 沒錯！它是一條水平線，與y軸垂直於點 $(0, \frac{9}{4})$。從這題，我們可以知道：方程式 $ax = c(a \neq 0)$的圖形是一條鉛直線，垂直於 x軸，通過點 $(\frac{c}{a}, 0)$；

方程式 $by = c(a \neq 0)$的圖形是一條水平線，垂直於y軸，通過點 $(0, \frac{c}{b})$。

例題五

說明：求圖形通過已知一點的二元一次方程式。已知方程式 $ax - 3y = 6$ 其圖形通過點 $(-3, 2)$，則$(-3, 2)$即為方程式的解。

求 a 值只需將 -3 和 2 代入方程式 $ax - 3y = 6$ 中，即可得到 a 的值。

（英文）The graph of the linear equation $ax - 3y = 6$ passes through the point $(-3, 2)$. Find the value of a.

（中文）已知二元一次方程式 $ax - 3y = 6$，其圖形通過點$(-3, 2)$，試求 a 值。

Teacher: The graph of the equation $ax - 3y = 6$ passes through $(-3, 2)$. We can use -3 and 2 to substitute the values of x and y in the equation.
So, we get $a \times (-3) - 3 \times 2 = 6$ (a times negative three, then minus three times two, and the value is equal to six.) Please find the value of a right now.
(After 30 seconds)

Teacher: What is the value of a?

Student: Negative four. ($a = -4$)

Teacher: Very good. Remember that the point in a line is also a solution to the equation. So, we can find the value of a when we know the x and y values from the given coordinates.

老師： 方程式$ax - 3y = 6$的圖形經過$(-3, 2)$這個點。我們可以把-3和2代入方程式中的x和y的值。因此，我們得到$a \times (-3) - 3 \times 2 = 6$。現在請算出$a$。
（30 秒後）a是多少？

學生： $a = -4$

老師： 很好。請記住，線上的點也是方程式的解，因此我們可以通過已知坐標來找到a的值。

例題六

說明：求圖形通過已知兩點的二元一次方程式，並判斷任一點是否也在該方程式的圖形上。已知方程式 $y = ax + c$，其圖形通過 $A(-1, 0)$及 $B(8, 6)$兩點。

將 $A(-1, 0)$及 $B(8, 6)$代入方程式 $y = ax + c$ 中，即可得到聯立方程組 $\begin{cases} 0 = -a + c \\ 6 = 8a + c \end{cases}$。

解出 a 與 c 之值，即可獲得代表該直線的二元一次方程式。接下來，將 C 點座標代入(a)小題中的直線方程式，找出 k 之值。

（英文）The graph of the linear equation $y = ax + c$ passes through the points $A(-1, 0)$ and $B(8, 6)$.

(a) Find the values of a and c.　(b) Find the value of k if $(0, k)$ is also on this line.

（中文）已知二元一次方程式 $y = ax + c$，其圖形通過 $A(-1, 0)$及 $B(8, 6)$兩點。

(a)試求 a、c 之值。

(b)如果點 $C(0, k)$也在方程式 $y = ax + c$ 的圖形上，求 k 的值是多少。

Teacher: The graph of the equation $y = ax + c$ passes through $A(-1, 0)$ and $B(8, 6)$. We can use these values to replace x and y in the equation.

So, we get $\begin{cases} 0 = -a + c \\ 6 = 8a + c \end{cases}$

Please find the values of a and c of the equation system.

(After a few minutes)

Teacher: What are the values of a and c?

Student: a and c are both equal to two-thirds ($a = c = \frac{2}{3}$).

Teacher: Very good. So, the equation of the line is $y = \frac{2}{3}x + \frac{2}{3}$. Then, we can use 0 and k as the values of x and y in the equation $y = \frac{2}{3}x + \frac{2}{3}$. The reason is that point C is on the graph of the equation. Find the answer to Part (b) now.

Student: The value of k is $\frac{2}{3}$.

Teacher: Excellent.

老師： 已知方程式 $y = ax + c$ 的圖形經過 A(-1,0) 和 B(8,6)。我們可以將座標代入方程式中的 x 和 y。

因此，可以列出 $\begin{cases} 0 = -a + c \\ 6 = 8a + c \end{cases}$。請找出聯立方程式中 a 和 c 的值。

（幾分鐘後）

老師： a 和 c 是多少？

學生： a 和 c 都等於 $\frac{2}{3}$。

老師： 非常好。所以，這條線的方程式是 $y=\frac{2}{3}x+\frac{2}{3}$。然後，我們可以使用 0 和 k 作為方程式的 x 和 y 值。原因是點 C 在方程式的圖形上。現在回答(b)小題。

學生： k 是 $\frac{2}{3}$。

老師： 不錯喔。

例題七

說明：求二元一次聯立方程式圖形的交點座標，並畫出兩個方程式的圖形及其交點。讓學生將兩個二元一次方程式 $y=\frac{3}{2}x$、$x+2y=8$ 的圖形畫在同一座標平面上。老師可以提醒學生任何方程式 $y=mx$ 的圖形都會通過原點(0, 0)，在分別找出每一方程式的兩組解，並畫出相交的兩條直線後，不難根據圖形判別交點座標(2, 3)。如果圖形不易判別交點座標，也可以用代入消去法來找出解答。最後找出兩個二元一次方程式的圖形與 x 軸所圍成三角形區域的底與高，求得該區域面積。

（英文）(a) Graph the system of linear equations $\begin{cases} y=\frac{3}{2}x \\ x+2y=8 \end{cases}$ and mark the coordinates of the intersection on the coordinate plane.

(b) Find the area of the triangular region, which is enclosed by the graphs of the two linear equations $y=\frac{3}{2}x$、$x+2y=8$ and the x-axis.

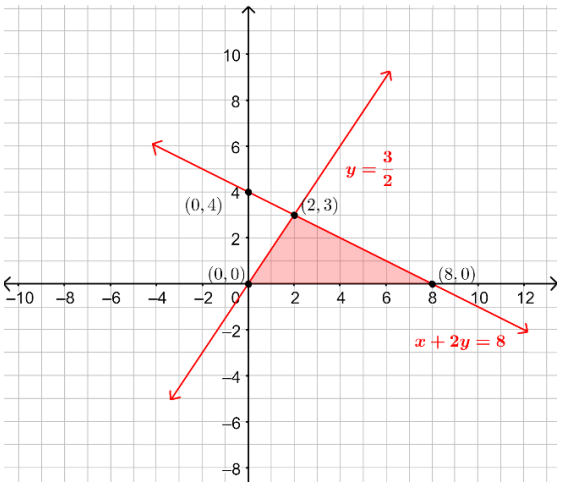
（中文）(a) 在座標平面上分別畫出二元一次方程式 $y=\frac{3}{2}x$、$x+2y=8$ 的圖形，並標記這兩條直線的交點座標。

(b) 求出這兩個方程式的圖形與 x 軸所為成三角形區域的面積。

Teacher: To graph the system of linear equations $\begin{cases} y=\frac{3}{2}x \\ x+2y=8 \end{cases}$, we need to find two different solutions to each equation. For equation $y=\frac{3}{2}x$, we can use (0, 0) and (2, 3). In fact,

the graph of the linear equation with the type "$y = mx$" always passes through the origin (0, 0). And we can choose (8, 0) and (0, 4) by setting $x = 0$ or $y = 0$ for the equation $x + 2y = 8$.

Now draw the graphs of the two linear equations on the coordinate plane. Check whether you can solve the system by graphing.



(After a few minutes)

Teacher: Time's up. Can you find the intersection point of the two lines from the graph?

Student: Yes, they intersect at a point (2, 3).

Teacher: Very good. Sometimes we can easily find the coordinates of the intersection point from a graph. If the graph is not clear, we can use algebra to find it.

First, let's use $\frac{3}{2}x$ (three over two x) to substitute y in the equation $x + 2y = 8$. Next, we get $x + 2 \times \frac{3}{2}x = 8$. Then $x + 3x = 8$ or $4x = 8$. So, $x = 2$ and $y = 3$.

Now, let's do part (b). From the graph, you can tell the base is 8 (from (0, 0) to (8, 0)) and that the height is 3. It's because (2, 3) is the vertex of the triangle. So, what is the area of the region formed by the two lines and the x-axis?

Student: 12.

老師： 要繪製二元一次聯立方程式 $\begin{cases} y = \frac{3}{2}x \\ x + 2y = 8 \end{cases}$ 的圖形，我們需要找到每個方程式的兩個不同解。我們可以找到方程式 $y = \frac{3}{2}x$的兩個不同解 $(0, 0)$ 和 $(2, 3)$。事實上，具有類型 "$y = mx$" 的線性方程式圖形始終通過原點 $(0, 0)$。

另外，我們可以設 $x = 0$ 或 $y = 0$ 來得到方程式 $x + 2y = 8$ 的解$(8, 0)$ 和 $(0, 4)$。

老師： 現在試著在座標平面上繪製這兩條線性方程式的圖形。看看是否可以在圖形找到該聯立方程式的解。

(過了幾分鐘) 時間到。你們能從圖形中找到兩條線的交點嗎？

學生： 它們相交於點 (2, 3)。

老師： 很好。有時我們可以輕鬆從圖形中找到交點的坐標。如果圖形不清晰，我們可以使用代數方法找到它。

首先，讓我們使用 $\frac{3}{2}x$ 來代替方程式 $x+2y=8$ 中的 y。接下來，我們得到

$x+2\times\frac{3}{2}x=8$，接著 $x+3x=8$；$4x=8$。因此，$x=2$，$y=3$。

老師： 現在，讓我們進行(b)部分。從圖形中，你可以看出三角形的底為 8 (從 (0, 0) 到 (8, 0))，高為 3。這是因為 (2, 3) 是該三角形的頂點。那麼，由這兩條線和 x 軸形成的區域的面積是多少？

學生： 12。

應用問題 / 會考素養題

例題一

說明：二元一次方程組求解。坐標平面上有兩直線相交於一點$(2, a)$，則$(2, a)$即為兩直線的方程式分別為 $2x+3y=7$ 和 $3x-2y=b$ 的解。

將$(2, a)$代入方程式 $2x+3y=7$ 可得 $a=1$，再將$(2, 1)$代入 $3x-2y=b$ 則得到 $b=4$。故 $a+b=5$，答案為(C)。

（英文）There are two lines intersecting at the point $(2, a)$ on a coordinate plane. If the equations of these lines are $2x+3y=7$ and $3x-2y=b$, what is the value of $a+b$?

(A) 1　(B) −1　(C) 5　(D) −5

（中文）已知坐標平面上有兩直線相交於一點 $(2, a)$，且兩直線的方程式分別為 $2x+3y=7$、$3x-2y=b$，其中 a、b 為兩數。求 $a+b$ 之值為何？

(A) 1　(B) −1　(C) 5　(D) −5

（106 年國中會考第 5 題）

Teacher: Since $(2, a)$ is the intersection point of the two lines, it is a solution to the two linear equations. When using $(2, a)$ to substitute (x, y) in both equations, we get $2\times2+3\times a=7$ and $3\times2-2\times a=b$. Simplify the two equations.

We get $4+3a=7$ and $6-2a=b$. Now find a and b.

Student: $a=1$ and $b=4$.

Teacher: You're correct. From the first equation, we find that the value of a is 1.

Now let's use "$x = 2$ and $y = 1$" in the second equation. We get $b = 4$. So, what is the answer to $a + b$?

Student: (C) 5

Teacher: Good job!

老師： 由於 $(2, a)$ 為兩條直線的交點，因此它是兩個線性方程式的解。當我們在兩個方程式中使用 $(2, a)$ 來取代 (x, y) 時，我們得到 $2 \times 2 + 3 \times a = 7$ 和 $3 \times 2 - 2 \times a = b$。簡化這兩個方程式。

我們得到 $4 + 3a = 7$ 和 $6 - 2a = \mathrm{b}$。現在找出 a 和 b 的值。

學生： $a = 1$，$b = 4$。

老師： 很好。從第一個方程式中，我們可以得到 a 的值為 1。

現在讓我們在第二個方程式中使用 $x = 2$ 和 $y = 1$。我們得到 $b = 4$。那麼 $a + b$ 的值是多少？

學生： (C) 5

老師： 很好！

例題二

說明：瞭解二元一次方程式的直線圖形與鉛垂線或水平線的交點，及所在位置之象限。

（英文）On a coordinate plane, the graph of a linear equation in two variables passes through the points $(-3, 0)$ and $(0, -5)$. Determine which of the following equation graphs and the linear equation graph intersect in the third quadrant.

(A) $x - 4 = 0$ (B) $x + 4 = 0$ (C) $y - 4 = 0$ (D) $y + 4 = 0$

（中文）坐標平面上有一個二元一次方程式的圖形，此圖形通過$(-3, 0)$、$(0, -5)$ 兩點。判斷此圖形與下列哪一個方程式的圖形的交點在第三象限？

(A) $x - 4 = 0$ (B) $x + 4 = 0$ (C) $y - 4 = 0$ (D) $y + 4 = 0$

（105 年國中會考第 11 題）

Teacher: From the given information, the line passes through $(-3, 0)$ and $(0, -5)$. When we plot the two points on a coordinate plane, this line goes through the second, third, and fourth quadrants. Now, let's draw the line of the first equation $x - 4 = 0$ Can you tell what kind of graph the equation is?

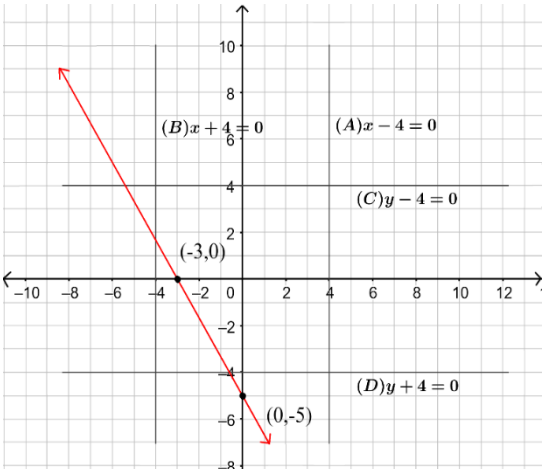
Student: A vertical line.

Teacher: Very good. On the coordinate plane, clearly, the graph of $x - 4 = 0$ intersects with the given line in the fourth quadrant. Then, let's draw the other vertical line of the part (B) $x + 4 = 0$. In which quadrant is the intersection point?

Student: In the second quadrant. (Quadrant II)

Teacher: You are right. Now, who wants to draw the graphs of (C) and (D) on the board to find the answer?

(After one student shows his/her work on the board.)



Teacher: Excellent. You can see that both graphs of (C) and (D) are horizontal lines. Clearly, the graph of choice (D) intersects with the given line in the third quadrant.

