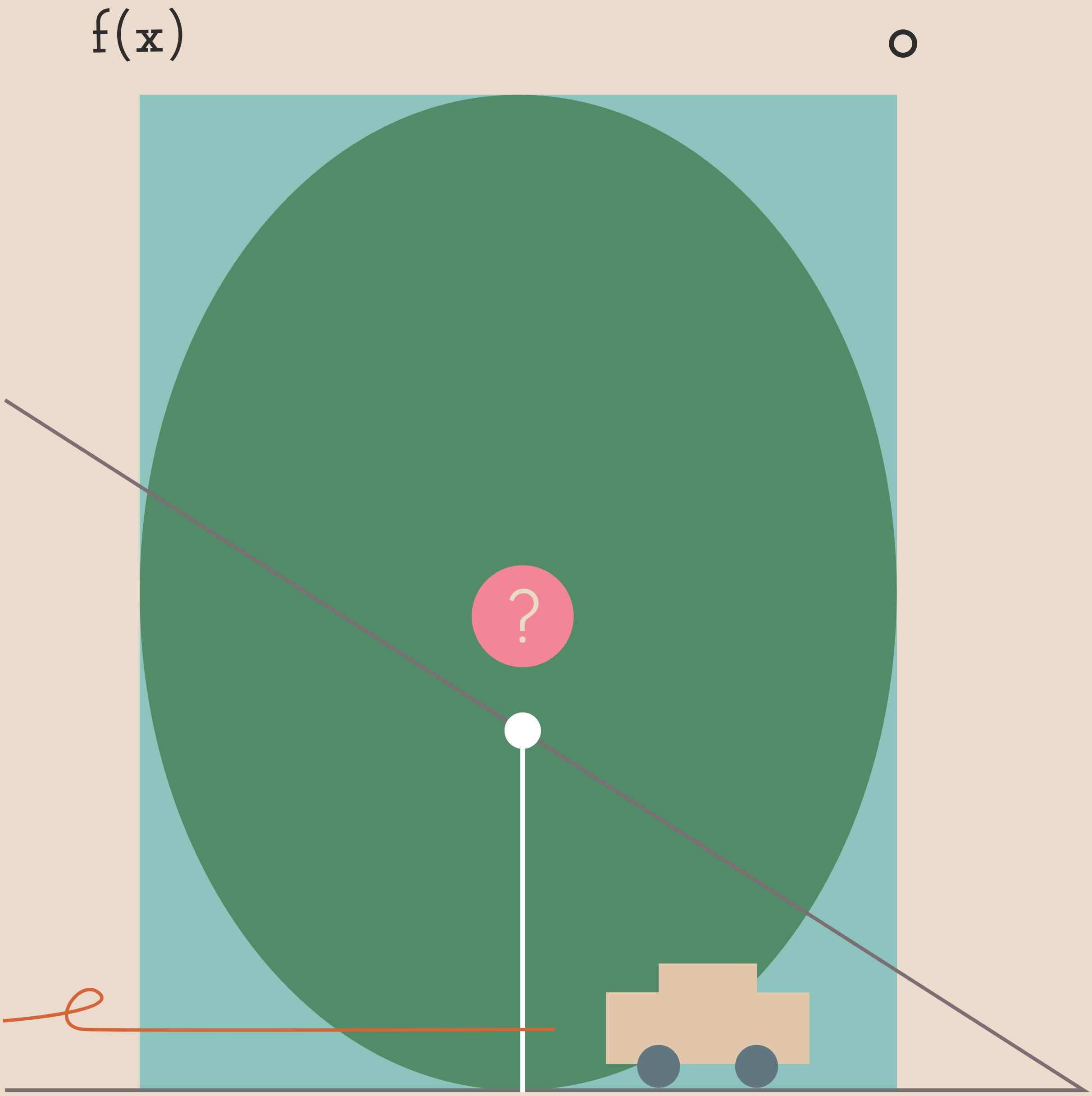
高中數學領域

雙語教學資源手冊 英語授課用語

A Reference Handbook for Senior High School Bilingual Teachers in the Domain of Mathematics: Instructional Language in English

〔高三下學期〕



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單元一　拋物線
Parabolas

國立臺灣師範大學附屬高級中學　蕭煜修老師

■ 前言 Introduction

我們在國中、高一時就曾經學過二次函數，包含了二次函數的圖形、平移與極值等問題，二次函數圖形其實就是拋物線，物理的斜拋運動、日常建築的設計等經常都可以見到拋物線的存在。除了用函數的方式描述拋物線，這個單元我們將說明如何用幾何的方式精準地描述拋物線。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
quadratic curve	**二次曲線**	cone	圓錐
quadratic function	二次函數	conic section	圓錐曲線
parabola	**拋物線**	directrix	準線
focus	**焦點**	axis of symmetry	對稱軸
focal length	**焦距**	chord	弦
latus rectum	**正焦弦**	projectile	拋體

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ The ________ follows ________.

例句：**The** path of a projectile **follow**s a parabolic trajectory under the influence of gravity.
在重力的影響下，拋體的路徑呈現拋物線的軌跡。

❷ Let's take a closer look at ________.

例句：**Let's take a closer look at** the definition of a parabola.
讓我們更仔細地看一下拋物線的定義。

❸ To derive ________, we use ________.

例句：**To derive** the equation of a parabola, **we use** the definition "a parabola is a curve generated by a moving point such that its distance from a fixed point is equal to its distance from a fixed-line."
為了推導拋物線的方程式，我們使用定義「拋物線是由一個點移動而生成的曲線，這個動點到一個固定點的距離等於它到一條固定直線的距離」。

❹ However, ________ we are mainly concerned with ________.

例句：**However**, in our high school curriculum, the symmetric axes of parabolas **we are mainly concerned with** are either parallel or perpendicular to the coordinate axes.
然而，在我們的高中課程中，我們主要關注拋物線的對稱軸要與坐標軸平行或垂直。

❺ In fact, we can achieve ________ by ________.

例句：**In fact, we can achieve** the desired result **by** applying the concept of translation.
事實上，我們可以透過應用平移的概念來達到所需要的結果。

■ 問題講解 Explanation of Problems

♋ 說明 ♌

I. Definition of the parabola

You learned the properties of quadratic functions ($f(x)=ax^2+bx+c$) in grade 10. The graphs of these quadratic functions are parabolas. In our daily lives, parabolas can be observed being applied in various fields. The following are some examples.

1. **<u>Projectile Motion</u>**

   The path of a projectile (throwing an object or launching a rocket) follows a parabolic trajectory under the influence of gravity (重力). Engineers and physicists use the equations of parabolas to predict and analyze the motion of projectiles.

2. **<u>Building Design</u>**

   Some architectural structures, such as the L'Oceanogràfic, incorporate parabolic shapes for both aesthetic and structural reasons.

L'Oceanogràfic (https://en.wikipedia.org/wiki/L%27Oceanogr%C3%A0fic#)

3. **<u>Telescope Mirrors</u>**

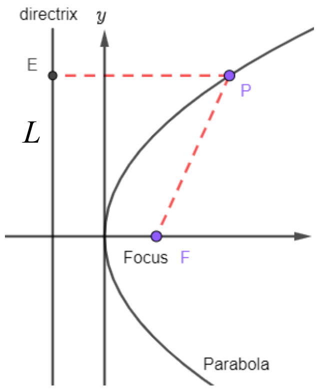
   The mirrors in reflecting telescopes are often shaped as segments of a parabola. This shape allows incoming parallel light rays to gather at a single point, improving the quality of the image.

From the examples above, you can see that we use parabolas in many different fields. How are they used? They use the definition and properties of the parabola. Let's take a closer look at the definition of a parabola.

Definition of the parabola

A parabola is a curve where any point is at an equal distance from:

1. a fixed point (focus), and
2. a fixed straight line (directrix) (This straight line won't pass through the focus.)



P is a point on the parabola.
F is the focus of the parabola.
L is the directrix of the parabola.

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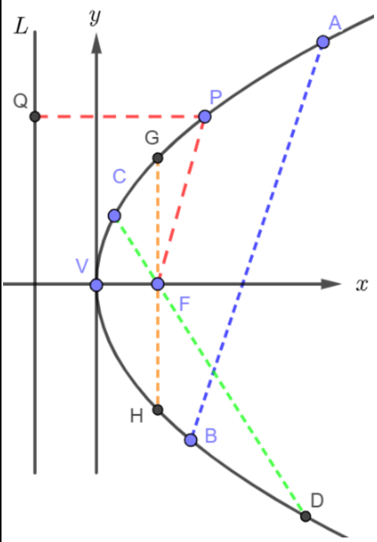


(1) **Axis of symmetry**:	The line passes through the focus F and perpendicular to the directrix L.
(2) **Vertex**: (point V on the graph)	The intersection of the parabola and its axis of symmetry.
(3) **Focal length**: (distance $\overline{VF}$)	The focal length is the distance between the vertex and the focus of the parabola.
(4) **Chords**: (segment $\overline{AB}$, $\overline{CD}$, $\overline{GH}$)	A chord of a parabola refers to the line segment that connects two distinct points on the parabola.
(5) **Focal chord** ($\overline{CD}$, $\overline{GH}$) and focal diameter:	A focal chord is the chord that passes through the focus. The focal diameter is the segment that passes through the focus and is perpendicular to the axis of symmetry.