老師： 已知一直線通過$(-3, 0)$ 和$(0, -5)$兩點。在座標平面上標出這兩點時，我們知道此線會穿過第二、第三和第四象限。

現在，讓我們畫出第一個方程式 $x - 4 = 0$，告訴我畫出來直線是什麼？

學生： 是一條鉛直線。

老師： 不錯喔。在座標平面上，顯然，$x - 4 = 0$ 的圖形與所給的直線在第四象限相交。

現在，讓我們畫出選項(B) $x + 4 = 0$ 的另一條鉛直線，他們的交點位於哪個象限？

學生： 在第二象限。

老師： 是的。現在，誰可以在黑板上畫出選項(C)和(D)的圖形？

學生： (一位學生在黑板上畫出選項(C)和(D)的圖形)

老師： 很好。可以看出選項(C)和(D)的圖形都是水平線。

顯然地，(D)的圖形與已知的直線相交於第三象限。

單元五 比例式、正比與反比
Proportion, Direct Proportion and Inverse Proportion

國立新竹科學園區實驗高級中等學校　周慧蓮老師

■ 前言 Introduction

本章教學用語涉及分數的英文說法，建議老師應先熟悉分數的英文說法，並提醒學生分數的英文說法與中文說法的差異。

關於「$a:b=c:d$」的說法，中文有前項、後項，或是內項、外項之分，其英文分別是 antecedent, consequent 或是 means, extremes。但教學上，建議教師可使用 the first term, the second term, the third term, and the fourth term 分別代表 a, b, c, d。

中文特意區分比和比值。英文雖然也可以用 ratio 和 the value of the ratio 來區別，但實務上，ratio 也可以代表比值。

比例式教學常用的內項乘積等於外項乘積，解說如下：

Means-extremes property of proportions: The product of the extremes equals the product of the means.

For "$a:b=c:d$", the first and last terms of a proportion are called extremes (a and d). The middle terms are the means (b and c).

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中文	單字	中文
ratio	比	**direct variation/proportion/ratio**	正比
value of a ratio	比值	**inverse variation/proportion**	反比
proportion	比例	**constant of variation/proportionality.**	**(變動)常數**

proportional	成比例	equivalent ratio	相等的比
cross multiply	**交叉相乘**	**relatively prime**	**互質**
fraction	**分數**	**rational number**	**有理數**
fraction reduction	約分	fraction expansion	擴分
the simplest form	最簡	complex fraction	繁分數

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ A ratio _____.

例句(1)：**A ratio** is a comparison of two quantities.

「比」是兩個數量的比較。

例句(2)：**Ratios** can be used to compare two numbers.

「比」可以用來比較兩個數量。

❷ The value of the ratio "*a* to *b*" (*a*: *b*) is _____.

例句(1)：**The value of the ratio "2 to 5" (2: 5) is** the fraction "two over five" ($\frac{2}{5}$).

2 比 5 的比值是五分之二。

例句(2)：**The value of the ratio "2 to 3" (2: 3) is** the quotient when two is divided by three.

2 比 3 的比值是 2 除以 3 的商數。

❸ Write/Find the ratio of _____ to _____.

例句(1)：**Write the ratio of** "baking soda" **to** "salt".

寫出小蘇打粉和鹽的比。

例句(2)：**Find the ratio of** the length **to** the width.

找出長和寬的比。

❹ A is twice / ____ times + B

例句(1)：The amount of cold water in the recipe **is** $\frac{5}{2}$ **times** the amount of soy sauce.

食譜上冷水量是醬油量的$\frac{5}{2}$倍。

例句(2)：The side length of the scale drawing **is** 5 **times** the actual side length of the computer chip.

比例圖的邊長是電腦晶片實際邊長的 5 倍。

❺ Multiply / Divide _____ by the same number.

例句(1)：**Multiply** each part of the ratio **by two**.

將比的各項同乘以 2。

例句(2)：**Divide** the numerator and denominator of the fraction **by 3**.

將分數的分子和分母同除以 3。

❻ _____ is proportional to ____.

例句(1)：As you grow older, **is** your age **proportional to** your father's age?

隨著年齡增長，你的年紀和父親的年紀成比例嗎？

例句(2)：The number of square meters you paint **is proportional to** the number of minutes it takes you.

你油漆的面積與耗費時間成比例。

❼ _____ and _____ are in proportion.

例句(1)：The bases **and** heights of the two similar triangles **are in proportion.**

相似三角形的底和高對應成比例。

例句(2)：Radius and circumference **are in proportion.**

半徑和圓周長成比例。

❽ Tell whether _____.

例句(1)：**Tell whether** *x* and *y* show direct variation, inverse variation, or neither.

判斷 *x* 和 y 成正比、反比，或都不是。

例句(2)：**Tell whether** *x* and *y* are proportional.

判斷 *x* 和 *y* 是否成比例。

❾ _____ varies directly/inversely with _____

例句(1)：The time (in hours) for a group of workers to paint a wall **varies inversely with** the number of workers.

一群工人油漆一面牆所需要時間與工人的數量成反比。

例句(2)：The amount of money Sue makes **varies directly with** the number of hours she works.

蘇賺的錢和她工作的時間成正比。

❿ 主詞 ＋ 動詞 ____ every/in/per ____.

例句(1)：You mix 0.25 cup of juice concentrate for **every** 2 cups of water to make 18 cups of juice.

你在每兩杯水中加入 0.25 杯的果汁濃縮液，製作出 18 杯的果汁。

例句(2)：I can complete 57 push-ups **in** 3 minutes.

我可以在 3 分鐘內完成 57 個伏地挺身。

例句(3)：The average speed of a metro car is 35 kilometers **per** hour.

捷運的平均速度是每小時 35 公里。

■ 問題講解 Explanation of Problems

☙ 運算問題的講解 ❧

例題一

說明：教師先從整數比講解比值。再說明當比的任一項出現分數或小數時，如何計算比值。

（英文）Write the value of each ratio in simplest form.

（中文）寫出下列各比的比值，並以最簡分數表示。

(a) 2 : 3

(b) 4 : 8

(c) 0.5 : 2.5

(d) $\frac{1}{3} : \frac{5}{6}$

Teacher: Ratio "*a* to *b*" is the same as the fraction "*a* over *b*" ($\frac{a}{b}$). Therefore, the value of "two to three" is $\frac{2}{3}$.

What is the value of ratio of 4 : 8?

Student: $\frac{4}{8}$ (four over eight)

(Students might say "eight over four" because we refer to $\frac{4}{8}$ as "八分之四" in Chinese. Teachers can remind students of the proper order to name a fraction in English.)

Teacher: How to simplify $\frac{4}{8}$?

Student: Divide the numerator and denominator both by 4. Then you will have the simplest form $\frac{1}{2}$.

Teacher: What is the ratio of 0.5 : 2.5?

Student: We can divide each part of the ratio by 0.5 to get the equivalent ratio 1 : 5.

Then, the value of the ratio is $\frac{1}{5}$.

Teacher: Good job! If you find it difficult to work with decimals, you can convert them to whole numbers by multiplying both terms by 10. This will give you the ratio of 5: 25, which is 1:5. For the last question, try to figure out how to write the value of the ratio in simplest form with your classmates.
(At this moment, the teacher encourages students to discuss with their classmates. She/he can walk around to scaffold students' discussion. Let students share their ideas.)

Student: It is $\frac{\frac{1}{3}}{\frac{5}{6}}$ ("one-third" over "five-sixths"). To simplify the complex fraction, we multiply both parts of the fraction by 6. Then we will have $\frac{2}{5}$.

Teacher: Any other suggestion to find the value of the ratio?

Student: I would like to multiply both parts of the ratio by 6 first. In this case, the ratio becomes 2 : 5. The value of the ratio will be $\frac{2}{5}$.

老師： 「a比b的比值」等同於分數「b 分之a 」($\frac{a}{b}$)。因此，2 : 3的比值為 $\frac{2}{3}$。那麼，4 : 8 的比值是多少？

學生： $\frac{4}{8}$。(學生英文可能會說"eight over four"。老師可以提醒學生英文分數念法的正確順序。)

老師： 如何化簡 $\frac{4}{8}$？

學生： 把分子和分母都除以 4。所以，最簡分數就是$\frac{1}{2}$。

老師： 0.5 : 2.5 的比值是多少？

學生： 我們可以將比的每個部分都除以 0.5，得到相同的比 1 : 5。所以，比值是$\frac{1}{5}$。

老師： 做得好！如果你覺得計算小數有困難，你可以把兩項都成以十換成整數。你會得到 5 : 25，也就是 1 : 5。
對於最後一個問題，請和你的同學討論如何用最簡分數來表示比值。
(此時，老師鼓勵學生與同學討論，並可以同時在班級中走動幫助學生。讓學生分享他們的想法。)

學生： 答案是 $\frac{\frac{1}{3}}{\frac{5}{6}}$。為了簡化繁分數，我們將分數的兩部分都乘以 6。那麼，我們就能得到 $\frac{2}{5}$。

老師： 有同學有其他做法嗎？

學生： 我會先將比 $\frac{1}{3}:\frac{5}{6}$ 都乘以6。這樣就變成了2：5。那麼，比值就是 $\frac{2}{5}$。

例題二

說明：利用比例式的外項乘積等於內項乘積求解未知數。

（英文）Find the missing value.
（中文）找出適當的數值填入 □ 中。

(1) 3：5 = □：10

(2) 3：(-5) = □：11

(3) $\frac{5}{□} = \frac{4}{9}$

Teacher: For the first question, we can apply means-extremes property to find the missing value. The product of the extremes equals the product of the means.
Therefore, $5 \times □ = 3 \times 10$ (five times the blank equals three times ten)

$□ = \frac{30}{5} = 6$ (the blank is six)

(Teachers are suggested to ask students other ways of solving this question if the time permits.)

Teacher: For the second question, we can apply means-extremes property to find the missing value. The product of the extremes equals the product of the means.
Therefore, $-5 \times □ = 3 \times 11$ (negative five times the blank equals three times eleven)

$□ = \frac{33}{-5}$ (the blank is thirty-three over negative five)

(Teachers are suggested to ask students the sign of each term if the time permits.)

Teacher: For the third question, we can apply cross-product property, then $5 \times 9 = 4 \times □$.

$□ = \frac{45}{4}$

老師： 對於第(1)小題，我們可以使用內項乘積等於外項乘積的性質。

因此，$5 \times \square = 3 \times 10$，$\square = \frac{30}{5} = 6$。（如果時間夠，建議老師詢問學生其他解決這個問題的方式。）

老師： 對於第(2)小題，我們一樣可以應用內項乘積等於外項乘積的性質。

因此，$-5 \times \square = 3 \times 11$，$\square = \frac{33}{-5}$。（如果時間允許，建議老師詢問學生每一項的正負號。）

老師： 對於第(3)小題，我們可以應用交叉乘積的性質，然後得到 $5 \times 9 = 4 \times \square$。

$\square = \frac{45}{4}$。

例題三

說明：教師引導學生指出適用正比的方程式，再利用該方程式求解。

（英文）The variables x and y vary directly, and $y = 3$ when $x = -12$.

(1) Write an equation that relates x and y.

(2) Then find y when $x = 3$.

（中文）設 y 與 x 成正比。當 $x = -12$ 時，$y = 3$。

(1) 寫出 x 與 y 的關係式

(2) 當 $x = 3$ 時，y =?

Teacher: Which equation can be used for direct variation?

Student: $y = kx$ (y is equal to the product of k and x)

Teacher: If you replace x with -12, replace y with 3, what is the constant of variation?

Student: $\frac{3}{-12} = \frac{-1}{4}$ (negative one-fourth)

Teacher: So, what is the equation relating x and y?

Student: $y = -\frac{1}{4}x$ (y equals the product of x and negative one-fourth)

Teacher: Now, we use this equation to solve for y when x equals 3.

Student: $y = -\frac{1}{4} \times 3 = \frac{-3}{4}$ (y equals the product of 3 and negative one-fourth, which is negative three-fourths).

老師： 哪個關係式可以用來表示正比關係？

學生： $y = kx$

老師： 如果將 x 替換為 -12，將 y 替換為 3，正比關係中的常數是多少？

學生： $\frac{3}{-12} = \frac{-1}{4}$。

老師： 那麼，x 和 y 之間的關係式是什麼？

學生： $y = -\frac{1}{4}x$。

老師： 現在，我們使用這個關係式來解出當 x 等於 3 時的對應 y 值。

學生： $y = -\frac{1}{4} \times 3 = \frac{-3}{4}$。

例題四

說明：教師引導學生指出適用反比的方程式，再利用該方程式求解。

（英文）The variables x and y vary inversely, and $y = 3$ when $x = -12$.

(1) Write an equation that relates x and y.

(2) Then find y when $x = 3$.

（中文）設 y 與 x 成反比。當 $x = -12$ 時，$y = 3$。

(1) 寫出 x 與 y 的關係式

(2) 當 $x = 3$ 時，y =?

Teacher: Which equation can be used for inverse variation?

Student: $xy = k$ (the product of x and y equals k)

Teacher: If you replace x with -12, replace y with 3, what is the constant of variation?

Student: -36 (negative thirty-six)

Teacher: So, what is the equation relating x and y?

Student: $xy = -36$ (the product of x and y equals negative thirty-six)

Teacher: Now, we use this equation to solve for y when x equals 3.

Student: $3 \times y = -36$ (the product of 3 and y equals negative thirty-six).

$y = -12$

老師： 哪個關係式可以用來表式反比關係？
學生： $xy = k$
老師： 如果你將 x 替換為 -12，將 y 替換為 3，反比關係中的常數是多少？
學生： -36。
老師： 那麼，x 和 y 之間的關係式是什麼？
學生： $xy = -36$。
老師： 現在，我們使用這個關係式來解出當 x 等於 3 時的對應 y 值。
學生： $3 \times y = -36$。$y = - - 12$

∞ 應用問題 / 會考素養題 ∞

例題一

說明：教師強調兩個變數成正比，代表這兩個變數要滿足方程式 $y = kx$。必要時，可代入實際數字計算。

（英文）Which of the following shows direct variation?

(A) The weight of the green beans sold in a grocery store and its unit price.
(B) Mother's age and his daughter's age
(C) The area and the side length of a square
(D) The circumference and radius of a circle

（中文）下列何種問題情境屬於正比關係？

(A) 雜貨行賣的綠豆重量和單價
(B) 媽媽的年齡和女兒年齡
(C) 正方形的面積和邊長
(D) 圓形的圓周長和半徑

（107 中區聯盟國小教師甄試 38）

Teacher: When two variables meet the equation $y = kx$ and k is a constant, they are directly proportional. Therefore, we need to figure out whether the variables in each choice satisfy this equation.
Student: What does it mean by "unit price"?

Teacher: A unit price is a price for one item or one measurement, such as a cup, a kilogram. Therefore, the unit price is constant. Do you think the weight of beans is in direct variation with the unit price?

Student: No. The unit price stays the same no matter what the weight is.

Teacher: How about the other choices?

Student: Choice B is in direct variation because as my age increases, my mom's age increases.

Teacher: Let's imagine this situation. You are 13 years old, and your mom is 39 years old now. Then we can list the ages as the following.

Your' age (x)	13	14	15	16	17
Mom's age (y)	39	40	41	42	43

Do you think x and y are in direct variation?

Student: No, because the values do not fit the equation $y = 3x$.

Teacher: Now, let's consider the Choice C. As the side length of a square increases, the area increases as well. Do they show a direct variation?

Student: It is a direct variation. If you assume the side length as x, then the area will be x^2. $x^2 = x \cdot x$. When the constant of variation k is x, the area and side length satisfy the equation of direct variation.

Teacher: Sorry, it is wrong. The constant of variation has to be a constant. In your reasoning, the constant of variation is a variable. Therefore, Choice C does not show a direct variation.

Student: Then, the answer must be D.

Teacher: Can you relate the two variables by applying the equation of direct variation?

Student: If the radius is r, the circumference will be $2\pi r$. $2\pi r = 2\pi \times r$

It means the constant of variation is 2π.

So, the circumference and radius of a circle show a direct variation.

老師： 當兩個未知數符合關係式 $y = kx$ 且 k 是一個常數時，它們就成正比關係。因此，我們需要確認每個選項中的未知數是否滿足這個關係式。

學生： "unit price"是什麼意思？

老師： 單價是一個單位物品的價格，例如一杯，一公斤。因此，單價是恆定的。你認為豆子的重量是否與單價成正比關係？

學生： 不是的。無論重量如何，單價都保持不變。

老師： 其他選項呢？

學生： 選項(B)是正比關係，因為隨著我的年齡增長，我媽媽的年齡也會增加。

老師： 讓我們假設這個情況。你現在 13 歲，你媽媽現在 39 歲。接著，我們可以將年齡列成下表。

你的年齡(x)	13	14	15	16	17
媽媽的年齡 (y)	39	40	41	42	43

你們覺得 x 和y是否成正比關係？

學生： 不是的，因為這些值不滿足關係式 $y = 3x$。

老師： 現在，讓我們來看選項 C。隨著正方形的邊長增加，面積也增加。它們是否成正比關係？

學生： 這是一個正比關係。如果你把邊長假設為 x，那麼面積就會是 x^2。$x^2 = x \cdot x$。當關係式中的變化常數 k 是 x 時，面積和邊長就會滿足正比的關係式。

老師： 抱歉，錯了。常數必須是定值，而不是一個變數。因此，選項(C)並不成正比關係。

學生： 那麼，答案一定是(D)。

老師： 你能否應用正比的關係式來描述這兩個未知數？

學生： 如果半徑是 r，那麼圓周長將是 $2\pi r$。$2\pi r = 2\pi \times r$，這意味著常數是 2π。因此，圓的周長和半徑成正比關係。

例題二

說明：教師引導學生寫出比例式，並利用內項乘積等於外項乘積的概念求解。

（英文）The following is a cake recipe. Suppose May has a hundred milliliters of milk at hand and decides to increase the ingredients in the recipe to make a proportional recipe. How many grams of flour are needed for making the cake in the new recipe?

egg	5 pieces
flour	60 grams
sugar	50 grams
milk	80 grams
salt	5 grams

（中文）下圖為簡易蛋糕食譜。如果小美家裡有 100 毫升的牛奶可以拿來做蛋糕。她決定按比例增加食材來製作蛋糕，請問她需要加入多少公克的麵粉？

蛋	5 顆
麵粉	60 公克
白糖	50 公克
牛奶	80 毫升
食鹽	5 公克

Teacher: What is the ratio of milk to flour according to the original recipe?

Student: 80 : 60 (eighty to sixty). We can simplify it to 4 : 3 (four to three).

Teacher: If May wants to increase the number of ingredients proportionally and the ratio of milk to flour stays the same, please list an equation to represent these conditions.

Student: $4 : 3 = 100 : x$

Teacher: Let's apply the means-extremes property.

Student: $300 = 4x$. Then $x = \frac{300}{4} = 75$ grams.

老師： 根據原來的食譜，牛奶和麵粉的比例是多少？

學生： 80 : 60。我們可以簡化為4 : 3。

老師： 如果小美想要按比例地增加食材的數量，且牛奶和麵粉的比例保持不變，請列出比例式來表示這些條件。

學生： $4 : 3 = 100 : x$

老師： 讓我們應用「內項乘積等於外項乘積」的性質。

學生： $300 = 4x$。$x = \frac{300}{4} = 75$ 克。

例題三

說明：教師引導學生透過題意列式得知蝦水餃和韭菜水餃的單價比。學生計算剩下的錢可以買幾顆蝦水餃，再換算可以買幾顆韭菜水餃。

（英文）Weili would like to buy dumplings. The amount of money he has is enough for either 15 shrimp dumplings or 20 leek dumplings. After buying 9 shrimp dumplings, how many leek dumplings can he buy with the remaining money?

（中文）威立到小吃店買水餃，他身上帶的錢恰好等於 15 粒蝦仁水餃或 20 粒韭菜水餃的價錢。若威立先買了 9 粒蝦仁水餃，則他身上剩下的錢恰好可買多少粒韭菜水餃？

（106 年國中會考第 15）

Teacher: First of all, how many shrimp dumplings can Weili buy after buying 9 shrimp dumplings?

Student: $15 - 9 = 6$. Weili can buy six more shrimp dumplings.

Teacher: We need an equation to relate the shrimp dumplings with leek dumplings.
You can assume the unit price of a shrimp dumpling is x, the unit price of a leek dumpling is y.

Student: According to the first sentence, the equation is $15x = 20y$.

Teacher: Now, let's rewrite this equation into equivalent ratios. What will be x to y?

Student: I have no idea.

Teacher: Let's fill in the missing values of the equation $\frac{x}{y} = \frac{\square}{\square}$ by transforming $15x = 20y$.

Student: I know. $\frac{x}{y} = \frac{20}{15}$

Teacher: How do you know?

Student: I find the answer by applying the cross-product property.

Teacher: "x over y" is equal to "twenty over fifteen." It means the unit price of a shrimp and a leek dumpling is "four to three."

Teacher: That's right. We can assume the unit price of a shrimp is $4k$, the unit price of a leek dumpling is $3k$.
If the remaining money is enough to buy 6 shrimp dumplings, it means that the amount of money Weili left is $6 \times 4k = 24k$.

Divide $24k$ by the unit price of a leek dumpling. $\frac{24k}{3k} = 8$

Weili can buy 8 leek dumplings with the remaining money.

老師： 首先，如果威立買了 9 個蝦仁水餃，那他還可以買多少個？

學生： $15-9=6$。威立還可以買六個蝦仁水餃。

老師： 我們需要一個關係式來描述蝦仁水餃和韭菜水餃的關係。你可以假設一個蝦仁水餃的單價是x，一個韭菜餃的單價是 y。

學生： 根據第一句話，我們知道他們的關係式是$15x=20y$。

老師： 現在，讓我們將這個方程式重寫成等價的比。x 與 y 的比是多少？

學生： 我不知道。

老師： 我們透過轉換 $15x=20y$，以填寫方程式 $\frac{x}{y}=\frac{\square}{\square}$ 中缺空的值。

學生： 我知道了。是 $\frac{x}{y}=\frac{20}{15}$。

老師： 你是怎麼知道的？

學生： 我是應用交叉相乘來得出答案。

老師： “x除以y” 等於 “二十除以十五”。這意味著一個蝦餃和一個韭菜餃的單價是“四比三”。

老師： 沒錯。我們可以假設一個蝦仁水餃的單價是 $4k$，一個韭菜水餃的單價是 $3k$。如果剩下的錢足夠買 6 個蝦仁水餃，那麼威立剩下的錢就是 $6\times 4k=24k$。

把 $24k$ 除以一個韭菜水餃的單價。$\frac{24k}{3k}=8$。

所以，威立用剩下的錢可以買 8 個韭菜水餃。

單元六 解一元一次式不等式
Linear Inequalities in One Unknown

國立新竹科學園區實驗高級中等學校　印娟娟老師

■ 前言 Introduction

解一元一次式不等式前，建議老師將常用語句（大於、小於等），寫於黑板上讓學生能即時掌握老師之教授內容。本節常用符號為：

＞ 大於	greater than
≥ 大於或等於	greater than or equal to (no less than)
＜ 小於	less than
≤ 小於或等於	less than or equal to (no more than)

其次，介紹不等式的四則運算規則並強調其與解一元一次方程式之不同在於不等式兩邊同乘或除以一個負數時，其不等號方向會改變。所以練習中亦會多加強乘或除一個負數之練習。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中文	單字	中文
linear inequality	**一次不等式**	number line	數線
one variable	一元	**least common multiple**	**最小公倍數**
solve	求解	symbol	符號
greater than	**大於**	like terms	同類項

less than	小於	no less than/no more than	大於或等於/小於或等於
combine	合併	graph	繪圖/圖形
flip/reverse	翻轉	direction	方向
within	在……之中	intersection	交集

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ Solve _____.

例句：Today we will talk about how to **solve** the linear inequalities with one variable.
今天我們將討論如何解一元一次不等式。

❷ Use similar patterns _____.

例句：We will **use similar patterns** to solve these inequalities as we did for solving linear equations.
我們將用解一元一次方程式的相似方法來解一元一次不等式。

❸ Subtract ____ from _____.

例句：**Subtract** 2*x* **from** both sides.
兩邊同時減去 2*x*。

❹ Apply _____ to _____.

例句：We will learn to **apply** the distributive property and the commutative property **to** the next questions.
我們將會學到將分配律和交換律應用於接下來的題目。

❺ Instead of _____.

例句：**Instead of** using equals in the linear equations, we use the following symbols.
我們用以下的符號來取代線性方程式中的等號。

❻ keep _____ in mind.

例句：Please **keep** that **in mind** as we solve inequalities.
當解不等式時要記住那個特點。

❼ sharpen your skills

例句：Let's do one more exercise of inequality to **sharpen your skills**.
讓我們再做一題不等式練習來加強你們解不等式的能力。

❽ reverse the direction/flip the direction

例句：Divide each side by −8 (negative eight) and **reverse the direction** of the inequality.
每邊各除以(−8)並反轉不等式的方向。

❾ be familiar with _____

例句：Let's do some exercises to help you **be familiar with** these skills.
讓我們多做幾題練習讓技巧更熟練。

❿ It was known that _____.

例句：**It was known that** she bought 10 boxes of cupcakes and spent not more than 2500 NT.
已知她購買 10 盒蛋糕，花費的金額不超過 2500 元。

■ 問題講解 Explanation of Problems

☙ 說明 ❧

We will use similar patterns to solve these inequalities as we did for solving linear equations. Instead of using equals in the linear equations, we use the following symbols:

$>$	greater than
$\geq$	greater than or equal to (no less than)
$<$	less than
$\leq$	less than or equal to (no more than)

Basically, the direction of the inequality symbols will not be changed if we add or subtract the same number to both sides of the inequalities. Also, the symbols stay the same when we multiply or divide both sides by a positive number. But symbols are flipped when we multiply or divide by a negative number. The rules are listed below:

If $a > b$, then $a + c > b + c$.

If $a > b$, then $a - c > b - c$.

If $a > b$, then $ac > bc$ $(c > 0)$ or $ac < bc$ $(c < 0)$.

If $a > b$, then $a \div c > b \div c$ $(c > 0)$ or $a \div c < b \div c$ $(c < 0)$.

By the end of this section, you will be able to:

1. Add or subtract from both sides of the inequality.
2. Multiply or divide (positive or negative numbers) on both sides of the inequality.
3. Graph the solutions of the inequalities on the number line.
4. Apply and solve inequality skills in real-life problems.

Now let's do some exercises to see whether you understand the concepts.

運算問題的講解

例題一

說明：老師引導學生求解例題一之不等式，運算規則和以前學過的解一元一次方程式大致相同，唯有在同乘或同除以一個負數時須改變不等號的方向。

（英文）Solve the inequality: $2x - 1 \leq 5$.
（中文）解一元一次不等式：$2x - 1 \leq 5$。

Teacher: As we just mentioned, we can add, subtract, multiply, or divide each side of the inequality. But if we multiply or divide by a negative number, the direction of the inequality symbol will be flipped. Please keep that in mind while we solve inequalities.

The first question is, "two x minus one is less than or equal to five." Use similar ways as we did in solving linear equations. Add one to both sides.

$2x - 1 + 1 \leq 5 + 1$

Combine the like terms and get "$2x \leq 6$."

Finally, divide both sides by two, then we will get the result "$x \leq 3$."

Is there any question?

Student: Nope.

Teacher: Ok! Great! Let's see the second exercise.

老師：正如我們剛才所提到的，我們可以對不等式的兩邊同時進行加、減、乘、除運算。但如果我們乘以或除以一個負數，不等式符號的方向將會改變。在求解不等式時要牢記這一點。

第一題解 $2x - 1 \leq 5$。如同解線性方程時的方法，將兩邊都加上 1。

$2x - 1 + 1 \leq 5 + 1$

合併同類項後得到 $2x \leq 6$。

最後，將兩邊都除以 2，我們將得到 $x \leq 3$。

有沒有問題？

學生：沒有。

老師：很好！現在我們來看例題二。

例題二

說明：例題二之不等式求解，著重在化簡分數時兩邊同乘最小公倍數 LCM，還有運用分配律。因為最小公倍數是正數，故同乘以一個正數時不等號的方向不變。

（英文）Solve the inequality: $\frac{1}{4}x+\frac{2}{3}>\frac{5}{6}$.

（中文）解一元一次不等式：$\frac{1}{4}x+\frac{2}{3}>\frac{5}{6}$。

Teacher: Example 2 is "one-fourth x plus two-thirds is greater than five-sixths." If you do not like the fractions, you can multiply both sides by the least common multiple to get rid of the fractions. So, what is the LCM (least common multiple) of four, three, and six?

Student: Twelve.

Teacher: Good.

Let's multiply both sides by the 12, and then you'll get "$12(\frac{1}{4}x+\frac{2}{3})>12\cdot\frac{5}{6}$"

(Read as: Twelve times the sum of one-fourth x and two-thirds is greater than twelve times five-sixths.)

There's no need to change the direction of the inequality since 12 is a positive number.

Here we can use the distributive property to get $3x+8>10$.

Then subtract eight from both sides to get $3x>2$.

After dividing both sides by three, you will get the answer is $x>\frac{2}{3}$.