II. Equation of a parabola

The equation of a parabola can help us solve many more complex problems. We use the definition to derive the equation of a circle $C:(x-h)^2+(y-k)^2=r^2$ (This circle is centered at (h,k) and has a radius of r). To derive the equation of a parabola, we use the definition "a parabola is a curve generated by a point moving such that its distance from a fixed point is equal to its distance from a fixed line." Let's take a look at the following example.

Example 1

Use the definition of a parabola to find the equation of a parabola with focus $F(1,1)$ and directrix $L: x+y+3=0$.

<illustration>

Let the point $P(x,y)$ be the point on the parabola, by definition, we have:

$$\sqrt{(x-1)^2+(y-1)^2}=\frac{|x+y+3|}{\sqrt{2}} \quad (\overline{PF}=d(P,L))$$

Square both sides, and we get: $x^2-2xy+y^2-10x-10y-5=0$ ◆

This is the equation of a parabola with a symmetric axis not parallel with the coordinate axes. However, in our high school curriculum, the symmetric axes of parabolas we are mainly concerned with are either parallel or perpendicular to the horizontal axis. Let's discuss this in more detail.

After reviewing the above example, you may wonder if there is a general method for solving quadratic equations and what commonalities exist among them. Let's take a look at the explanatory example together.

Example 2

Find the equation of the parabola with focus $F(0,c)$, $c\neq 0$ and directrix $L: y=-c$

<illustration>

Let $P(x,y)$ be the point on the parabola, by definition, we have:

$$d(P,L)=\overline{PF}$$

Then we use the distance formula to get:

$$d(P,L)=|y+c|=\sqrt{x^2+(y-c)^2}=\overline{PF}$$

Square both sides, and we have:

$$(y+c)^2 = x^2 + (y-c)^2$$

Then we can simplify it to get the final result:

$$x^2 = 4cy \blacklozenge$$

Conversely, if a point $P(x, y)$ satisfies the equation $x^2 = 4cy$, then the point P lies on the parabola. $x^2 = 4cy$ is a parabola with the y-axis as the axis of symmetry. Now, what if the parabola has the y-axis as the axis of symmetry? Let's look at the next example.

Example 3

Find the equation of the parabola with focus $F(c, 0),\ c \neq 0$ and directrix $L: x = -c$

<illustration>

We can use the same process as in Example 2.

Let point $P(x, y)$ be the point on the parabola, so by definition we have:

$$d(P, L) = \overline{PF}$$

Then we use the distance formula to get:

$$d(P, L) = |x+c| = \sqrt{(x-x)^2 + y^2} = \overline{PF}$$

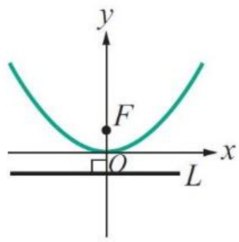
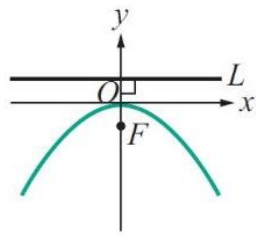
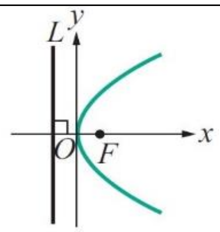
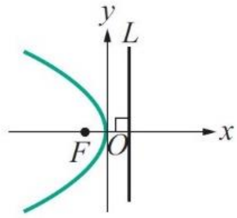
Square both sides, and we have:

$$(x+c)^2 = (x-c)^2 + y^2$$

Then we can simplify it to get the final result:

$$y^2 = 4cx \blacklozenge$$

With the examples above, we can draw the following table:

<table>
<tr><th>Equation</th><th>Vertex</th><th colspan="2">Symmetric axis</th><th>Focal length</th></tr>
<tr><td rowspan="3">$x^2 = 4cy$</td><td>(0,0)</td><td colspan="2">y-axis</td><td>$|c|$</td></tr>
<tr><td colspan="2">$c > 0$ (concave upward)</td><td colspan="2">$c < 0$ (concave downward)</td></tr>
<tr><td colspan="2">
</td><td colspan="2">
</td></tr>
<tr><th>Equation</th><th>Vertex</th><th colspan="2">Symmetric axis</th><th>Focal length</th></tr>
<tr><td rowspan="3">$y^2 = 4cx$</td><td>(0,0)</td><td colspan="2">x-axis</td><td>$|c|$</td></tr>
<tr><td colspan="2">$c > 0$ (concave to the right)</td><td colspan="2">$c < 0$ (concave to the left)</td></tr>
<tr><td colspan="2">
</td><td colspan="2">
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In fact, if we already have the vertex $(0,0)$ and symmetric axis $(y = 0$ or $x = 0)$ of a parabola, we only need to know the focal length and the direction of opening, then we can write the equation of the parabola. Let's try some more examples.

Now, we know how to find a parabola's equation when the parabola has a vertex at the origin and symmetry to one of the coordinate axes. What if the vertex of our parabola is not at the origin, and the symmetry axis is not the x-axis or the y-axis? In fact, we can achieve the desired result by applying the concept of translation, like what we learned in the operations on linear functions, quadratic functions, circles, and cubic functions. Using the same method, we can determine the equation we need. Let's try the following examples:

Example 4

Given a parabola $\Gamma_1 : x^2 = 4cy\ (c \neq 0)$, translating Γ_1 along the vector $\vec{v} = (-2,3)$ results in a new parabola Γ_2. Find the following:

(1) the vertex of Γ_2

(2) the equation of Γ_2

<illustration>

(1) The vertex of Γ_1 is $(0,0)$. All the points on Γ_1 move along the vector $\vec{v} = (-2,3)$, so the vertex of Γ_2 is $(0,0)+(-2,3)=(-2,3)$

The equation of Γ_2 is $(x+2)^2 = 4c(y-3)$

☙ 運算問題的講解 ❧

例題一

說明：利用拋物線的定義求出拋物線的要素。

（英文）Given the parabola with focus $F(1, 1)$ and directrix $L: y = x + 2$. Find:

(1) the function of the symmetric axis,

(2) the vertex of the parabola,

(3) the focal length of the parabola.

（中文）給定拋物線的焦點 $F(1, 1)$ 及準線 $L: y = x + 2$，試求：

(1) 對稱軸的方程式　(2) 拋物線的頂點　(3) 拋物線的焦距

Teacher: Now, let's try to see how we can use the definition of a parabola to determine its equation and various components. Can anyone tell me what conditions we have already grasped in this question?

Student: The focus and the directrix.

Teacher: Very good. Where should we start?

Student: Find the line that passes through the focus $F(1,1)$ and perpendicular to the directrix $L: y = x + 2$. Hence this line has gradient (slope) -1.

Teacher: Well done! Actually, this is the symmetric axis of the parabola. The equation of this axis is $L':(y-1)=-(x-1)$. The answer to the first question is: $y=-x+2$. How about the next question?

Student: We should find the intersection of the symmetric axis and the directrix. This intersection is $A(0,2)$. The vertex of the parabola is the midpoint between $A(0,2)$ and the focus $F(1,1)$. So, the vertex of the parabola is $V(\frac{1+0}{2},\frac{2+1}{2})=(\frac{1}{2},\frac{3}{2})$.

Teacher: Good job. How about the focal length of the parabola?

Student: To find the focal length of the parabola, we should find the distance between the vertex and the focus. It is $\overline{VF}$.

Teacher: Yes, to find the distance $\overline{VF}$ we'll use the distance formula: $\overline{VF}=\sqrt{(1-\frac{1}{2})^2+(1-\frac{3}{2})^2}=\sqrt{\frac{1}{4}+\frac{1}{4}}=\frac{\sqrt{2}}{2}$. I hope everyone finds this question easy, Next, you can practice on your own.

Student: Okay, no problem!

老師：現在我們來試試看，如何利用拋物線的定義求出拋物線的相關要素。(頂點、焦距、對稱軸等）由於這一題的拋物線的對稱軸不平行 x 軸或是 y 軸，這樣的拋物線方程式在現在的課綱中並不會特別強調，所以在這一題當中我們也不會去求出拋物線的方程式。請問誰可以說說看，這一題已經給了我們那些條件呢？

學生：拋物線的焦點與準線。

老師：非常好，我們要從哪裡開始呢？

學生：找出通過焦點 $F(1,1)$ 且與準線 $L:y=x+2$ 垂直的直線。由於 $L:y=x+2$ 的斜率為 1，所求的直線斜率為-1。

老師：沒錯，這就是拋物線的對稱軸，這個對稱軸的方程式為 $L':(y-1)=-(x-1)$ 經過整理後，可得第一題的直線方程式為 $y=-x+2$。那下一題呢？

學生：我們要先找到對稱軸與準線的交點，這個交點為 $A(0,2)$。此拋物線的頂點就是點 $A(0,2)$ 與焦點 $F(1,1)$ 的中點。因此可知拋物線頂點坐標為 $V(\frac{1+0}{2},\frac{2+1}{2})=(\frac{1}{2},\frac{3}{2})$。

老師：很好，那麼最後拋物線的焦距要怎麼做呢？

學生：焦距我們要找拋物線的頂點到焦點的距離（或是頂點到剛剛求出的點 $A(0,2)$ 的距離），也就是 $\overline{VF}$。

老師：　是的，我們如果要求出$\overline{VF}$的距離，需要使用距離公式：

$\overline{VF}=\sqrt{(1-\frac{1}{2})^2+(1-\frac{3}{2})^2}=\sqrt{\frac{1}{4}+\frac{1}{4}}=\frac{\sqrt{2}}{2}$。希望所有的同學可以覺得這一題不會很困難，回去之後你們也可以自己練習這些題目。

學生：　謝謝老師，沒有問題！

例題二

說明：利用方程式與拋物線的基本定義求解問題。

（英文）In the coordinate plane, a straight line passes through $F(1,0)$ and intersects $\Gamma: y^2=4x$ at two points P and Q. P is on the upper-half plane (the y-coordinate of point P is positive.) Given that $3\overline{PF}=5\overline{QF}$, find the x-coordinate of point P.