老師： 例題二題目是「$\frac{1}{4}x+\frac{2}{3}>\frac{5}{6}$」。如果你不喜歡分數，你可以把兩邊同乘最小公倍數，可以消去分數。4、3 和 6 的最小公倍數是多少？

學生： 12。

老師： 很好。把兩邊同乘以 12，就會得到 $12(\frac{1}{4}x+\frac{2}{3})>12\cdot\frac{5}{6}$。由於 12 是正數，所以不需要改變不等式符號的方向。

接著，我們可以使用分配律得到 $3x+8>10$。然後，兩邊同減 8，得到 $3x>2$。

將兩邊都除以 3 後，你會得到答案是 $x>\frac{2}{3}$。

例題三

說明：例題三之不等式求解，著重在兩邊同乘或同除以一個負數時必須改變不等號的方向。並以口頭問題問學生，期以問答方式加深學生的思考與記憶。

（英文）Solve the inequality: $-2(x+2) \leq 10$.
（中文）解一元一次不等式：$-2(x+2) \leq 10$。

Teacher: Ok, let's see the next question "$-2(x+2) \leq 10$ negative two times the sum of x and two is less than or equal to ten."
To undo the grouping symbol "( )" parenthesis, you need to apply the distributive property first. Hence, you'll get $-2x-4 \leq 10$.
Next, move the like terms on the same side or you can think we add four on both sides. So, you get $-2x \leq 14$.
The next step is to divide both sides by **negative two**. Remember that we've emphasized that all the procedures are similar to the ways in solving linear equations except when multiplying or dividing by a negative number. What do you need to do when dividing both sides of an inequality by a negative number?
Student: Change the direction of the inequality symbol.
Teacher: Correct! Then can you tell me what you should change "less than or equal" when dividing by a negative number?
Student: Greater than or equal.
Teacher: Bravo! Yes, just like Maria mentioned that the flipped direction of less than is greater than. Please keep that in mind as you solve inequalities.

老師：好的，下一題是：$-2(x+2) \leq 10$
先應用分配律把括號去掉，得到 $-2x-4 \leq 10$。接著，將同類項移至同一邊，或者是在兩邊都加上 4。得到 $-2x \leq 14$。
接下來將兩邊都除以-2。請記住，我們強調過所有解不等式的過程都與解線性方程的方式相似，除了乘或除以負數時例外。當你將不等式的兩邊都除以負數時，你需要做什麼？
學生：改變不等式符號的方向。
老師：正確！那你能告訴我，在除以負數時，你應該將 $\leq$ 改成什麼？
學生：$\geq$。

老師： 太好了！就像剛才同學提到的，乘或除負數時小於要變成大於。在解不等式時要記得這一點。

例題四

說明：例題四之不等式求解，在於強化例題三之兩邊同乘或同除以一個負數時必須改變不等號的方向的題型。之後要求學生做三題隨堂練習，並選出三位志願學生在黑板上演示。

（英文）Solve the inequality: $-4x + 21 \leq 3(2x - 3)$.
（中文）解一元一次不等式：$-4x + 21 \leq 3(2x - 3)$。

Teacher: Let's do one more inequality exercise to sharpen your skills. What should we do first?

Student: Undo the parenthesis.

Teacher: Yes, undo the parenthesis by using the distributive property. So, you will get $-4x + 21 \leq 6x - 9$.

Next, combine the like terms. Usually, we put x terms on the left side. So, we'll get $-4x - 6x \leq -9 - 21$. (negative four x minus six x is less than or equal to negative nine minus twenty-one.)

Next, the left side "$-4x - 6x$" is "$-10x$" and the right side "$-9 - 21$" is "-30." We'll get $-10x \leq -30$. Then divide both sides by "-10".

What will you get?

Student: $x \geq 3$.

Teacher: Great! Please flip the symbols. Let's do some exercises together. And I like to choose three volunteers to do these questions on the board. Who likes to try these questions on the blackboard?

(Please choose three students to demonstrate their work on the blackboard.)

The rest of you, please do these questions in your class notebook within 10 minutes.

Exercises:

(1) $4x + 3 > -x + 8$

(2) $\frac{3}{2}(-x + 8) \leq 4x + 2$

(3) $-2(-x + 8) \geq 3(x - 9) + 2$

(10 minutes later…)

Teacher: Ok, now let's see the three questions on the board. The answers for (1) is $x > 1$, (2) $x \geq -\frac{20}{11}$, and (3) $x \leq 9$. Who got all the questions right? (Teacher can see how many students got the questions right and say something to encourage the students.) Now, let's talk about the next topic: solve the inequality and graph the solutions on the number line. Let's see example 5.

老師： 讓我們再做一題不等式的練習，讓大家更熟練。首先要做什麼？

學生： 去掉括號。

老師： 是的，使用分配律來消去括號。所以，你會得到 $-4x + 21 \leq 6x - 9$。
接下來，合併同類項。我們通常將 x項放在左邊。所以得到
$-4x - 6x \leq -9 - 21$。左邊的 $-4x - 6x$ 是$-10x$，右邊的 $-9 - 21$ 是-30。
我們得到 $-10x \leq -30$。然後，兩邊都除以 -10。答案是什麼？

學生： $x \geq 3$.

老師： 很好！要改變不等式的符號。讓我們再多做一些練習題。我需要三位同學志願來黑板上做問題。誰想嘗試來黑板上做這些問題？
(選三位學生在黑板上作答。)
其他的人，請在 10 分鐘內在課堂筆記本中完成這些問題。
練習題：

(1) $4x + 3 > -x + 8$

(2) $\frac{3}{2}(-x + 8) \leq 4x + 2$

(3) $-2(-x + 8) \geq 3(x - 9) + 2$

(10 分鐘後...)

老師： 好的，現在讓我們來看看黑板上的三個問題。第(1)小題答案是$x > 1$，第(2)小題是$x \geq -\frac{20}{11}$，第(3)小題是$x \leq 9$。有人所有的問題都做對了嗎？（老師可以看到有多少學生做對了問題，並鼓勵他們。）
現在，讓我們來談談下一個主題：解不等式並將解繪製在數線上。我們來看例題五。

例題五

說明：例題五同樣是不等式求解的題型，但可以兩邊同乘分數之最小公倍數來簡化分數。同時增加在數線上圖示其解。

（英文）Solve and graph the solutions of the inequality: $\frac{-2x+1}{3} \leq \frac{x-4}{2}$.

（中文）解一元一次不等式：$\frac{-2x+1}{3} \leq \frac{x-4}{2}$

Teacher: First, time the LCM 6 on both sides to get rid of the fractions. So, you'll get $2(-2x+1) \leq 3(x-4)$. Apply the distributive property, and then you'll get $-4x+2 \leq 3x-12$
Move the like terms together and get $-4x-3x \leq -12-2$.
Next, simplify both sides and get $-7x \leq -14$.
If we divide both sides by -7, then the answer is $x \geq 2$.
Now, how do we graph the solutions?
First, draw a number line and find the position "two."
Second, "x is greater than or equal to two" means that the solutions include the point $x = 2$ and all the points at the right side of 2.

老師： 首先，將兩邊乘以最小公倍數 6 以消去分數。這樣，你會得到 $2(-2x+1) \leq 3(x-4)$。然後用分配律化簡成 $-4x+2 \leq 3x-12$。
合併同類項得到 $-4x-3x \leq -12-2$，接下簡化兩邊，得到 $-7x \leq -14$。
如果我們將兩邊同除以-7，就能得到答案是 $x \geq 2$.。

老師： 現在，我們如何將解繪製成圖形呢？
首先，畫一條數線，找到 2 的位置。其次，$x \geq 2$ 表示解答包括$x = 2$ 這個點和 2 右側所有的點。

∞ 應用問題 / 會考素養題 ∞

說明：這個部分請老師可參考會考題數學題，或其他老師可找到的資源，提供一至兩個例題，題目用英文和中文寫出來之後，依照老師上課的實際經驗，寫出讓現場數學老師用英文講解這一題時，可參考的講稿、以及這段講稿的中文大意。

例題一

說明：此題是 107 年會考題目，老師需要解釋收入 income、支出 cost 及獲利 profit 之關係。同時強調在這題應用問題中的解必須是整數。

（英文）Nina plans to order cards for Mother's Day from the Lake Printing Company. The design fee is NT$1000 and the printing costs NT$5 per card. If Nina is going to sell the card at the price of NT$15 each, then how many cards does she need to print and sell them out to make the profit which exceeds 20% of the total expenses?

(A) 112　　(B) 121　　(C) 134　　(D) 143

（中文）圖中的宣傳單為萊克印刷公司設計與印刷卡片計價方式的說明，妮娜打算請此印刷公司設計一款母親節卡片並印刷，她再將卡片以每張 15 元的價格販售。若利潤等於收入扣掉成本，且成本只考慮設計費與印刷費，則她至少需印多少張卡片，才可使得卡片全數售出後的利潤超過成本的 2 成？

(A) 112　　(B) 121　　(C) 134　　(D) 143

（107 年國中會考 13）

Teacher: Assume that Nina needs to order x cards, then the total costs will be the design fee 1000 plus the printing fee $5x$ for x cards with 5 dollars each.
Cost $=1000+5x$.
And the income of selling x cards for 15 dollars each is equal to $15x$.
Now, who knows the relationship between the cost, income, and profit?

Student: Profit is equal to the income minus the cost. Profit = Income − Cost.

Teacher: Great! Therefore, the profit is equal to $15x-(1000+5x)$.
Let's convert the word expression "the profit is more than 20% of the total cost" to the math inequality. We will get

$$15x-(1000+5x) > 0.2\,(1000+5x)$$

The profit is more than 20% of the total cost.
Did you get it?

Student: Yes.

Teacher: Ok! Now let's see how to solve this inequality.
First, simplify both sides and you'll get $15x-1000-5x>0.2\times1000+0.2\times5x$ by the distributive property.
Then you'll get $10x-1000>200+x$.
Subtract x from both sides, and then add 1000 on both sides.
You'll get $9x>1200$ or $x>133.33$ after dividing 9 on both sides.
Remember that the number of cards is an integer.
So, what's the answer?

Student: (C) 134.

Teacher: Bravo! Please copy down the numbers of your homework questions on the chalkboard. And hand in your homework tomorrow.

老師： 假設妮娜需要訂購 x 張卡片，那麼總成本將包括 1000 元的設計費和每張 5 元的 $5x$ 元印刷費。成本$=1000+5x$。
而以每張 15 元的價格出售 x 張卡片的收入等於 $15x$ 元。
現在，有誰知道成本、收入和利潤之間的關係？

學生： 利潤等於收入減去成本。

非常好！因此，利潤等於 $15x-(1000+5x)$。讓我們將「利潤超過成本的 2 成」這個句子轉換成數學不等式。我們會得到

$$15x-(1000+5x) > 0.2\,(1000+5x)$$

利潤超過總成本的 20%。這樣清楚嗎？

學生：清楚。

老師：好的！現在讓我們來看看該如何解這個不等式。

首先，通過分配律簡化兩側，會得到 $15x - 1000 - 5x > 0.2 \times 1000 + 0.2 \times 5x$。

然後，你會得到 $10x - 1000 > 200 + x$。

將兩側減去 x，然後兩側加上 1000。

你會得到 $9x > 1200$，或者將兩側都除以 9 後，得到 $x > 133.33$

請記住，卡片的數量會是整數。那麼，答案是什麼？

學生：(C)134。

老師：太好了！作業題號在黑板上，大家抄下來回家練習，明天交。

例題二

說明：因為題型包含兩個不等式，老師可先做第一項不等式，提醒學生不超過和至少在不等式符號的轉換。並選出一位志願學生在黑板上演示如何做第二項不等式。並衍申如何解聯立不等式，並以數線圖示求解。

（英文）A-Hui bought two different kinds of cupcakes from the souvenir shop in Yilan. It was known that she bought 10 boxes of cupcakes and spent not more than NT$2500. If she distributed the cupcakes to her 75 colleagues, then every colleague could get at least one cupcake. How much did A-Hui spend on these cupcakes?

(A) NT$2,150　(B) NT$2,250　(C) NT$2,350　(D) NT$2,450

flavor	Price/box
Dried Longan	NT$350 for 12 cupcakes/box
Kumquat	NT$200 for 6 cupcakes/box

（中文）阿慧在店內購買兩種蛋糕當伴手禮，圖為蛋糕的價目表。已知阿慧購買 10 盒蛋糕，花費的金額不超過 2500 元。若他將蛋糕分給 75 位同事，每人至少能拿到一個蛋糕，則阿慧花多少元購買蛋糕？

(A) 2,150 元　(B) 2,250 元　(C) 2,350 元　(D) 2,450 元

（108 年會考第 12 題）

Teacher: As we know that A-Hui bought 10 boxes for two different kinds of cupcakes. Suppose we assume x boxes for the dried longan flavor, then $(10 - x)$ for the kumquat flavor. The total price for the dried longan cakes will be 350 times x, and the price for the kumquat cakes will be 200 times $(10 - x)$.

According to the given information, the total amount that A-Hui spent was not greater than 2500 NT.

Notice that "not greater than" means "less than or equal to." Hence, we can get the inequality: $350x + 200(10 - x) \leq 2500$.

Also, according to the table, the numbers of cupcakes are different for both different flavors. For the dried longan flavor, each box contains 12 cupcakes; however, there are only 6 cupcakes in each box of kumquat flavor. Therefore, the total number of cupcakes of the two different flavors will be

$12x + 6(10 - x) \geq 75$

By the given condition, they could be distributed to 75 people, and each person could get at least one cupcake.

Which inequality symbol does "at least" represent?

Student: "at least" means "greater than or equal to."

Teacher: Yes, Great! Can you list the inequality for the total number of cupcakes?

Student: $12x + 6(10 - x) \geq 75$

Teacher: Correct! Now, we get two inequalities, and the solutions need to satisfy both of them. Let's see the first one $350x + 200(10 - x) \leq 2500$.

Apply the distributive property, and then you get

$350x + 2000 - 200x \leq 2500$.

Move the like terms together and get $150x \leq 500$.

Next, divide both sides by 150 and get $x \leq \frac{500}{150}$ or $\frac{10}{3}$.

Now, let's go through the second one, "$12x + 6(10 - x) \geq 75$."

Ok, now, who can answer this question and demonstrate the show-work on the board?

Student: I do!

Teacher: Very good. Please come over the blackboard and do it now. The rest of you, please also solve this inequality by yourselves now. Three minutes.

(After three minutes…)

Ok. Time's up. Let me see your work now. (Teachers please patrol the classroom and check students' work.) Mmmm… Most of you did a great job. So, Nina, please

explain your work.

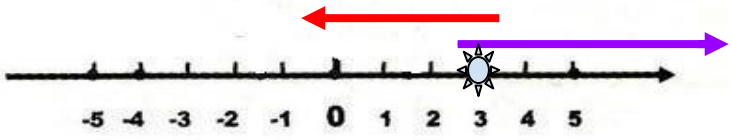
Nina: I use a similar method as the former inequality.

From $12x + 6(10 - x) \geq 75$, we will get $6x \geq 15$ and $x \geq \frac{15}{6}$ or $\frac{5}{2}$ after dividing by 6.

Teacher: Excellent! Please give Nina a round of applause. After we get the solutions for both inequalities, how do we graph the solutions?

First, draw a number line and "$x \leq \frac{10}{3}$" means all the points at the left side of the point $\frac{10}{3}$. Please notice that $\frac{10}{3}$ or $3\frac{1}{3}$ is included.

Second, "$x \geq \frac{5}{2}$ or $2\frac{1}{2}$" means that the solutions include the point $x = 2\frac{1}{2}$ and all the points at the right side of $2\frac{1}{2}$.



Teacher: According to the graph, x is within the intersection of two rays, and also x is an integer.

Check the number line, and you'll see that only the integer three is qualified.

When $x = 3$, the total amount is $350x + 200(10 - x)$ which equals to

$350 \times 3 + 200 \times 7 = 1050 + 1400 = 2450$ NT.

So, the answer is D.

老師： 我們知道阿惠買了 10 盒兩種不同口味的蛋糕，假設有 x 盒桂圓蛋糕，那麼就有 $(10 - x)$ 盒金棗蛋糕。

桂圓蛋糕的總價格是 350 乘以 x，金棗蛋糕的總價格是 200 乘以$(10 - x)$。

根據所給的條件，阿惠花費的金額不超過 2500 元。注意，"不超過"代表 "小於或等於"。

因此，我們可以得到不等式：$350x + 200(10 - x) \leq 2500$。

老師： 此外，表格上可看出兩種口味不同的蛋糕每盒的數量不同。對於桂圓蛋糕，每盒裡有 12 個；然而，每盒金棗蛋糕只有 6 個。因此，兩種不同口味的蛋糕的總數量是 $12x + 6(10 - x) \geq 75$。

根據題目，蛋糕被分配給 75 個人，且每個人至少能拿到一個蛋糕。

"至少"代表什麼不等式符號？

學生： “至少”代表“大於或等於”。

老師： 很好，你能列出蛋糕總數的不等式嗎？

學生： $12x + 6(10 - x) \geq 75$

老師： 正確！現在，我們要求兩個不等式的解。

先看第一個不等式 $350x + 200(10 - x) \leq 2500$

。應用分配律得到 $350x + 2000 - 200x \leq 2500$。

合併同類項後得到 $150x \leq 500$。將兩邊都除以150，得到$x \leq \frac{500}{150}$，約分後變$\frac{10}{3}$。

第二個不等式為：$12x + 6(10 - x) \geq 75$。

誰志願在黑板上解這一個不等式？

學生： 我！

老師： 很好。現在請上來示範。其餘的同學，請在三分鐘內找出這個不等式的解。

（三分鐘後...）好了，時間到了。現在讓我看看你們的答案。

（老師們開始巡視教室，檢查學生的筆記）嗯...大部分同學都做得很好。示範同學 Nina，請解釋一下你的解法。

Nina： 我的方法與解前一個不等式的相似。

從 $12x + 6(10 - x) \geq 75$ 開始，我們得到 $6x \geq 15$，然後除以 6 得到 $x \geq \frac{15}{6}$，即

$x \geq \frac{5}{2}$。

老師： 太好了！請給 Nina 一個掌聲。在得到兩個不等式的解之後，我們如何繪製解的圖形？

首先，繪製一條數線，$x \leq \frac{10}{3}$ 代表在 $\frac{10}{3}$ 這個點的左側的所有點都是解。請注意，

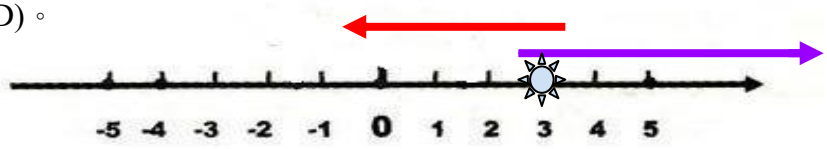
$\frac{10}{3}$（或 $3\frac{1}{3}$）是包含在解中的。

其次， $x \geq \frac{5}{2}$（或 $2\frac{1}{2}$）代表解包括$x = 2\frac{1}{2}$這個點及其右側的所有點。

老師： 根據圖表，x在兩條射線的交集範圍內，且 x 為整數。檢查數線，你會發現只有整數 3 符合條件。當$x = 3$時，總金額為 $350x + 200(10 - x)$，即

$350 \times 3 + 200 \times 7 = 1050 + 1400 = 2450$ 元

所以，答案是 (D)。



單元七 統計圖表
Data Display

國立新竹科學園區實驗高級中等學校　陳立業老師

■ 前言 Introduction

打開電視，翻開報紙或是瀏覽網頁都是我們得到最近資訊的方式。舉凡社會，時事，經濟或是健康生活，任何我們關注的議題都會吸引我們的注意，而許多有用的資訊及知識是靠統計圖表來傳達。在這一節中會讓學生認識多種常用的統計圖表，並能學習從不同的圖表中去解讀，分析數據。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中文	單字	中文
bar chart	**長條圖**	**two-way table (contingency table)**	**列聯表**
line graph	**折線圖**	column	欄
pie chart (circle graph)	**圓形圖**	row	列
frequency	次數	bin	直條
histogram	**直方圖**	bin width/class interval	組距
frequency table	次數分配表	categorical variable	類別變數
quantitative variable	量化變數		

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ Draw a _____.

例句：**Draw a** bar graph.
繪製長條圖。

❷ Rank the top three _____.

例句：**Rank the top three** celebrities you like the most.
列出你最喜愛的前三名藝人。

❸ According to the bar graph _____, work out _____

例句：**According to the bar graph** on the right, **work out** how many people who take a bus to school.
根據右方的長條圖，判斷有多少學生搭公車上學。

❹ How many kinds of _____ are displayed?

例句：**How many kinds of** transportation students take to school **are displayed** in the bar graph?
在長條圖上列出幾種學生通勤的方式？

❺ What kind of _____?

例句：**What kind of** transportation do the fewest students use to take to school?
哪一種學生通勤的學生人數最少？

❻ Between which _____ was there the greatest increase in _____?

例句：**Between which** two years **was there the greatest increase in** the birth rate?
哪兩個年度之間有最大的嬰兒出生率的成長？

❼ How many people surveyed _____?

例句：**How many students surveyed** have math scores between 60 and 70?
有多少學生成績介於 60 分到 70 分？

■ 問題講解 Explanation of Problems

說明

In daily life, we collect information that catches our attention and analyze the data to form some conclusions. One way to analyze data is to draw graphs. Why do we draw graphs? It is because graphs can easily help people understand information quickly.
統計圖表能讓讀者更快的了解作者文字想表達的意思。

Suppose we know what category or group this data belongs to, like gender or hair colors, we can use a bar chart or pie chart to show the frequency or relative frequency of each category. For example, we can do a survey during class, asking students, “What is your favorite food in the night market?” Then we tally the counts of each type of food. After that, we will have a brief picture of what kind of food teenagers like to eat in the night market.
在繪製圖表時，若這個資料是歸類於某一個族群，或是可以用一個名稱來將它涵蓋在內，我們會用長條圖來繪製。

If data can be quantified, measured (with a unit), or ranked from least to greatest, we can show the frequency or relative frequency in each bin. A bin is a part of the histogram and plays a role in collecting data between the lower bound (inclusive) and the upper bound (exclusive). The height of the bin shows the counts of data collected in this bin. The bin width or class interval is the difference between the upper bound and lower bound. Usually, each bin has the same bin width as the other one.
若這個資料是可以被量化的，可以從小到大排序的，那我們會將其分組並用直方圖來呈現。

∞ 運算問題的講解 ∞

例題一

說明：我們可以用長條圖來呈現不同的物種遭到路殺的數量，並了解如何解讀列聯表。

（英文）A study done by volunteers in Taiwan reported the following numbers of animals killed by motor vehicles in October, 2021.

turtledoves: 6　sparrows: 5　dogs: 4　pigeons: 2　cats: 3

a. Draw a bar graph that represents the data.

（中文）在 2021 年十月一份由台灣志工進行的研究報告指出，動物在路上被車輛撞死的數量統計如下：

斑鳩：6 隻、麻雀：5 隻、狗：4 隻、鴿子：2 隻、貓：3 隻

a. 繪製代表該資料的長條圖。

（參考資料：路殺社資料庫 https://roadkill.tw）

Teacher: A lot of animals died on the highway because they were killed by motor vehicles. To prevent animals being killed, the Highway Bureau uses several ways such as building fences along the roadsides and digging tunnels for them.

Student: I also feel very sorry for those poor animals.

Teacher: I am glad you sympathize with those little creatures. To know whether the government's methods work, we can track roadkill numbers.
Draw a table with two columns. The first column is the species of animals killed on highways.

turtledoves	
sparrows	
dogs	
pigeons	
cats	

Now, draw the table in your notebook.

Student: I'm done.

Teacher: Great. In the second column are the counts of the animal being killed. Fill in the numbers now.

Student: My table looks like this. Is that correct?

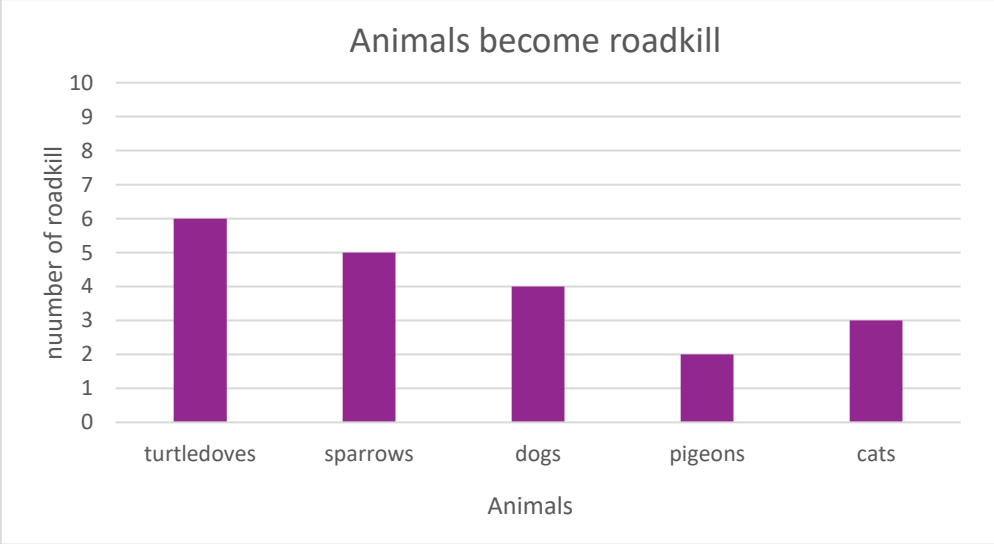
turtledoves	6
sparrows	5
dogs	4
pigeons	2
cats	3

Teacher: Excellent. We are ready to draw a bar graph using a grid. On the horizontal line, we label the category of the animals. How many kinds of animals are displayed?

Student: Five.

Teacher: Correct. For the vertical axis, one unit represents one animal. From each animal category, we will draw a bar with a height equal to its corresponding counts. Now, carefully draw the bars for turtledoves, sparrows, dogs, pigeons, and cats.

Student: I am finished. Please check my bar graph. Does the width of the rectangle mean anything?



Teacher: Good question! In a bar graph, the width of the bar doesn't represent anything in this context. And each bar isn't connected to the other bars. In the roadkill study, the researcher also recorded the gender of each animal killed on the highway. The table was displayed like this:

	Females	Males	Total
turtledoves	3	3	6
sparrows	1	4	5
dogs	3	1	4
pigeons	0	2	2
cats	1	2	3
Total	8	12	20

This is a two-way table, which is also called a contingency table. Suppose we can use two different characteristics to describe one group of subjects. In that case, we can organize our data in a table like this. Use a row to show the species and a column to show the gender. How many female sparrows died?

Student: One.

Teacher: How did you find that number?

Student: I drew a horizontal line through sparrows and a vertical line through females. I found the intersection point is 1.

Teacher: Awesome! You can read two-way tables now.

老師： 很多動物死在高速公路上是因為被汽機車輛撞死的。公路局使用多種方式避免動物再被撞死，例如在路邊建造柵欄，或為動物挖掘穿越的隧道等等。

學生： 我也為那些可憐的動物感到難過。

老師： 我很高興你能同情這些小生命。

要了解政府的這些方法是否有效，我們可以追蹤路殺數據。

繪製一個有兩欄的表格。第一欄是在公路上被撞死的動物種類。

斑鳩	
麻雀	
狗	
鴿子	
貓	

現在，在筆記本上畫出這個表格。

學生： 畫好了。

老師： 很好。在第二欄填入被殺死的數量。

學生： 我的表格長這樣，是正確的嗎？

斑鳩	6
麻雀	5
狗	4
鴿子	2
貓	3

老師： 非常好。我們現在可以在格線紙上繪製一個長條圖。在橫軸上，我們標上動物的類別。這邊總共有多少種動物？

學生： 五種。

老師： 正確。至於縱軸的部分，一個單位代表數量一隻。在每個動物類別，我們畫出一個高度等於其相應計數的長條形。現在，仔細地畫出斑鳩、麻雀、狗、鴿子和貓的長條圖。

學生： 我完成了，請看看我的長條圖。
對了，老師，長條形的寬度有什麼意義嗎？

老師： 很好的問題！在長條圖中，長條形的寬度在這個情況下並沒有任何意義。並記住每個長條形不跟其他長條形相連。
接著，在路殺研究中，研究人員還記錄了每隻在公路上死亡的動物的性別。表格如下：

	雌性	雄性	總數
斑鳩	3	3	6
麻雀	1	4	5
狗	3	1	4
鴿子	0	2	2
貓	1	2	3
總數	8	12	20

老師： 這是一個二維表，也稱作列聯表。假設我們可以使用兩個不同的特徵來描述一群被調查的對象，那麼我們可以像這樣將資料整理成表格。
用列來標示物種，欄來顯示性別。請問有幾隻母的麻雀死掉了？

學生： 一隻。

老師： 你是怎麼得出這個數字的？

學生： 我在“麻雀”這列畫了一條水平線，雌性這欄畫了一條垂直線，找到的交集點是1。

老師： 太棒了！現在你已經會辨讀列聯表了。

（英文）Extra questions:

I. What percentage of dogs killed on the highway were female?

II. What percentage of female animals killed on highways were cats?

（中文）延伸問題：

(1) 在公路上被撞死的狗中，雌性占多少百分比？

(2) 在公路上被撞死的雌性動物中，貓占多少百分比？

Teacher: The question is, “What percentage of dogs killed on the highway were female?”. First, underline the dogs and consider them as the major groups we focus on right now. Make the number of dogs the denominator. Then count how many dogs are female and make that number the numerator. What is the fraction?

Student: Is the fraction “$\frac{3}{4}$”?

Teacher: Good! Convert $\frac{3}{4}$ to a percentage now.

Student: It is 50%. I can try to do Question b. First I find the total number of female animals and make it the denominator.
Next, I count the number of female cats and place it in the numerator. So, the fraction is $\frac{1}{8}$, which is 12.5%.

Teacher: Fantastic! Since you know how to find a percentage, you will learn how to solve this problem if the question is different in any way.