（中文）在坐標平面上，通過點$F(1,0)$的直線交拋物線$\Gamma: y^2=4x$於P、Q兩點，其中P在上半平面（P點的y坐標為正），且已知$3\overline{PF}=5\overline{QF}$，試求$P$點的$x$坐標。

Teacher:　This question is slightly more challenging compared to the previous one. Let's all read the question together and identify which conditions we can use.

Student:　In the coordinate plane…

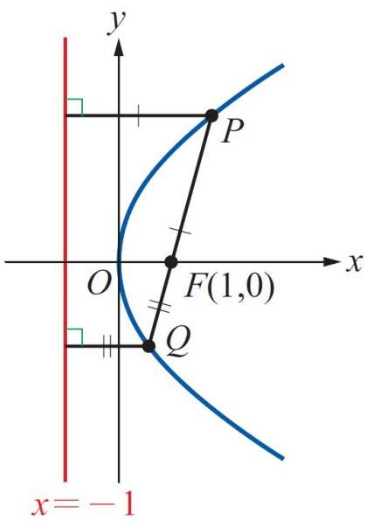
Teacher:　Okay, what conditions do we already have?

Student:　The equation of the parabola, a fixed point $F(1,0)$ on the plane and the proportion of the intersections ($3\overline{PF}=5\overline{QF}$).

Teacher:　With these conditions, we know that the directrix of the parabola is $L: x=-1$, and for $\overline{PF}:\overline{QF}=5:3$, let $\overline{PF}=5k$ and $\overline{QF}=3k$. What should we do next?

Student:　For $d(P,L)=\overline{PF}=5k$, we know that the x coordinate of point P is $5k-1$.

For $d(Q,L)=\overline{QF}=3k$, we know that the x coordinate of point Q is $3k-1$.



Teacher: Good, like in the figure,

we can apply the formula for internal division.

$\dfrac{3(5k-1)+5(3k-1)}{8}=1$, so, $k=\dfrac{8}{15}$.

We have the x coordinate of point P is

$5\times\dfrac{8}{15}-1=\dfrac{5}{3}$.

(The x coordinate of point Q is $3\times\dfrac{8}{15}-1=\dfrac{3}{5}$.)

If the ratio of $\overline{PF}$, $\overline{QF}$ changes in the problem, we can still get the result by adjusting the preceding numbers accordingly. The key to this question lies in being able to use the fact that F is the focus of the parabola and can deduce the directrix through the given equation of the parabola. With this information, the question can be solved.

老師： 這一題相較於前面的問題稍微困難一些，請各位同學一起把這題的題目讀一遍，並找找看我們有哪些可以使用的條件。

學生： 在坐標平面上…

老師： 很好，題目中已經給定了我們哪些條件？

學生： 拋物線的方程式、一個固定點（拋物線的焦點）$F(1,0)$，以及通過焦點的直線與拋物線交點的比例關係 $3\overline{PF}=5\overline{QF}$ 。

老師： 是的，利用這些條件，我們知道這個拋物線的準線方程式為 $L:x=-1$，此外，由於 $\overline{PF}:\overline{QF}=5:3$，我們可以假設 $\overline{PF}=5k$ 以及 $\overline{QF}=3k$。接下來我們可以做些甚麼？

學生： 因為 $d(P,L)=\overline{PF}=5k$，我們知道 P 點的 x 坐標為 $5k-1$ 。

因為 $d(Q,L)=\overline{QF}=3k$，我們知道 Q 點的 x 坐標為 $3k-1$。

老師：很好，如同下面的圖形所示：

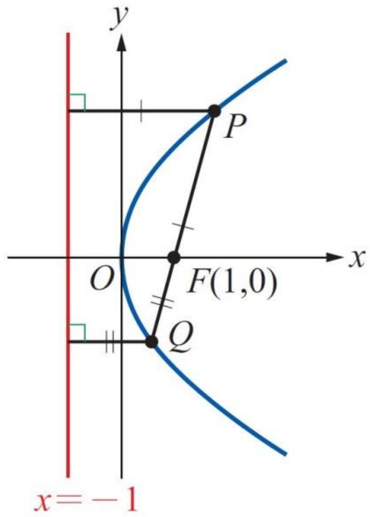
我們可以利用內分點公式，

$\frac{3(5k-1)+5(3k-1)}{8}=1$ 因此 $k=\frac{8}{15}$。

我們知道 P 點的 x 坐標為 $5\times\frac{8}{15}-1=\frac{5}{3}$。

（Q 點的 x 坐標為 $3\times\frac{8}{15}-1=\frac{3}{5}$。）

如果題目改變 $\overline{PF}, \overline{QF}$ 的比例，我們只要調整前面的數字一樣可以得出結果。



這題的關鍵在於，同學們要可以利用 F 就是拋物線的焦點、並且可以透過給定的拋物線方程式推論出準線，就可以把這題給解出來了！

應用問題 / 學測指考題

例題一

說明：利用拋物線性質求解應用問題。

（英文）In a situation where there is only a tape measure and no ladder, measure the height of a parabolic arch. It is known that the vertical line passing through the highest point of the parabola is the axis of symmetry. Persons A and B measured the width of the arch at the bottom with a tape measure to be 6 meters, and at a height of $\frac{3}{2}$ meters above the bottom, the width is 5 meters. Using the data above, calculate the height of the arch to be ______ meters. Express your answer in its simplest fraction.

（中文）在只有皮尺沒有梯子的情形下，想要測出一拋物線形拱門的高度。已知此拋物線以過最高點的鉛垂線為對稱軸。現甲、乙兩人以皮尺測得拱門底部寬為 6 公尺，且距底部 $\frac{3}{2}$ 公尺高處其寬為 5 公尺。利用這些數據可推算出拱門的高度為______公尺。（化成最簡分數）

（92 年學測數學填充題 7）

Teacher: The following is a word problem with a lengthy description. Please read through the question first.

Student: In a situation where…

Teacher: This is a measurement problem in real life where we need to measure the height of a parabolic arch. We can make assumptions and utilize a coordinate-based approach for this.

Student: Ah… this seems a bit challenging. Where can we start?

Teacher: To start, we can assume the center of the imaginary arch at the bottom as the origin of the coordinate axis. So, the coordinates of the right bottom point of the arc are $A(3,0)$, and the left bottom point of the arc are: $B(-3,0)$. What are the coordinates of the top left and top right points? ($\frac{3}{2}$ meters above the bottom.)

Student: The top left is $C(\frac{-5}{2},\frac{3}{2})$ and the top right: $D(\frac{5}{2},\frac{3}{2})$.

Teacher: Well done. Since the parabola is concave downward, let the equation of the parabola be: $y=ax^2+b$.

Student: Then we substitute the known coordinates into the equation to solve for the equation. We can substitute the coordinates of points $A(3,0)$ and $C(\frac{-5}{2},\frac{3}{2})$ we got earlier.

Teacher: Wonderful. Now we have the simultaneous equations $\begin{cases} 0=9a+b \\ \frac{3}{2}=\frac{25}{4}a+b \end{cases} \Rightarrow \begin{cases} a=-\frac{6}{11} \\ b=\frac{54}{11} \end{cases}$.

So, the equation of the parabola is: $y=-\frac{6}{11}x^2+\frac{54}{11}$. Since the center of the arch at the bottom is the origin, we simply substitute $x=0$ to find the corresponding y value, which represents the height of the arch.

Student: The height of the arch is $\frac{54}{11}$ meters.

Teacher: Correct. In problems like this, all students need more practice in making assumptions. Everyone should make good use of the properties of parabolas and the origin.

老師：接下來是一個應用問題，題目敘述較長，請同學們先把題目讀過一遍。

學生：在只有皮尺沒有梯子…

老師：這是一個生活中的測量問題，我們要測量拋物線的拱門高度，我們可以利用坐標化的方式進行假設。

學生：哦…這好像有點困難，老師，我們應該要從哪一部分開始下手？

老師：一開始，我們可以假設拱門底部中心為坐標軸的原點，所以拱門的右底部的點坐標為 $A(3,0)$、左底部的點坐標為 $B(-3,0)$。那麼左上與右上兩個角的點坐標呢？（距離底部 $\frac{3}{2}$ 公尺高的地方）

學生：左上角是 $C(\frac{-5}{2},\frac{3}{2})$、右上角是 $D(\frac{5}{2},\frac{3}{2})$。

老師：很好，因為這個拋物線是一個凹口向下的上下型拋物線，我們可以假設這個拋物線的方程式為：$y=ax^2+b$。

學生：然後我們再把已知的點坐標代入解出方程式，我們可以把前面求出的點 $A(3,0)$ 與點 $C(\frac{-5}{2},\frac{3}{2})$ 代入。

老師：太棒了，因此我們得到聯立方程式：$\begin{cases} 0=9a+b \\ \frac{3}{2}=\frac{25}{4}a+b \end{cases} \Rightarrow \begin{cases} a=-\frac{6}{11} \\ b=\frac{54}{11} \end{cases}$。

可以求出拋物線方程式為 $y=-\frac{6}{11}x^2+\frac{54}{11}$，因為拱門底部中心為原點，所以我們只要把 $x=0$ 代入，所對應的 y 值就是拱門的高度了。

老師：拱門的高度是 $\frac{54}{11}$ 公尺。

學生：沒錯，這樣的問題同學們還要再多加練習如何假設，要善用拋物線的性質與坐標軸中的原點。

例題二

說明：利用拋物線的方程式與直角坐標解題。

（英文）Let $A(1,0)$, $B(b,0)$ be two points on the coordinate plane and $b>1$. If there is a point P on the parabola $\Gamma: y^2=4x$ such that ΔABP is an equilateral triangle, find $b=$_____.

（中文）設 $A(1,0)$, $B(b,0)$ 為坐標平面上的兩點，其中 $b>1$。若拋物線 $\Gamma: y^2=4x$ 上有一點 P 使得 ΔABP 為一正三角形，則 $b=$_____。

（92 年學測數學填充題 3）

Teacher: This question is also from the same year's entrance exam. Unlike the previous one, this one provides us with the equation of the parabola in advance. As long as we make good use of the given conditions, it shouldn't be difficult to solve the problem. How can we get started?

Student: Since point P is on the parabola, we can assume that the coordinates of point P are $P(a^2,2a)$, and ΔABP is an equilateral triangle. How can we make this assumption?

Teacher: Since ΔABP is an equilateral triangle and $\overline{AB}$ lies on the x-axis, we have the following:

1. $\overline{PA}=\overline{AB} \Rightarrow (a^2-1)^2+4a^2=(b-1)^2 \ldots(*)$

2. Point P's x-coordinate is the average of the x-coordinates of points A and B.

$$\Rightarrow a^2=\frac{b+1}{2} \ldots(**)$$

Plug (**) into (*) then we get: $(\frac{b+1}{2}-1)^2+4\cdot\frac{b+1}{2}=(b-1)^2$

So, $3b^2-14b-5=0 \Rightarrow (b-5)(3b+1)=0$.

There are two possibilities. Are both of these correct, or is only one of them correct?

Student: The condition given in the problem is that $b>1$,so for $b=5$ and $b=-\frac{1}{3}$, we can only accept the result $b=5$.

Teacher: Well done! In problems like this, where there are given conditions or constraints, it's important to remember to substitute the answers back into the original equations to confirm whether the results we've obtained fall within the specified range. Skipping this step can often lead to extra solutions being included. You must pay special attention to this.

Student: No problem!

老師： 這一題也是同一年的大考題，但是與前面問題不一樣的地方在於這一題預先給了我們拋物線的方程式，同學們如果可以妥善利用方程式與所給定的條件應該不難把問題解出來。我們可以從哪邊開始下手呢？

學生： 因為點 P 在拋物線上，我們可以假設 P 點的坐標為 $P(a^2, 2a)$，此外$\triangle ABP$ 是一個正三角形…這個我們應該要怎麼進行假設呢？

老師： 沒關係，老師幫大家一下，因為 $\triangle ABP$ 是一個正三角形，而且線段 $\overline{AB}$ 落在 x 軸上，我們可以有以下的關係：

1. $\overline{PA} = \overline{AB} \Rightarrow (a^2 - 1)^2 + 4a^2 = (b-1)^2 ...(*)$

2. 點 P 的 x 坐標是點 A 與點 B 的 x 坐標的平均值$\Rightarrow a^2 = \dfrac{b+1}{2}...(**)$

接著我們再把(**)的結果代入(*)，我們可以得到 $(\dfrac{b+1}{2} - 1)^2 + 4 \cdot \dfrac{b+1}{2} = (b-1)^2$

因此：$3b^2 - 15b - 5 = 0 \Rightarrow (b-5)(3b+1) = 0$

不過這裡有兩個可能的解。這兩個解都是正確的嗎？或是只有其中一個是正確的？為什麼呢？

學生： 這一題的條件中說了 $b > 1$，所以在這兩個解 $b = 5$ 以及 $b = -\dfrac{1}{3}$ 中，我們只可以取 $b = 5$ 是正確的。

老師： 做得好！在這樣有給定範圍條件的問題中，最後求出答案時，別忘了要把答案再代回去確認一遍，看看我們所求出來的結果是否有符合題目所設定的範圍限制。如果少了這一步，常常會多算出一些不符合的答案，同學們一定要特別注意。

學生： 好的老師，沒問題！

單元二　橢圓
Ellipses

臺北市陽明高級中學　吳柏萱老師

■ 前言 Introduction

我們將從幾何定義來討論橢圓的圖形，學會橢圓的方程式與圖形特徵，進一步將橢圓進行平移、伸縮與旋轉變換，並解決生活中的應用問題。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
ellipse	**橢圓**	vertex (複數形：vertices)	頂點
focus (複數形：foci)	**焦點**	horizontal	水平的
center	中心	vertical	垂直的
major axis	**長軸**	parametric	參數的
minor axis	**短軸**	rotation	旋轉

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ To figure out ________, we have to find ________.

例句：**To figure out** point P, **we have to find** point P' first.

要了解點 P，我們要先找到點 P'。

❷ Applying ________, we obtain ________.

例句：**Applying** the distance formula, **we obtain** $\sqrt{(x-c)^2+y^2}+\sqrt{(x+c)^2+y^2}=2a$.

利用距離公式，我們得到 $\sqrt{(x-c)^2+y^2}+\sqrt{(x+c)^2+y^2}=2a$。

❸ ________ is centered at ________.

例句：The ellipse **is centered at** the origin.

橢圓中心在原點。

❹ Plug in ________ for ________.

例句：**Plug in** $\left(\frac{x}{2}, \frac{y}{3}\right)$ **for** equation $\Gamma_1 : \frac{x^2}{3}+\frac{y^2}{4}=1$.

將 $\left(\frac{x}{2}, \frac{y}{3}\right)$ 代入式子 $\Gamma_1 : \frac{x^2}{3}+\frac{y^2}{4}=1$。

■ 問題講解 Explanation of Problems

說明

[Definition of an Ellipse]

Recall that a circle is the set of all points where the distance from *a fixed point* is a constant. Similarly, an ellipse is the set of all points where the sum of the distances from *two fixed points* is a constant.

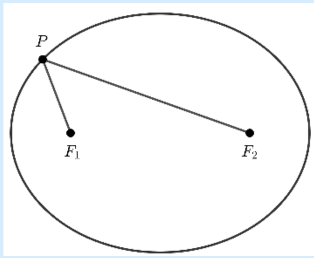
Definition of the Ellipse

An ellipse is the set of all points $P(x, y)$ in a plane, the sum of whose distances from two distinct fixed points, F_1 and F_2, is constant $2a$, where $2a > \overline{F_1F_2}$.

That is

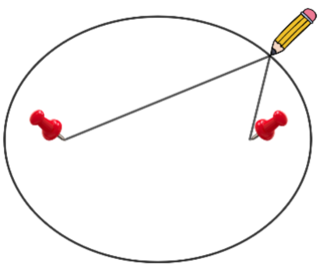
$$\overline{PF_1} + \overline{PF_1} = 2a$$

Then F_1 and F_2 are called **foci**.

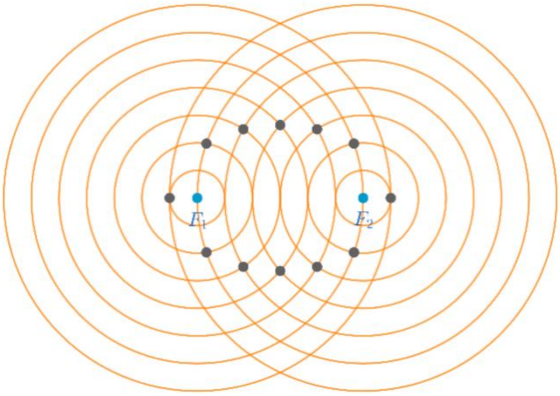


Note: A circle is also an ellipse, where the foci are at the same point, which is the **center** of the circle.

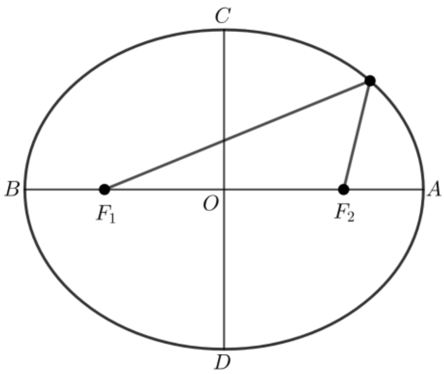
We can draw an ellipse using a piece of cardboard, two thumbtacks, a pencil, and a piece of string. Place the thumbtacks in the cardboard to form the foci of the ellipse. Cut a piece of string longer than the distance between the two thumbtacks. The length of the string represents the constant in the definition. When the ends of the string are fastened to the thumbtacks and the string is pulled tight with a pencil, the path traced by the pencil is an ellipse.