老師： 這個問題是“公路上被撞死的狗中，雌性占多少百分比？”。首先，把「狗」畫底線，將狗視為我們現在專注的主要群體。把狗的總數當作分母，然後計算有多少隻狗是雌性，把這個數字當作分子。答案是多少？

學生： 是 $\frac{3}{6}$ 嗎？

老師： 很好！現在把 $\frac{3}{6}$ 轉換成百分比。

學生： 是 50%。
我可以嘗試做第(2)小題。首先，找出所有雌性動物的總數，把它當作分母。然後，計算有多少隻母貓，把它放在分子中。因此，這個分數是 $\frac{1}{8}$ ，即 12.5%。

老師： 太好了！既然你知道如何找到百分比，就算題目給的條件不同，你也可以學著去解它。

（英文）b. Draw a pie chart.

（中文）b.繪製圓形圖。

Teacher:

	Frequency	Proportion
turtledoves	6	$\frac{6}{20}$
sparrows	5	
dogs	4	
pigeons	2	
cats	3	
Total	20	

There is another way to show this data. It's called a pie chart. In a pie chart, we have to find the proportion of each category first. Find the total number of the animals, which is 20. To find the proportion of turtledoves, we use 6 divided by 20, which is $\frac{6}{20}$. Now I will leave the rest of the proportions of each animal for you to do.

Student: The proportion of sparrows is $\frac{5}{20}$. The proportion of dogs is $\frac{4}{20}$. The proportion of pigeons is $\frac{2}{20}$. The proportion of cats is $\frac{3}{20}$.

Teacher: Next, multiply each of the proportions by 360^{o} to find the central angle of the sector.

The central angle of the sector for turtledoves is $\frac{6}{20} \times 360^{o} = 108^{o}$.

I will leave the rest of the central angles of the sector for you to do.

Student: Here are my calculations:

For sparrows: $\frac{5}{20} \times 360^{o} = 90^{o}$

For dogs: $\frac{4}{20} \times 360^{o} = 72^{o}$

For pigeons: $\frac{2}{20} \times 360^{o} = 36^{o}$

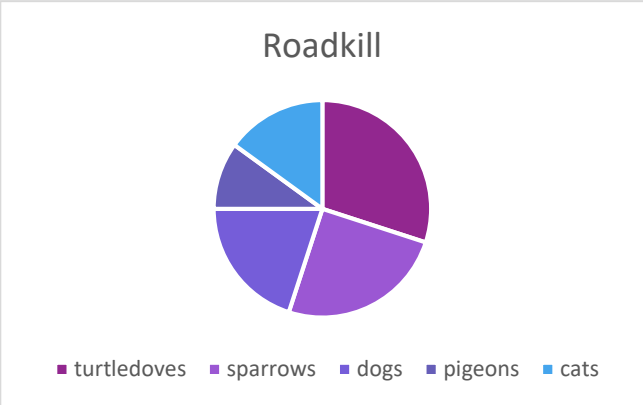
For cats: $\frac{3}{20} \times 360^{o} = 54^{o}$

Teacher: Well done. Use a compass to draw a circle and use a protractor to measure the angle of 108^{o}. Label this sector as turtledoves. Now I will check your progress.

Student: I have finished the sector of turtledoves.

Teacher: Repeat the process to draw and label the other sectors. You can also check if the sum of the measure of the angles is 360. Then you will know if there are any mistakes in your calculations.

Student: Here are my results.



Teacher: Terrific! Your pie chart is completed.

老師：

	次數	比例
斑鳩	6	$\frac{6}{20}$
麻雀	5	
狗	4	
鴿子	2	
貓	3	
總數	20	

還有另一種表示這些資料的方法，叫做「圓形圖」。在圓形圖中，我們必須先找出每個項目所佔的比例。找出所有動物的總數，即 20。要找出斑鳩佔全體動物的比例，就是用 6 除以 20，得到$\frac{6}{20}$。

現在，大家來計算其他動物的比例。

學生：麻雀的比例是$\frac{5}{20}$；狗的比例是$\frac{4}{20}$；鴿子的比例是$\frac{2}{20}$；貓的比例是$\frac{3}{20}$。

老師：接下來，將每個比例乘上 360^o，求出扇形的中心角度數。

舉例來說，斑鳩的度數為 $\frac{6}{20} \times 360^o = 108^o$。其他的中心角度數留給你們算。

學生：這是我的計算過程：

麻雀是 $\frac{5}{20} \times 360^o = 90^o$；狗是 $\frac{4}{20} \times 360^o = 72^o$；鴿子是 $\frac{2}{20} \times 360^o = 36^o$；貓是

$\frac{3}{20} \times 360^{o} = 54^{o}$

老師： 很好。接著用圓規畫一個圓，用量角器量108^{o}的角度，並將此扇形標記為斑鳩。現在我下去看看大家畫的進度。

學生： 我已經畫好斑鳩的扇形了。

老師： 重複這個過程，畫出其他扇形並標記。你也可以檢查角度的和是否為360^{o}，這樣你就會知道是否有計算錯誤。

學生： 我的結果如下：

老師： 太棒了！你完成圓形圖了。

例題二

說明：本題是跟學生介紹如何繪製直方圖及折線圖。

（英文）Assume this is the data of the ages of people who visited a museum, which are recorded on a random day in September.

Age: 5, 29, 41, 25, 26, 8, 38, 59, 72, 14, 15, 35, 7.

Draw a histogram that shows this data.

（中文）假設這是九月份某一天參觀博物館的人的年齡數據。

數據如下：5、29、41、25、26、8、38、59、72、14、15、35、7。

請畫出該數據的直方圖。

Teacher: If you are the curator of a museum, you might want to know which age group of people visits your museum the most. From the numbers above, we can't really answer this question. One way to think about it is to collect the numbers from different groups, or bins.

We can divide those ages into 8 bins like this:

0-10, 10-20, 20-30, 30-40, 40-50, 50-60, 60-70, 70-80.

The bin width is the difference between its upper bound and its lower bound.

We draw a table with two columns. Put those 8 bins in the left column and name them the "Ages" column. Check which bin the age of the visitor belongs to and make a tally. Check all the visitors and count the frequency in each bin. This table is called a frequency table.

Ages	Frequency
0-10	3
10-20	2
20-30	3
30-40	2
40-50	1
50-60	1
60-70	0
70-80	1

How many people fall into the 0-to-10-years-old bin?

Student: Let me count. One. Two. Three. There are three people.

Teacher: Correct. Write 3 next to the 0-to-10-year-old bin. I will leave the rest of the frequencies for you to complete.

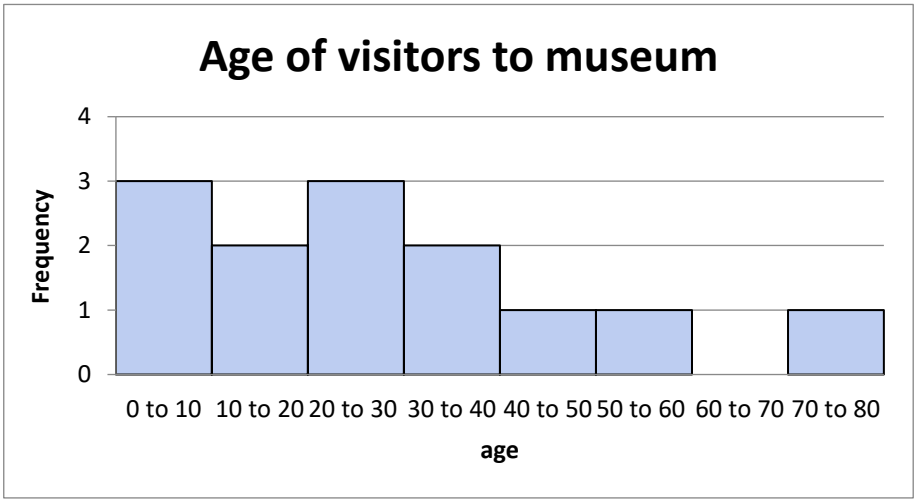
Student: I have a question. If the visitor is 40 years old, which bin will the data fall into? Is that the 30-to-40 bin or 40-to-50 bin?

Teacher: 40-year-olds fall into the 40-50 bin because a bin collects numbers from the lower bound, inclusively, to the upper bound, exclusively.

Student: Oh, I see. Thanks for explaining. I should complete this now.

Teacher: Great. Next, we will draw a histogram with a grid. On the horizontal axis, let's label 0, 10, 20, 30, 40, 50, 60, 70, and 80. On the vertical axis, let's label an appropriate scale. Mark off 1, 2, 3, and 4. Draw a bar one by one. For the first bin, there are three people in this bin, so we draw a bar with a width of 10 and a height of 3. Be careful. When we draw a histogram, there is no space left between one bar and another bar. It is your turn to finish the rest.

Student: Here is my histogram.



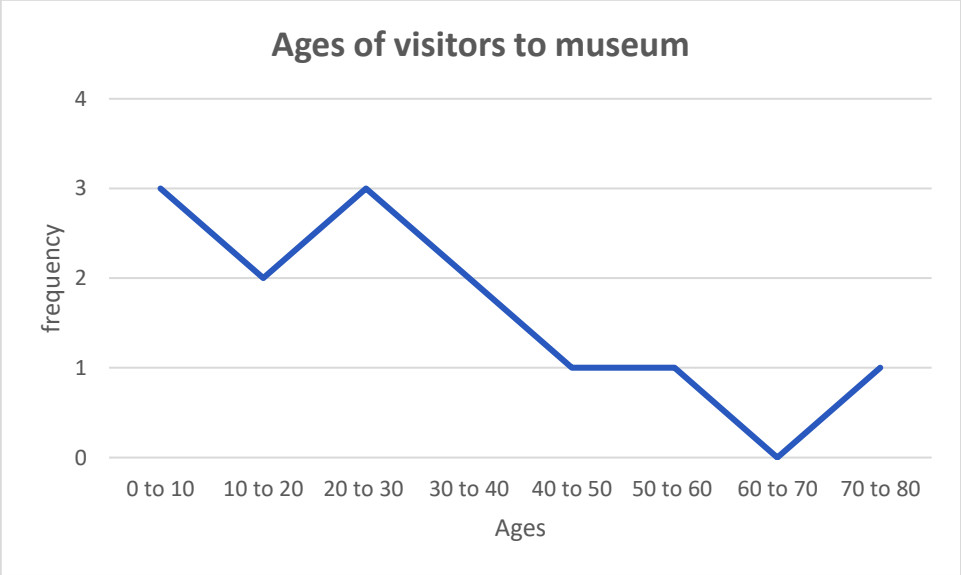
Teacher: Good job. Next, we can draw a line graph. Use the existing table and add one more column. Calculate the midpoint in each bin.

Ages	Midpoint	Frequency
0-10	5	3
10-20	15	2
20-30	25	3
30-40	35	2
40-50	45	1
50-60	55	1
60-70	65	0
70-80	75	1

Consider ages as *x*-axis and frequency as *y*-axis, and label the points (5, 3), (15, 2), (25, 3), (35, 2), (45, 1), (55, 1), (65, 0), and (75, 1).

Student: Can we just draw a dot in the middle of the top of each bar?

Teacher: Yes, that will come to the same result. And the graph you see is called a line graph.



老師： 如果你是一個博物館的館長，你可能會想知道哪個年齡層的人最常來參觀你的博物館。單看上面的數字，我們無法真正回答這個問題。一種思考方式是觀察來自不同組距的數字。

我們可以將這些年齡分成 8 個組距，如下所示：

0-10、10-20、20-30、30-40、40-50、50-60、60-70、70-80。

組距是其上限和下限之間的差。我們繪製一個帶有兩欄的表格，將這 8 個組距放在左欄，並將其命名為“年齡”欄。對照遊客的年齡屬於哪個組距，並進行統計。計算每個組距的遊客數量，這個表格就叫做次數分配表。

年齡	數量
0-10	3
10-20	2
20-30	3
30-40	2
40-50	1
50-60	1
60-70	0
70-80	1

0 到 10 歲的組距有多少人？

學生： 讓我數一下。1, 2, 3。有三個人。

老師： 正確。在 0 到 10 的組距旁邊寫上 3。剩下的數量留給你們完成。

學生： 我有一個問題。如果遊客是 40 歲，那會位於哪個組距？是 30 到 40 歲還是 40 到 50 歲？

老師： 40 歲的人屬於 40 到 50 歲的組距，因為一個組距包含從下限（包括）到上限（不包括）之間的數字。

學生： 哦，我明白了，謝謝老師。現在我應該可以完成這個表格了。

老師： 很好，接下來我們要來在格線紙上畫直方圖。在橫軸，標記 0、10、20、30、40、50、60、70 和 80。在縱軸，標記適當的刻度，標示 1、2、3 和 4，然後逐一畫出每一條長條。第一個組距，有兩個人在這個組距中，所以我們畫一條寬度為 10，高度為 2 的長條。請注意，畫直方圖時，長條和長條之間沒有空隙喔。接下來輪到你們完成其餘部分。

學生： 這是我的直方圖。

老師： 畫得很好！接下來，我們可以來畫一個折線圖。在現有的表格中添加一欄，計算每個組距的中點。

年齡	組中點	數量
0-10	5	3
10-20	15	2
20-30	25	3
30-40	35	2
40-50	45	1
50-60	55	1
60-70	65	0
70-80	75	1

將年齡視為 x 軸，數量視為 y 軸，並標記點 (5, 3)、(15, 2)、(25, 3)、(35, 2)、(45, 1)、(55, 1)、(65, 0) 和 (75, 1)。

學生： 我們可以在剛剛的每個長條的頂部中央畫一個點嗎？

老師： 當然可以，將這些點連線之後得到的結果一樣。你所看到的圖形稱為折線圖。

ꕥ 應用問題 / 會考素養題 ꕥ

例題一

（英文）A city was divided into two regions: the southern and northern regions. The bar graph displays the population of each region in that city from 2016 to 2018. Describe the change in the total population in that city from 2016 to 2018.

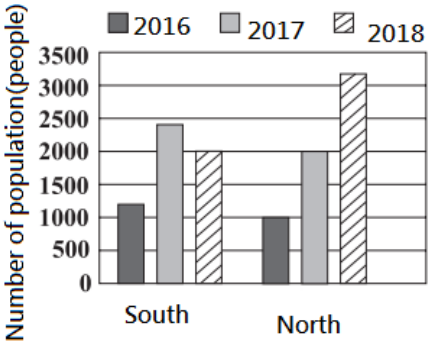
（中文）某城市分為南、北兩區，圖（一）為 105 年到 107 年該城市兩區的人口數量長條圖。根據圖（一）判斷該城市的總人口數量，從 105 年到 107 年的變化情形為下列何者？

(A) 逐年增加 (Increase.)

(B) 逐年減少 (Decrease.)

(C) 先增加，再減少 (Increase, then decrease.)

(D) 先減少，再增加 (Decrease, then increase.)



（108 年國中會考第 2 題）

Teacher: This is a question from the Comprehensive Assessment Program for Junior High School Students in 2019. First, let's read this question.

Student: A city was divided into two regions: the southern and northern regions. The bar graph displays the population of each region in that city from 2016 to 2018. Describe the change in the total population in that city from 2016 to 2018.

Teacher: Thank you. Let's find the total population in 2016. How many people were there in the southern region and the northern region?