Consider two points, F_1 and F_2, as the centers of concentric circles with radii 1, 2, 3, 4, 5, 6, and 7, as shown in the figure. All the black dots satisfy the condition that the sum of distances from each black dot to F_1 and from the same black dot to F_2 is 8. We can construct more concentric circles to create additional black dots, enabling us to sketch a smoother curve, which is an ellipse.



The midpoint of the foci is the *center* O of the ellipse. The line through the foci intersects the ellipse at two points A and B. The chord joining A and B is the ***major axis***. The line perpendicular bisector of the major axis intersects the ellipse at two points C and D. The chord joining C and D is the ***minor axis*** of the ellipse. The points A, B, C, and D are called **vertices** of the ellipse.



Key features of an ellipse:

(1) A Center: The midpoint of $\overline{F_1F_2}$ is called the center. Let $\overline{F_1F_2}$ be $2c$, then $\overline{OF_1} = \overline{OF_2} = c$.

(2) Vertices: The line F_1F_2 intersects the ellipse at the two points A and B, which are called vertices. The perpendicular bisector of $\overline{F_1F_2}$ intersects the ellipse at two points C and D, which are also called vertices. So, there are four vertices of an ellipse.

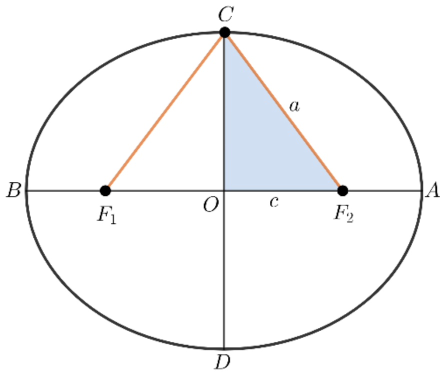
(3) A Major Axis: $\overline{AB}$ is called the major axis, and its length is $2a$. The illustration is as follows: Since point A is on the ellipse, $\overline{AF_1}+\overline{AF_2}=2a$. Also,

$$\begin{aligned}\overline{AF_1}+\overline{AF_2}&=\left(\overline{AO}+\overline{OF_1}\right)+\overline{AF_2}\\&=\overline{AO}+\left(\overline{OF_2}+\overline{AF_2}\right)=2\overline{AO}\end{aligned}$$

Thus, $\overline{AO}=a$, and likewise $\overline{BO}=a$.

Therefore, center O is the midpoint of major axis $\overline{AB}$, and the major axis of length $\overline{AB}$ is $2a$.

(4) A Minor Axis: $\overline{CD}$ is called the minor axis, and its length is denoted as $2b$, where $b=\overline{CO}=\overline{DO}$ satisfies $b^2=a^2-c^2$. The illustration is as follows:



Since point C is on the ellipse, $\overline{CF_1}+\overline{CF_2}=2a$.

Also, $\overline{CD}$ is the perpendicular bisector of $\overline{F_1F_2}$, then

$\overline{CF_1}=\overline{CF_2}=a$.

In other words, the hypotenuse in the right triangle OCF_1 is a, as shown in the figure.

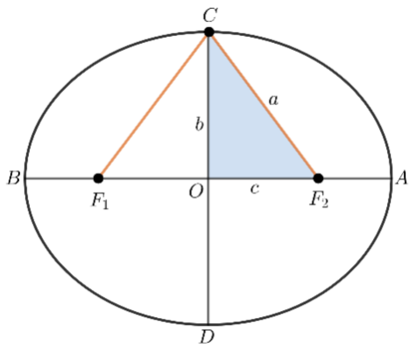
Using Pythagoras' theorem, we have $\overline{CO}=\sqrt{a^2-c^2}$, and similarly, $\overline{DO}=\sqrt{a^2-c^2}$.

Therefore, center O is the midpoint of the minor axis $\overline{CD}$. And, $b=\overline{CO}=\overline{DO}$, this gives a minor axis of length $\overline{CD}$, which is $2b$, and which satisfies $a^2=b^2+c^2$.

Properties of an Ellipse

Let F_1 and F_2 be the foci of the ellipse, and the sum of the distances between any point and the two foci is a constant.

Let O be the center, $\overline{AB}$ be the major axis, and $\overline{CD}$ be the minor axis, as shown in the figure to the right.



(1) Center O is the midpoint of $\overline{F_1F_2}$, the major axis $\overline{AB}$, and minor axis $\overline{CD}$.

(2) The major axis of length $\overline{AB}$ is $2a$.

(3) When the minor axis of length $\overline{CD}$ is $2b$ and $\overline{F_1F_2}$ is $2c$, the constants a, b, and c satisfy the equation $a^2=b^2+c^2$.

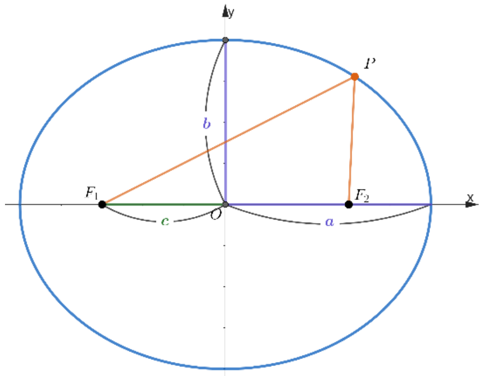
[Equations of an Ellipse Centered at the Origin]

Based on the definition and properties of an ellipse, we can derive the standard equation of an ellipse centered at the origin with the foci on the *x*-axis and *y*-axis.

(1) Horizontal Major Axis

Given the major axis of length as $2a$ and minor axis as $2b$, we determine c using the equation $a^2=b^2+c^2$. This gives the values for the foci and provides one and only one equation of the ellipse.

To derive the equation of the ellipse centered at the origin with a horizontal major axis, we consider the ellipse in the figure with the points listed below.



Center: $(0,0)$ Vertices: $(\pm a,0)$ and $(0,\pm b)$ Foci: $(\pm c,0)$

Let $P(x,y)$ be any point on the ellipse, then the sum of the distances between $P(x,y)$ and the two foci is $2a$. That is $\overline{PF_1}+\overline{PF_2}=2a$. Applying the distance formula, we have

$$\sqrt{(x-c)^2+y^2}+\sqrt{(x+c)^2+y^2}=2a.$$

After subtracting $\sqrt{(x+c)^2+y^2}$ from both sides, we get

$$\sqrt{(x-c)^2+y^2}=2a-\sqrt{(x+c)^2+y^2}.$$

Square and expand, and then we get

$$\left(x^2-2cx+c^2\right)+y^2=4a^2-4a\sqrt{(x+c)^2+y^2}+\left(x^2+2cx+c^2\right)+y^2.$$

Reduce it, and we get

$$a\sqrt{(x+c)^2+y^2}=a^2+cx.$$

Again, square and expand it, and we get

$$a^2x^2+2a^2cx+a^2c^2+a^2y^2=a^4+2a^2cx+c^2x^2.$$

Reduce and regroup it, we get

$$\left(a^2-c^2\right)x^2+a^2y^2=a^2\left(a^2-c^2\right).$$

From the equation $a^2 = b^2 + c^2 \Rightarrow b^2 = a^2 - c^2$

It means that the equation of the ellipse is

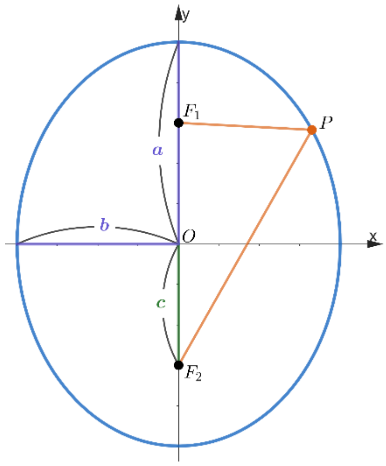
$$b^2x^2 + a^2y^2 = a^2b^2.$$

Dividing a^2b^2 on both sides, we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

(2) Vertical Major Axis

To derive the equation of the ellipse centered at the origin with the vertical major axis, we consider the ellipse in the figure with the points listed below.



Center: $(0,0)$ Vertices: $(\pm b,0)$ and $(0,\pm a)$ Foci: $(0,\pm c)$

We would obtain a similar equation in the derivation by starting with a vertical major axis.

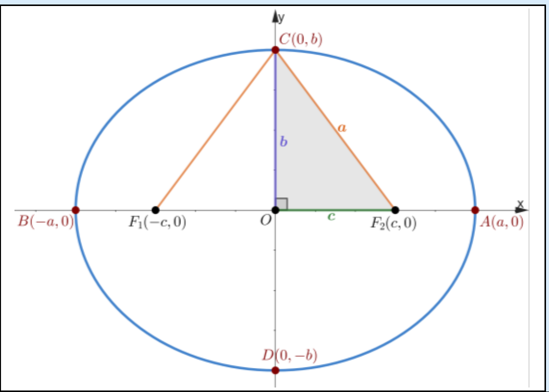
$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

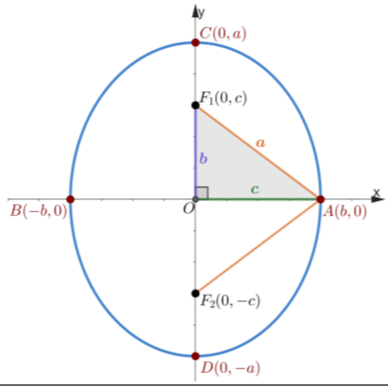
As shown in the figure, *a* and *b* still represent the half-lengths of the major and minor axes, respectively.

A summary of these results is given below.

Standard Equation of an Ellipse Centered at the Origin

Below is the standard form of the equation of an ellipse with its center at the origin and major and minor axes of lengths 2*a* and 2*b*, respectively. This is where $0 < b < a$.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	The major axis is horizontal.	

$\dfrac{x^2}{b^2}+\dfrac{y^2}{a^2}=1$	The major axis is vertical.	
The foci lie on the major axis, c units from the center, with $c^2=a^2-b^2$.		

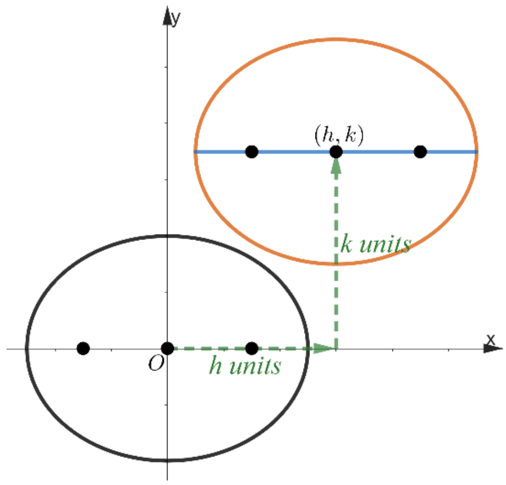
[Standard Equation of the Ellipse]

If the center of the ellipse is any point (h,k), and the major axis is either horizontal or vertical, we can vertically and horizontally shift the graph of the ellipse centered at the origin.

Consider shifting an ellipse centered at the origin with a horizontal major axis, $\Gamma_h:\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, by h units to the right and by k units up.

That is, replace x with $x-h$ and y with $y-k$, which means the equation $\Gamma_h:\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is transformed to

$$\Gamma_h':\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1.$$

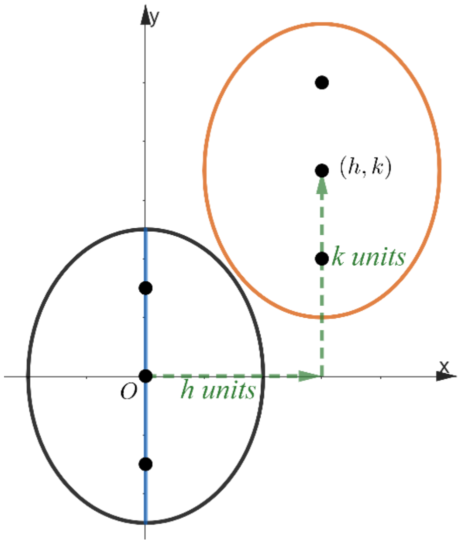


Meanwhile, the center of the new ellipse Γ' is (h,k), and the foci are $(h+c,k)$ and $(h-c,k)$.

Similarly, by shifting the ellipse centered at the origin with a vertical major axis, $\Gamma_v : \frac{x^2}{b^2}+\frac{y^2}{a^2}=1$, by h units to the right and k units up, we get the new equation:

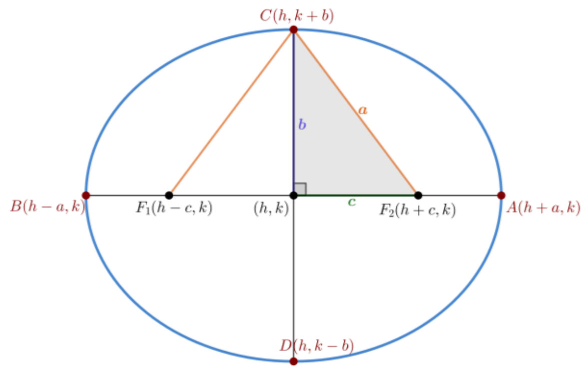
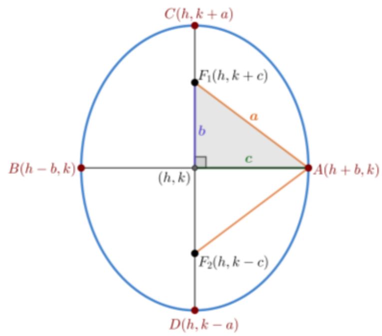
$$\Gamma_v' : \frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1.$$

A summary of the standard form of an ellipse is given below.



Standard Equation of an Ellipse

Below is the standard form of the equation of an ellipse with center (h,k) and major and minor axes of lengths 2*a* and 2*b*, respectively. This is where $0<b<a$.

$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$	The major axis is horizontal.	
$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$	The major axis is vertical.	

The foci lie on the major axis, *c* units from the center, with

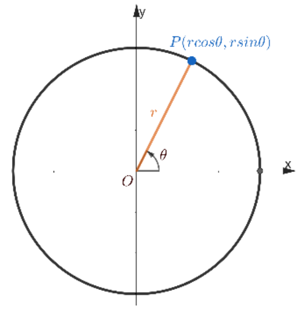
$c^2=a^2-b^2$.

[Applications of an Ellipse]

(1) Parametric Equation of an Ellipse

Let $P(x,y)$ be a point on circle $C: x^2+y^2=r^2$ ($r>0$), and the terminal side of θ passes through point P. Using the definition of trigonometric functions, we have

$$\cos\theta=\frac{x}{r},\ \sin\theta=\frac{y}{r}\quad (0\le\theta<2\pi)$$



which can be rewritten as:

$$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}\quad (0\le\theta<2\pi),$$

as shown in the figure.

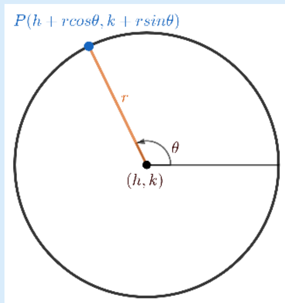
Thus, every point on the circle can be represented as $(x,y)=(r\cos\theta, r\sin\theta)$ Conversely, every point that can be represented as $(r\cos\theta, r\sin\theta)$ lies on the circle. This form is called a parametric equation which is different from a rectangular equation, and the angle θ is called a parameter.

Parametric Equations of a Circle

The parametric equation of a circle $C:(x-h)^2+(y-k)^2=r^2$ is

$$\begin{cases} x-h=r\cos\theta \\ y-k=r\sin\theta \end{cases}\quad (0\le\theta<2\pi),$$

rewritten as $\begin{cases} x=h+r\cos\theta \\ y=k+r\sin\theta \end{cases}\quad (0\le\theta<2\pi)$.



Applying the parametric equation of a circle derives the parametric equation of an ellipse. Let $P(x,y)$ be a point on the ellipse $\Gamma:\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, and point *P* satisfies the equation $\Gamma:\left(\frac{x-h}{a}\right)^2+\left(\frac{y-k}{b}\right)^2=1$. So, point $\left(\frac{x-h}{a},\frac{y-k}{b}\right)$ lies on the unit circle, and according to the parametric equation of a circle we get

$$\frac{x-h}{a}=\cos\theta,\ \frac{y-k}{b}=\sin\theta\ \ (0\le\theta<2\pi),$$

rewritten as

$$\begin{cases}x-h=a\cos\theta\\ y-k=b\sin\theta\end{cases}\ (0\le\theta<2\pi).$$

Thus, every point on the ellipse can be represented as $(x,y)=(h+a\cos\theta,k+b\sin\theta)$.

Conversely, every point that can be represented as $(h+a\cos\theta,k+b\sin\theta)$ lies on the ellipse.

It is called the parametric equation of an ellipse and the angle θ is called as parameter.

A summary of foregoing results is given below.

Parametric Equations of an Ellipse

The parametric equation of an ellipse $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ is

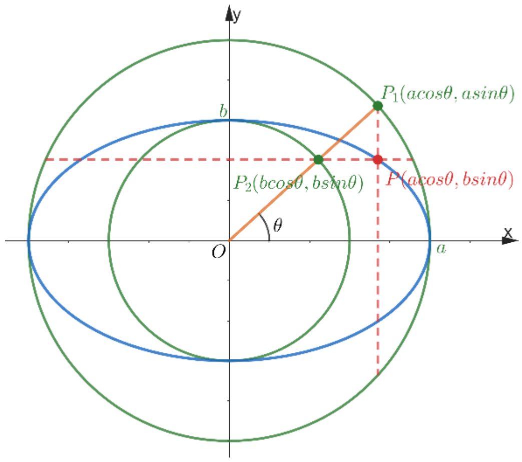
$$\begin{cases}x-h=a\cos\theta\\ y-k=b\sin\theta\end{cases}\ (0\le\theta<2\pi),\text{ rewritten as }\begin{cases}x=h+a\cos\theta\\ y=k+b\sin\theta\end{cases}\ (0\le\theta<2\pi).$$

Note that, the angle θ of the given point on the ellipse $P(a\cos\theta,b\sin\theta)$ is not the angle between $\overline{OP}$ and the *x*-axis.