Student: There were about 1200 people in the southern region and 1000 people in the northern region. The total is 2200.

Teacher: Correct. Now, find the city's total population in 2017 and 2018.

Student: There were about 4400 people in 2017 and 5200 people in 2018.

Teacher: Let's put the number in sequence. 2200, 4400, 5200. How would you describe this pattern?

Student: The numbers get larger. The city's total population increases from the year 2016 to the year 2018.

Teacher: Well done.

老師： 這是 108 年國中會考的題目。同學先唸一次題目。

學生： 某城市分為南、北兩區，圖為 105 年到 107 年該城市兩區的人口數量長條圖。根據判斷該城市的總人口數量，從 105 年到 107 年的變化情形為下列何者？

老師： 謝謝同學。我們先找到 105 年的總人口，南區和北區分別有多少人？

學生： 南區大約有 1200 人，北區有 1000 人，總共是 2200 人。

老師： 沒錯。現在，找到 106 年和 107 年的城市總人口。

學生： 106 年有大約 4400 人，107 年有 5200 人。

老師： 我們將數字排序，2200、4400、5200。數字是怎麼變化的？

學生： 數字變大，該城市的總人口從 105 年到 107 年逐年增加。

老師： 非常好。

例題二

說明：繪製折線圖。

（英文）Draw a line graph to illustrate the given data.

The average temperatures of 13 stations in Taiwan from 2010 to 2019										
2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	
23.87	23.3	23.71	23.91	23.95	24.3	24.4	24.4	24.22	24.56	^oC

（中文）根據下列數據繪製出一個折線圖。

台灣 13 個氣象站自 2010 年到 2019 年的平均氣溫。										
2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	
23.87	23.3	23.71	23.91	23.95	24.3	24.4	24.4	24.22	24.56	^oC

(參考資料：交通部 中央氣象局 https://www.cwb.gov.tw/)

Teacher: Can you feel it getting hotter and hotter every year? It is said that if the average temperatures rise by 1^oC, the sea levels will rise by 2.3 meters. The radical climate change is causing the extinction of many animals, like polar bears.

Student: Poor polar bears!

Teacher: If we don't do anything to slow down the changes in the climate, human beings will be like another species that loses their homes and becomes extinct. Let's check the data. From the data, you might not see the trend so easily. Let's draw a line graph. Take out your grid paper. Label the horizontal axis with the years listed on the table.

Student: Done.

Teacher: Choose a scale for the data. What is the least number and the greatest number?

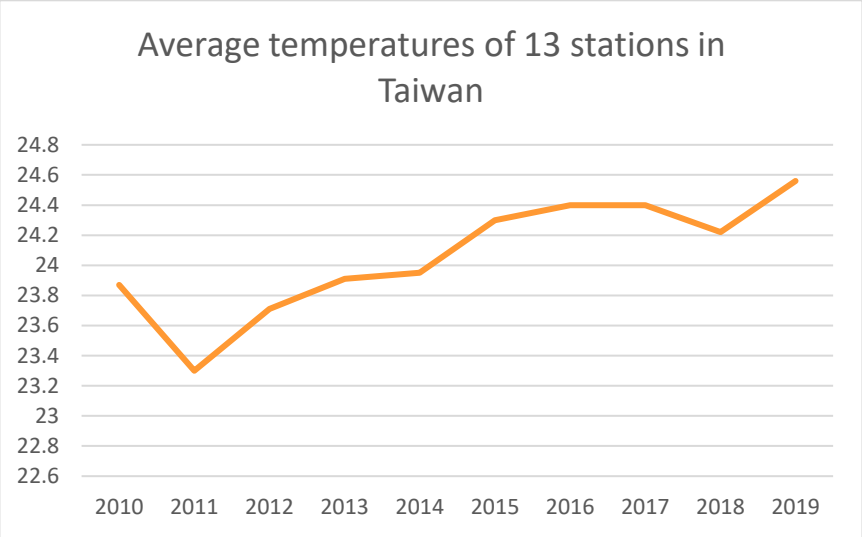
Student: The least number is 23.87 and the greatest is 24.56.

Teacher: So, we can label the vertical axis in increments of 0.2 from 23.6 to 24.8. Now consider the data in the table as ordered pairs: (2010, 23.87), (2011, 23.3), (2012, 23.71), and so on. What is the ordered pair for 2018?

Student: It is (2018, 24.22).

Teacher: Right. Now plot these points on the set of coordinate axes. Use line segments to join these points in order from the left to the right.

Student: Ok, please check my graph below.



Teacher: Between which two years was there the greatest increase in temperature? From the line graph, we can find which line segment is the steepest and rises to the right.

Student: I would say either the line segment from 2011 to 2012 or the line segment from 2018 to 2019.

Teacher: To confirm your results, find the differences in the temperature of the years you just mentioned. Use "+" to show that the temperatures rise. Make a conclusion.

Student: The difference between 2011 and 2012 is +0.41, and the difference between 2018 and 2019 is +0.34. So, the temperature increased the most from 2011 to 2012.

Teacher: Wonderful. That is correct.

老師： 你有感覺到每年都越來越熱了嗎？據說如果平均溫度上升 1 度，海平面就會上升 2.3 公尺。極端的氣候變遷正導致許多動物滅絕，像北極熊。

學生： 可憐的北極熊！

老師： 如果我們不採取任何行動來減緩氣候變化，人類就會像其他失去家園並滅絕的物種一樣。

我們來觀察數據。單看題目給的數據，不容易看出它的趨勢。讓我們來畫一張折線圖。拿出你的格線紙，橫軸標示表中列出的年份。

學生： 畫好了。

老師： 為這些數據選設一個刻度。最小值和最大值分別是多少？

學生： 最小值是 23.87，最大值是 24.56。

老師： 因此，我們可以將縱軸以 0.2 刻度為單位標示，從 23.6 標到 24.8。現在將表中的數據看作是數對：(2010，23.87)，(2011，23.3)，(2012，23.71)……等等。那 2018 年的數對是多少？

學生： 是(2018，24.22)。

老師： 沒錯，現在將這些點在坐標軸上標出，用線段將這些點從左到右照順序連起來。

學生： 好的，老師請看我畫的圖。

老師： 哪兩年之間溫度增加最多？從折線圖中，我們可以去找哪個線段是向右上升且最陡峭的。

學生： 應該是 2011 到 2012，或 2018 到 2019 的線段。

老師： 要確認你的答案，請算出你們剛才說的那兩個線段的溫度差。使用“+”表示溫度上升。

學生： 2011 到 2012 年之間的差異是+0.41，2018 到 2019 年之間的差異是+0.34。因此，從 2011 年到 2012 年，溫度上升得最多。

老師： 很好，答對了。

單元八 資料分析
Data Analysis

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■ 前言 Introduction

數據分析是統計的基礎，與日常生活息息相關。上一個章節，我們透過圖表來解讀數據的分布狀況。此章節中，我們希望以單一的指標（數值）來說明數據的集中趨勢。本單元介紹平均值、中位數和眾數。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中文	單字	中文
mean	**平均**	arithmetic mean	算術平均
average	**平均**	**mode**	**眾數**
median	**中位數**	**outlier**	**離群值**
total	**總和**	**weighted**	**加權**
affect	影響	minimum	最小值
maximum	最大值	combined	聯合的
center	中心	represent	代表
middle	中間	odd	奇數
even	偶數		

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ The _____ of a numerical data set is _____.

例句(1)：**The** (arithmetic) mean **of a numerical data set is** the sum of the data divided by the number of values.
（算術）平均數就是資料的總和除以資料的個數。

例句(2)：**The** median **of a numerical data set is** the middle number when the values of the data set are put in order.
中位數是資料依據大小順序排列後正中間的那個數。

例句(3)：**The** mode **of a numerical data set is** the number that happens most often.
眾數是資料集裡最常出現的數值。

❷ Find the mean/median/mode _____.

例句(1)：**Find the median** of the hourly wages.
找出時薪的中位數。

例句(2)：**Find the mean** heights without outliers.
找出未包含離群值的身高數據平均數。

❸ The median (of ___ numbers) is the _____ number.

例句(1)：**The median of these** 40 **numbers is the** average of the 20^{th} and 21^{st} **numbers.**
這 40 個數字的中位數是第 20 個和第 21 個數字的平均。。

例句(2)：There are 103 numbers. **The median is the** 52^{nd} **number.** Fifty-two equals the quantity one hundred three plus one divided by two ($\frac{(103+1)}{2}$).
這 103 個數字的中位數是第 52 個數字。五十二等於 103 加 1 後再除以 2。

❹ Arrange _____ in ascending order.

例句(1)：First, we **arrange** these numbers **in ascending order.**
首先，我們將這些數字由小到大排列。

例句(2)：When we make a stem and leaf plot, we usually **arrange** the leaves in each stem row **in ascending order.**
製作莖葉圖時，慣例上會將每個主幹對應的分枝數據由小到大排列。

❺ The (frequency) table below summarizes _____.

例句(1)：**The table below summarizes** Ming's daily sleeping hours in a week.
下表為小明一週內每天的睡眠時數。

例句(2)：**The table below summarizes** the responses of 30 students in Class 701 from a survey on their public service hours last month.
下表摘要七年一班 30 個學生問卷調查結果顯示的上個月公共服務時數。

❻ What is the difference in _____?

例句(1)：According to the table for math scores, **what is the difference in the** means between male and female students?
根據數學成績摘要表，男學生和女學生的數學成績均值差異為何？

例句(2)：**What is the difference in** the mean ages of the two groups?
這兩個族群的平均年齡差異為何？

❼ Compare _____ to/with _____.

例句(1)：Here is a table for math scores. Find the mean score and **compare** it **with** the median.
此為數學成績摘要表，找出均值並與中位數做比較。

例句(2)：**Compare** the science scores of students who joined after-school activities **to** the scores of those who did not. Which group's scores are higher?
比較有參加課後活動和沒有課後活動學生的自然科成績，哪一個比較高？

❽ _____ is affected by _____.

例句(1)：The median **is** less **affected by** outliers.
中位數較不受離群值影響。

例句(2)：The mean **is** easily **affected by** outliers.
平均值容易受到離群值影響。

■ 問題講解 Explanation of Problems

☙ 說明 ❧

A descriptive statistic is the foundation of data analysis. In this section, we will use a single value to talk about the central tendency of the data. The mean, median, and mode are often used. If outliers exist in the data, the median can show the data better than the mean value. The reason is that the median is resistant.

❧ 運算問題的講解 ☙

例題一

說明：此題為關於數據集中趨勢的基本運算。

（英文）The table below summarizes the daily internet usage (hours) for the students in Class 702 on the weekend.

a.) What is the mean daily internet usage for the students in Class 702?

b.) What is the median daily internet usage for the students in Class 702?

c.) What is the mode of the daily internet usage for the students in Class 702?

d.) Compare the mean to the median. Is there any difference?

（中文）下表為七年二班學生週末每日網路使用時間（小時）。

a.) 七年二班學生每日的平均網路使用時間是多少？

b.) 七年二班學生每日網路使用時間的中位數是多少？

c.) 七年二班學生每日網路使用時間的眾數為何？

d.) 將平均值和中位數進行比較。它們是否有任何差異？

Daily internet usage (hr.)	2	3	4	5	6	7
Frequency	2	10	7	6	4	1

Teacher: The mean of a numerical data set is the sum of the data divided by the number of data values. Let's list these numbers one by one. How many data values are here?

Student: $2 + 10 + 7 + 6 + 4 + 1 = 30$

Teacher: What is the sum of the data?

Student: $2 \times 2 + 3 \times 10 + 4 \times 7 + 5 \times 6 + 6 \times 4 + 7 \times 1 = 123$

Teacher: Divide the sum by the number of data values. Then you will have the mean value.

Student: $\frac{123}{30} = 4.1$ (Read as: One hundred twenty-three divided by thirty is four point one.)

Teacher: Right! Finding the mean of a list of numbers with frequencies is the same as finding the weighted mean. The weights act the same way as frequencies.
The number of data values is even. So, finding the median is the same as finding the average of the two middle numbers. What is the position of the middle number?

Student: The average of the 15^{th} and 16^{th} numbers.

Teacher: The data value of the first and second positions is 2. The data value of the third to the twelfth positions is 3. What are the data values of the 15^{th} and 16^{th}numbers?

Student: They are both 4. So, the median is 4.

Teacher: From the table, which internet usage is the highest?

Student: Three. Because of 10 times.

Teacher: So, 3 is the mode of daily internet usage for the students in Class 702.
Let's compare the mean and the median. Is there any difference?

Student: Four point one (4.1) is close to four. I think they are similar.

Teacher: Right. The mean is similar to the median because there's no outlier in the data.

老師： 數字資料的平均值是資料的總和除以數據的數量。讓我們一一列出這些數字。這裡有多少數據值？

學生： $2 + 10 + 7 + 6 + 4 + 1=30$

老師： 數據的總和是多少？

學生： $2 \times 2 + 3 \times 10 + 4 \times 7 + 5 \times 6 + 6 \times 4 + 7 \times 1 = 123$

老師： 把總和除以數據值的數量，就可以得到平均值。

學生： $\frac{123}{30} = 4.1$。

老師： 好極了！找出具有權重的數列的平均數與找出其加權平均數是一樣的，權重的作用方式就像次數一樣。
數據值的數量是偶數。所以找出中位數就等同於兩個中間數的平均值。中間數的位置在哪裡？

學生： 第 15 個和第 16 個數的平均值。
老師： 第一個和第二個位置的數據值是 2。第三到第十二個位置的數據值是 3。第 15 和 16 個數的數據值是什麼？
學生： 它們都是 4。因此，中位數是 4。
老師： 從表格中可以看出，哪一個網路使用量的權重最高？
學生： 3 小時，因為出現10次
老師： 所以，3 是 702 班學生每日網路使用量的眾數。
讓我們比較平均值和中位數。是否有差異？
學生： 4.1 接近 4。我認為它們很相似。
老師： 對。平均值與中位數相似，因為數據中沒有離群值。

例題二

說明：此題為數據合併的均值計算。

（英文）The averages of the math exams are respectively 71 for Class 701 and 74 for Class 702. There are 29 students in Class 701. There are 28 students in Class 702. What is the combined average of the students in the two classes?

（中文）七年一班有 29 名學生，數學考試平均成績為 71 分；七年二班有 28 名學生，數學考試平均成績是 74 分。請問這兩個班級合併後的平均成績是多少分？

Teacher: The mean of a numerical data set is the sum of the data divided by the number of data values. Seventy-one is the sum of math grades divided by 29 students in Class 701. In this way, what is the sum of the math grades for Class 701?
Student: $71 \times 29 = 2059$
Teacher: In the same way, what is the sum of math scores for Class 702?
Student: $74 \times 28 = 2072$.
Teacher: $2059 + 2072 = 4131$
So, the total math grades of the two classes are 4131.
What is the total number of students in these two classes?
Student: $29 + 28 = 57$
There are 57 students in total.
Teacher: What is the combined mean? Talk about it in detail.
Student: Divide the total grades by the number of students.
$4131 \div 57 = 72.47$.