Take ellipse $\Gamma:\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ for example, the ellipse Γ intersects with large circle $C_1:x^2+y^2=a^2$ and small circle $C_2:x^2+y^2=b^2$ at vertices of the major and minor axes, as shown in the right figure. Let P_1 and P_2 intersect with the terminal sides of θ and two circles, and we have the parametric equation of circles

$$P_1(a\cos\theta,a\sin\theta)\ \text{ and }\ P_2(b\cos\theta,b\sin\theta).$$

Now, construct a vertical line and horizontal line which pass through P_1 and P_2, respectively. The intersection of the two lines is $P(a\cos\theta, b\sin\theta)$.



(2) Horizontal and Vertical Stretches of an Ellipse

Recall that stretching point $P(s,t)$ horizontally by a factor p ($p>0$) and vertically by a factor q ($q>0$) results in the new point $Q(x,y)$, which is

$$\begin{cases} x = ps \\ y = qt \end{cases}.$$

In general, stretching a graph horizontally by a factor p and vertically by a factor q results in a new graph, which is

$$x \to \frac{x}{p} \text{ and } y \to \frac{y}{q}.$$

Thus, stretching $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ horizontally by a factor p and vertically by a factor q results in new equations of ellipse, which gives

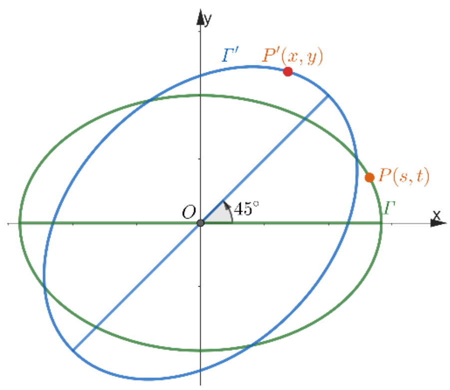
$$\frac{\left(\frac{x}{p}\right)^2}{a^2}+\frac{\left(\frac{y}{q}\right)^2}{b^2}=1 \text{ and } \frac{\left(\frac{x}{p}\right)^2}{b^2}+\frac{\left(\frac{y}{q}\right)^2}{a^2}=1.$$

(3) Rotation of an Ellipse

Recall that using the rotation matrix to transform point $P(x,y)$ by an angle θ counterclockwise around the origin results in the equation of new point $P'(x',y')$ and original point $P(x,y)$:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

For instance, rotate the ellipse $\Gamma: \frac{x^2}{2}+\frac{y^2}{1}=1$ by $45°$ counterclockwise around the origin. Let $P(s,t)$ be any point on the original ellipse Γ, and $P'(x,y)$ be the point on the new ellipse Γ' that is rotated by $45°$ counterclockwise around the origin, as shown in the figure. Using the rotation matrix, we get



$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}.$$

By rewriting and reducing, we get

$$\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}(x+y) \\ \frac{1}{\sqrt{2}}(-x+y) \end{bmatrix}.$$

Substituting (s,t) for Γ, we get $\frac{s^2}{2}+\frac{t^2}{1}=1$. Thus, this gives

$$\frac{\left(\frac{1}{\sqrt{2}}(x+y)\right)^2}{2}+\frac{\left(\frac{1}{\sqrt{2}}(-x+y)\right)^2}{1}=1,$$

Expanding and simplifying it, we have

$$\frac{x^2+2xy+y^2}{4}+\frac{x^2-2xy+y^2}{2}=1\Rightarrow 3x^2-2xy+3y^2=4$$

Therefore, rotating ellipse $\Gamma:\frac{x^2}{2}+\frac{y^2}{1}=1$ by $45°$ counterclockwise around the origin, we obtain the new ellipse $\Gamma':3x^2-2xy+3y^2=4$.

運算問題的講解

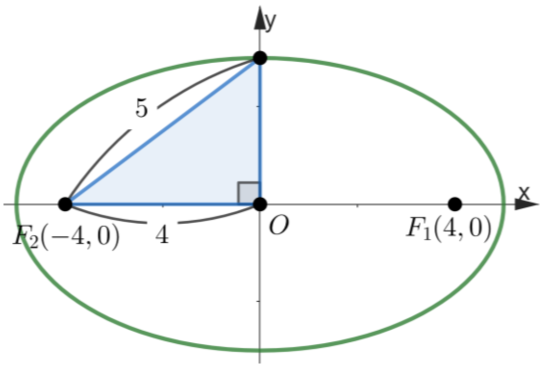
例題一

說明：已知橢圓的一些元素時，求出其方程式。

（英文）Write the equation for the given ellipse that satisfies the following conditions.
Foci: $F_1(4, 0)$ and $F_2(-4, 0)$; length of the major axis: 10.

（中文）求滿足下列各條件的橢圓方程式。焦點為 $F_1(4, 0)$ 與 $F_2(-4, 0)$，長軸長為 10。

Teacher:



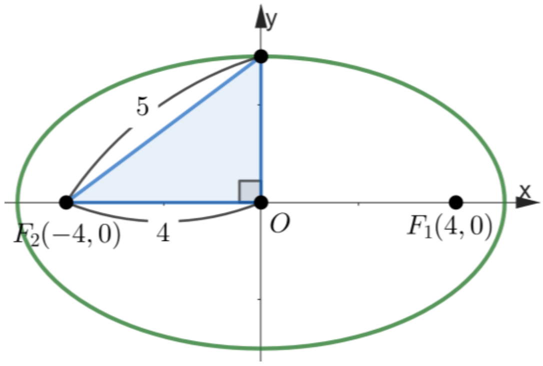
As shown in the figure, it's a horizontal ellipse centered at the origin. Given that $2a=10$ and $2c=\overline{F_1F_2}=8$, we get $a=5$ and $c=4$. How can we find the value of b?

Student: Using the definition of an ellipse $a^2=b^2+c^2$, we get $b=3$.

Teacher: By applying the standard equation of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, what can we get?

Student: The equation of an ellipse is $\frac{x^2}{25}+\frac{y^2}{9}=1$.

老師：



如圖所示，可得橢圓的長軸在 x 軸上，且其中心為原點。而由題目可知，長軸長為 $2a=10$，$2c=\overline{F_1F_2}=8$，所以 $a=5$ 及 $c=4$。我們要如何得知 b 之值？

學生：由橢圓的定義 $a^2=b^2+c^2$，得 $b=3$。

老師：橢圓的方程式形如 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$，所以我們可以得到？

學生：其方程式為：$\frac{x^2}{25}+\frac{y^2}{9}=1$。

例題二

說明：本題練習由橢圓的方程式求出其圖形的各元素。

（英文）Find the vertices and foci of the ellipse $4x^2+y^2+16x+2y+13=0$.

（中文）求橢圓 $4x^2+y^2+16x+2y+13=0$ 的頂點與焦點坐標。

Teacher: Completing the squares, what can we get?

Student: $4x^2+y^2+16x+2y+13=0$

$\Rightarrow 4(x^2+4x+4)+(y^2+2y+1)=4$

$\Rightarrow 4(x+2)^2+(y+1)^2=4$

Teacher: And, dividing both sides by 4, we get…

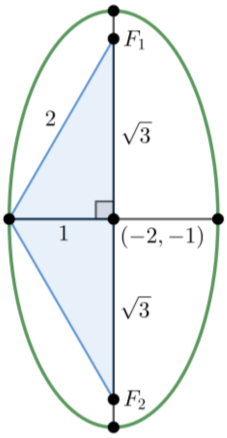
Student: By using the standard equation $\frac{(x+2)^2}{1^2}+\frac{(y+1)^2}{2^2}=1$, we get $a=2$ and $b=1$.

Teacher: And the center is…

Student: It is a vertical ellipse centered at $(-2,-1)$.

Using the definition of the ellipse $a^2=b^2+c^2$, we obtain $c=\sqrt{3}$.

Teacher:



As the figure shows, we can find the vertices and foci, which are…?

Student: Vertices: $(-3,-1)$, $(-1,-1)$, $(-2,1)$, and $(-2,-3)$.

Foci: $\left(-2,-1+\sqrt{3}\right)$ and $\left(-2,-1-\sqrt{3}\right)$.