老師： 數值資料集的平均數是數據總和除以數據值的個數。這裡七年一班 29 位學生數學成績平均是 71，那麼七年一班數學成績的總和是多少呢？

學生： $71 \times 29 = 2059$

老師： 同樣的方法，七年二班的數學成績總和是多少？

學生： $74 \times 28 = 2072.$

老師： $2059 + 2072 = 4131$。因此，這兩個班級的數學總成績為 4131。這兩個班級的學生總數是多少？

學生： $29 + 28 = 57$。總共有 57 名學生。

老師： 請詳細說明數據合併後的平均數。

學生： 將總成績除以學生人數。$4131 \div 57 = 72.47$ 分。

例題三

說明：此題為針對已分組資料判斷數據的集中趨勢。教師可以藉此說明長條圖「對稱」、「右偏」與「左偏」。並進一步說明對應的均值和中位數的關係。

（英文）The table below sums up the monthly income of 68 employees in a company.

a.) Make a histogram. Talk about the shape of the histogram.

b.) What is the mean monthly income?

c.) In which interval does the median monthly income fall?

d.) Compare the mean and median. Talk about what results in the difference.

Monthly income	20000-30000	30000-40000	40000-50000	50000-60000	60000-70000	70000-80000	80000-90000	90000-100000	100000-1100000	1100000-1200000
frequency	7	16	15	12	8	3	4	1	0	2

（中文）表格為某公司 68 名員工的月收入。

a.) 製作一個直方圖，談談直方圖的形狀。

b.) 這些員工的月平均收入是多少？

c.) 中位數的月收入落在哪個區間內？

d.) 比較平均數和中位數。什麼原因造成它們之間的差異。

Teacher: To scale the x-axis to be consistent with the intervals of the table, let x minimum be 0, x maximum be 1,200,000, and x scale be 100,000.
The maximum frequency is 16. So, we can set the y-axis from 0 to 17. Anyone can make a histogram on the board?

Student: Me.

(After the student makes the histogram, the teacher reminds students to label the x-axis and y-axis.)

Teacher: The shape of this histogram is called "skewed right." It has a longer "tail" to the right. To find the mean monthly income, we use the midpoint in each bin as the representative monthly income in the interval. We then can find the weighted mean according to its frequencies. What's the expression of the weighted mean?

Student: $(25000 \times 7 + 35000 \times 16 + 45000 \times 15 + 55000 \times 12 + 65000 \times 8 + 75000 \times 3 + 85000 \times 4 + 95000 \times 1 + 115000 \times 2)/68$

Teacher: The above expression is about 51176.47. So, the mean monthly income is around 51176.47. How do you find the interval where the median monthly income falls?

Student: The sample size is 68, and the middle two numbers are the 34^{th} and 35^{th} numbers. Both of them fall in the interval of 40000 – 50000. So, the median falls in the interval of 40,000 to 50,000.

Teacher: Let's compare the mean and median. Which one is greater?

Student: The mean is greater than the median.

Teacher: Yes. For the right-skewed distribution, the mean is greater than the median. The greater values will result in a great sum because there's a tail on the right side of the distribution. It also shows the resistance of the median. The median is less affected by the outliers.

老師： 為了使 x 軸與表格的區間一致，令 x 最小值為 0、x 最大值為 1,200,000、x 間格為 100,000。權重最大是 16，那麼，我們可以將 y 軸設置為 0 至 17。誰可以在黑板上繪製直方圖？

學生： 我可以。

（學生繪製完直方圖後，老師提醒學生標記 x 軸和 y 軸。）

老師： 這個直方圖的形狀被稱為"右偏"。它的右邊有一個長的"尾巴"。要找到平均每月收入，我們將每個直方中點作為該間隔的月收入。然後，我們可以根據其權重找到加權平均數。加權平均數的表達式是什麼？

學生： $(25000 \times 7 + 35000 \times 16 + 45000 \times 15 + 55000 \times 12 + 65000 \times 8 + 75000 \times 3 + 85000 \times 4 + 95000 \times 1 + 115000 \times 2)/68$

老師： 上述表達式的值約為 51176.47。因此，平均每月收入約為 51176.47。你如何找到中位數月收入所在的間隔？

學生： 樣本大小為 68，中間兩個數字是第 34 和第 35 個數字。它們都落在 40000 –

50000 的間隔內。因此，中位數落在 40,000 到 50,000 的間隔內。

老師： 讓我們比較平均值和中位數。哪一個數比較大？

學生： 平均值大於中位數。

老師： 是的。對於右偏分佈，平均值會大於中位數。因為在分佈的右側尾巴代表有較大的數值，這使得總和增大，平均值對應增加。這還顯示了中位數的不易變動性。中位數受到離群值的影響較小。

➿應用問題 / 會考素養題➿

例題一

說明：此題教師可以結合機率與樹狀圖引導學生計算均值。

（英文）There are four balls in a box. They are labeled as 1, 2, 3, and 4 respectively. Lily draws balls in succession (with replacement). The table below records the result of the first 8 drawings. If each ball is equally going to be drawn and its labeled number is the score, what are the answers to the following questions?

(a.) What is the average score of the first 8 drawings?

(b.) What is the probability to have the average score of the ten drawings between 2.2 and 2.4 inclusive? Show your work clearly. Show all the steps you take to reach your solution

（中文）一個箱子內有 4 顆相同的球，將 4 顆球分別標示號碼 1、2、3、4，今翔翔以每次從箱子內取一顆球且取後放回的方式抽取，並預計取球 10 次，現已取了 8 次，取出的結果如表（二）所列：

表（二）

次數	第1次	第2次	第3次	第4次	第5次	第6次	第7次	第8次	第9次	第10次
號碼	1	3	4	4	2	1	4	1		

若每次取球時，任一顆球被取到的機會皆相等，且取出的號碼即為得分，請回答下列問題：

(1) 請求出第 1 次至第 8 次得分的平均數。

(2) 承(1)，翔翔打算依計畫繼續從箱子取球 2 次，請判斷是否可能發生「這 10 次得分的平均數不小於 2.2，且不大於 2.4」的情形？若有可能，請計算出發生此情形的機率，並完整寫出你的解題過程；若不可能，請完整說明你的理由。

（107 年國中會考）

Teacher: To find the average score of the first 8 drawings, we have to add up the scores first. What are the total scores of the first 8 drawings?

Student1: $1 + 3 + 4 + 4 + 2 + 1 + 4 + 1 = 20$

Student2: Divide 20 by 8. Then we will have the average, 2.5.

Teacher: As for the second problem, let's draw a tree diagram to show the result. For the 9^{th} drawing, we can have four branches, labeled as 1, 2, 3 and 4. Each of them has four branches as the 10^{th} drawing. So, there are $4 \times 4 = 16$ possible outcomes. Assume the sum of the 9^{th} and 10^{th} drawings is x, we can have the inequality $2.2 \leq (20 + x) \div 10 \leq 2.4$. It means $22 \leq 20 + x < 24$.
In other words, $2 \leq x \leq 4$.How do we relate the tree diagram to (b.)?

Student: We have to look for the branch with the sum between 2 and 4 inclusive.

Teacher: Exactly! What are the outcomes with the sum between 2 and 4 inclusive?

Student: (1, 1), (1, 2), (1, 3) and (2, 1), (2, 2) and (3, 1).

Teacher: What is the probability?

Student: Six out of sixteen. It is three-eighths.

老師：要找出前八次的平均數，我們必須先加總這些數。前八次的總數是多少？

學生 1：$1 + 3 + 4 + 4 + 2 + 1 + 4 + 1 = 20$

學生 2：將 20 除以 8。這樣就會得到平均值，也就是 2.5。

老師：至於第二個問題，讓我們畫一個樹狀圖來展示結果。對於第九次取球，我們會有四個分支，記為 1、2、3 和 4。每個分支都有四個分支，作為第十次取球對應的樹狀圖。因此，有 $4 \times 4 = 16$ 個可能的結果。假設第九次和第十次的和是 x，我們可以得到不等式

$2.2 \leq (20 + x) \div 10 \leq 2.4$

$= 22 \leq 20 + x < 24$

$= 2 \leq x \leq 4$。

我們如何用樹狀圖算出第(2)小題？

學生： 我們必須尋找和在 2 到 4 之間的分支。

老師： 完全正確！哪些結果的和是落在 2 到 4 之間？

學生： (1, 1)、(1, 2)、(1, 3) 和(2, 1)、(2, 2) 和 (3, 1)。

老師： 機率是多少？

學生： 16 個結果中的 6 個。所以是八分之三。

例題二

說明：教師先引導學生判斷長條圖的中位數落點，再計算小於中位數的次數。

（英文）There are 36 students joining the basketball-shooting competition. The bar graph sums up the number of shots made by the students. How many students make fewer than the median shots?

（中文）圖（五）為甲班 36 名學生參加投籃測驗的投進球數長條圖。

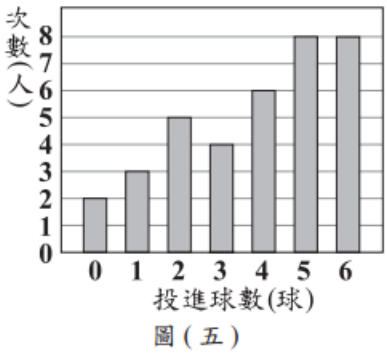
判斷甲班學生中，有多少人的投進球數小於該班學生投進球數的中位數？

(A) 10

(B) 14

(C) 17

(D) 18



圖（五）

（109 年國中會考第 13 題）

Teacher: To find the median, we have to find the middle number. What is the middle number among the 36 numbers?

Student: $36 \div 2 = 18$. The middle number is the average of the 18^{th} and 19^{th} numbers.

Teacher: Exactly. What are the 18^{th} and 19^{th} numbers?

Student: $2 + 3 + 5 + 4 = 14$ and $2 + 3 + 5 + 4 + 6 = 20$. So, both the 18^{th} and 19^{th} numbers are 4.

Teacher: How many people made fewer than 4 shots?

Student: $2 + 3 + 5 + 4 = 14$. Fourteen people made fewer than 3 shots.

老師： 這個數列有 36 個數字，要找中位數，必須找到中間的數字。36 個數字的中間是哪一個數字呢？

學生： 36 ÷ 2 = 18。中位數是第 18 個數字和第 19 個數字的平均數。

老師： 正確。第 18 個和第 19 個數字是什麼？

學生： 2 + 3 + 5 + 4 = 14 和 2 + 3 + 5 + 4 + 6 = 20，因此第 18 個和第 19 個數字都是 4。

老師： 有多少人投進不到 4 個球？

學生： 2 + 3 + 5 + 4 = 14 有 14 個人投進不到 4 球。

國內外參考資源 More to Explore

國家教育研究院樂詞網 查詢學科詞彙 https://terms.naer.edu.tw/search/	
教育雲：教育媒體影音 為教育部委辦計畫雙語教學影片 https://video.cloud.edu.tw/video/co_search.php?s=%E9%9B%99%E8%AA%9E	
Oak Teacher Hub 國外教學及影音資源，除了數學領域還有其他科目 https://teachers.thenational.academy/	
CK-12 國外教學及影音資源，除了數學領域還有自然領域 https://www.ck12.org/student/	
Twinkl 國外教學及影音資源，除了數學領域還有其他科目，多為小學及學齡前內容 https://www.twinkl.com.tw/	

Khan Academy 可汗學院，有分年級數學教學影片及問題的討論 https://www.khanacademy.org/	
Open Textbook (Math) 國外數學開放式教學資源 http://content.nroc.org/DevelopmentalMath.HTML5/Common/toc/toc_en.html	
MATH is FUN 國外教學資源，還有數學相關的小遊戲 https://www.mathsisfun.com/index.htm	
PhET: Interactive Simulations 國外教學資源，互動式電腦模擬。除了數學領域，還有自然科 https://phet.colorado.edu/	
Eddie Woo YouTube Channel 國外數學教學影音 https://www.youtube.com/c/misterwootube	

國立臺灣師範大學數學系陳界山教授網站	
國高中數學雙語教學相關教材 https://math.ntnu.edu.tw/~jschen/index.php?menu=Teaching_Worksheets	
2023 年第四屆科學與科普專業英文(ESP)能力大賽	
科學專業英文相關教材，除了數學領域，還有其他領域 https://sites.google.com/view/ntseccompetition/%E5%B0%88%E6%A5%AD%E8%8B%B1%E6%96%87%E5%AD%B8%E7%BF%92%E8%B3%87%E6%BA%90/%E7%9B%B8%E9%97%9C%E6%95%99%E6%9D%90?authuser=0	

國中數學領域雙語教學資源手冊：英語授課用語

[七年級下學期]

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[7th grade 2nd semester]

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- 指導單位：教育部師資培育及藝術教育司

- 撰稿：印娟娟、吳珮蓁、陳立業、周慧蓮
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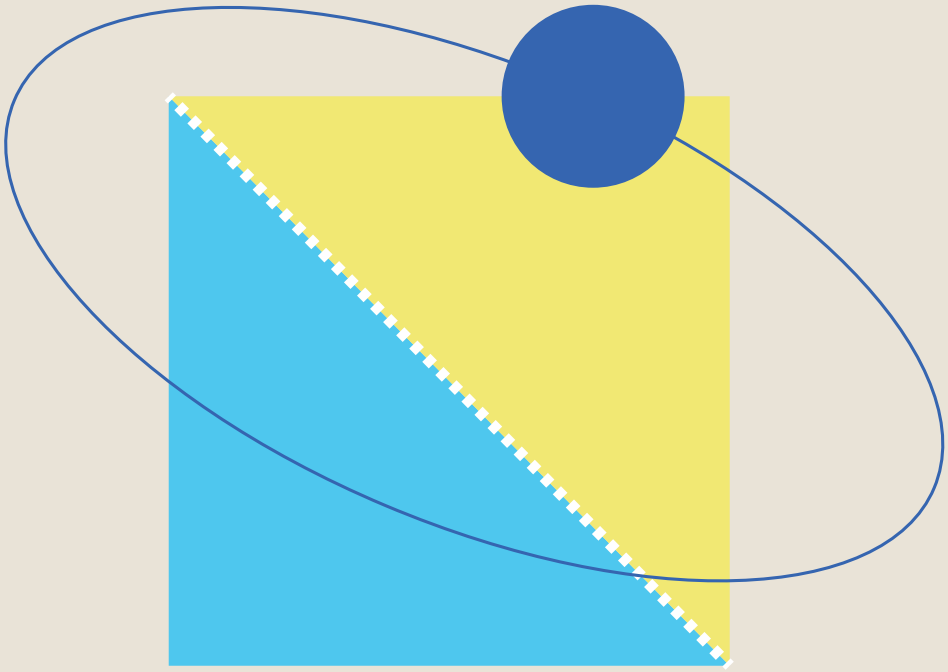
- 語言諮詢：李壹明

- 綜合規劃：王宏均
- 編輯排版：吳依靜
- 封面封底：JUPE Design



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