老師： 利用配方法，然後我們會得到什麼？

學生： $4x^2+y^2+16x+2y+13=0$

$\Rightarrow 4(x^2+4x+4)+(y^2+2y+1)=4$

$\Rightarrow 4(x+2)^2+(y+1)^2=4$

老師： 然後，兩邊同除 4 得到？

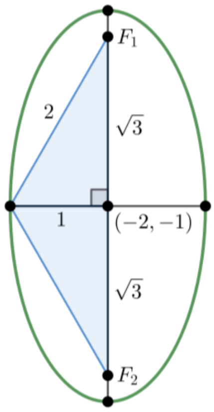
學生： 得到標準式 $\frac{(x+2)^2}{1^2}+\frac{(y+1)^2}{2^2}=1$，得知 $a=2$ 及 $b=1$。

老師： 其中心坐標為？

學生： 我們得到一直橢圓，且其中心 $(-2,-1)$。

而從橢圓的定義 $a^2=b^2+c^2$，得知 $c=\sqrt{3}$。

老師： 從圖可知及上述條件，我們可以得到頂點及焦點坐標。



學生： 頂點：$(-3,-1)$、$(-1,-1)$、$(-2,1)$ 且 $(-2,-3)$。

焦點：$\left(-2,-1+\sqrt{3}\right)$ 且 $\left(-2,-1-\sqrt{3}\right)$。

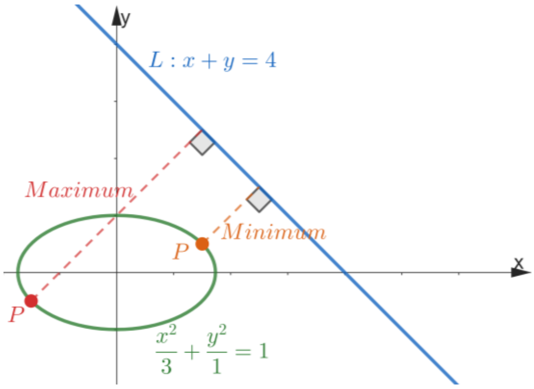
例題三

說明：本題利用橢圓的參數式可以解決一些幾何問題。

（英文）Given that P is a point on ellipse $\frac{x^2}{3}+\frac{y^2}{1}=1$, find the maximum and minimum distances from point P to line $L: x+y=4$.

（中文）已知 P 點為橢圓 $\frac{x^2}{3}+\frac{y^2}{1}=1$ 上的一點，求 P 點到直線 $L: x+y=4$ 的距離之最大值與最小值。

Teacher:



Applying the parametric equation of an ellipse, let point P be

$$\left(\sqrt{3}\cos\theta, \sin\theta\right),\ 0 \le \theta < 2\pi.$$

By using the distance formula

$$d(P, L) = \frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}},$$

we get the distance from point P to line L is…?

Student: Plugging in $a=1,\ b=1,\ c=-4,\ x_0=\sqrt{3}\cos\theta$, and $y_0=\sin\theta$, then we get

$$d(P, L) = \frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}} = \frac{|1\cdot\sqrt{3}\cos\theta+1\cdot\sin\theta-4|}{\sqrt{1^2+1^2}} = \frac{|\sqrt{3}\cos\theta+\sin\theta-4|}{\sqrt{2}}.$$

Teacher: To simplify the expression, we apply angle sum formula for sine

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

Simplify the expression

Student:

$$d(P, L) = \frac{|\sqrt{3}\cos\theta+\sin\theta-4|}{\sqrt{2}} = \sqrt{2}\left|\frac{\sqrt{3}}{2}\cos\theta+\frac{1}{2}\sin\theta-2\right|.$$

We can plug in the angle sum formula for sine by $\alpha=\frac{\pi}{3},\ \beta=\theta$, and this gives

$$d(P, L) = \sqrt{2}\left|\frac{\sqrt{3}}{2}\cos\theta+\frac{1}{2}\sin\theta-2\right| = \sqrt{2}\left|\sin\left(\frac{\pi}{3}+\theta\right)-2\right|.$$

Teacher: To figure out the range of $d(P, L)$, we can use the range of $\sin\theta$, which is $-1 \le \sin\theta \le 1$.

Student: It yields

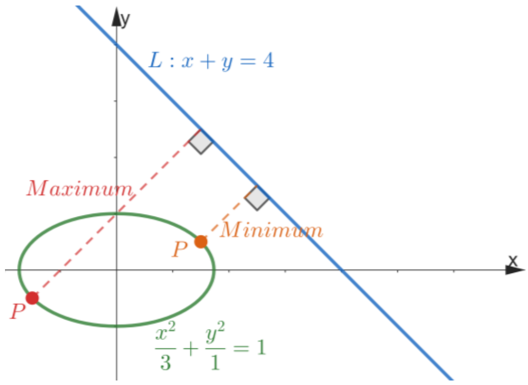
$$-1 \le \sin\theta \le 1$$

$$\Rightarrow 1 \le \left|\sin\left(\frac{\pi}{3}+\theta\right)-2\right| \le 3$$

$$\Rightarrow \sqrt{2} \le \sqrt{2}\left|\sin\left(\frac{\pi}{3}+\theta\right)-2\right| \le 3\sqrt{2}.$$

Student: Therefore, we know that the distance from point P to line L has maximum $3\sqrt{2}$; minimum $\sqrt{2}$.

老師：



利用橢圓的參數式，令 P 點為 $(\sqrt{3}\cos\theta, \sin\theta)$, $0 \le \theta < 2\pi$

利用點到直線的距離公式

$$d(P, L) = \frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}},$$

得 P 點到直線 L 的距離為？

學生： 代入$a=1$、$b=1$、$c=-4$、$x_0=\sqrt{3}\cos\theta$ 及 $y_0=\sin\theta$，我們得到

$$d(P, L) = \frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}} = \frac{|1\cdot\sqrt{3}\cos\theta+1\cdot\sin\theta-4|}{\sqrt{1^2+1^2}} = \frac{|\sqrt{3}\cos\theta+\sin\theta-4|}{\sqrt{2}}。$$

老師： 化簡表達式之後，我們可以使用疊合成正弦函數

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta。$$

學生： 化簡之後，

$$d(P, L) = \frac{|\sqrt{3}\cos\theta+\sin\theta-4|}{\sqrt{2}} = \sqrt{2}\left|\frac{\sqrt{3}}{2}\cos\theta+\frac{1}{2}\sin\theta-2\right|。$$

代入疊合正弦函數 $\alpha=\frac{\pi}{3}$ 及 $\beta=\theta$，得到

$$d(P, L) = \sqrt{2}\left|\frac{\sqrt{3}}{2}\cos\theta+\frac{1}{2}\sin\theta-2\right| = \sqrt{2}\left|\sin\left(\frac{\pi}{3}+\theta\right)-2\right|。$$

老師： 使用 $\sin\theta$ 的範圍 $-1 \le \sin\theta \le 1$ 找 $d(P, L)$ 的範圍。

學生： $-1 \le \sin\theta \le 1$

$$\Rightarrow 1 \le \left|\sin\left(\frac{\pi}{3}+\theta\right)-2\right| \le 3$$

$$\Rightarrow \sqrt{2} \le \sqrt{2}\left|\sin\left(\frac{\pi}{3}+\theta\right)-2\right| \le 3\sqrt{2}。$$

學生： 最後，我們得到 P 點到直線 L 的距離之最大值：$3\sqrt{2}$；最小值：$\sqrt{2}$

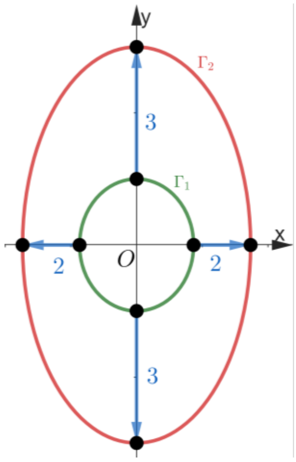
例題四

說明：本題求橢圓經過伸縮之後的方程式。

（英文）Given ellipse $\Gamma_1 : \frac{x^2}{3}+\frac{y^2}{4}=1$ centered at the origin, stretch it horizontally with factor 2 and vertically with factor 3 obtaining a new ellipse Γ_2. Find the equation of ellipse Γ_2.

（中文）已知將橢圓 $\Gamma_1 : \frac{x^2}{3}+\frac{y^2}{4}=1$ 以原點 O 為中心，沿著 x 軸方向伸縮 2 倍、沿著 y 軸方向伸縮 3 倍，得到橢圓 Γ_2，求橢圓 Γ_2 的方程式。

Teacher:



Let point (x, y) be any point lying on ellipse Γ_2. From the question we know that point (x, y) is stretched from point $\left(\frac{x}{2}, \frac{y}{3}\right)$ lying on ellipse Γ_1.

Student: Plugging in $\left(\frac{x}{2}, \frac{y}{3}\right)$ for equation $\Gamma_1 : \frac{x^2}{3}+\frac{y^2}{4}=1$, we obtain

$$\Gamma_2 : \frac{\left(\frac{x}{2}\right)^2}{3}+\frac{\left(\frac{y}{3}\right)^2}{4}=1.$$

Teacher: Simplify it, and get the standard equation.

It gives

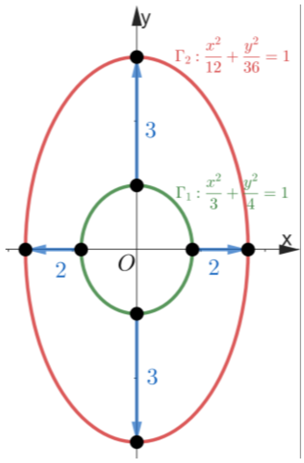
Student:

$$\Gamma_2 : \frac{\left(\frac{x}{2}\right)^2}{3} + \frac{\left(\frac{y}{3}\right)^2}{4} = 1$$

$$\Rightarrow \frac{x^2}{4} \cdot \frac{1}{3} + \frac{y^2}{9} \cdot \frac{1}{4} = 1$$

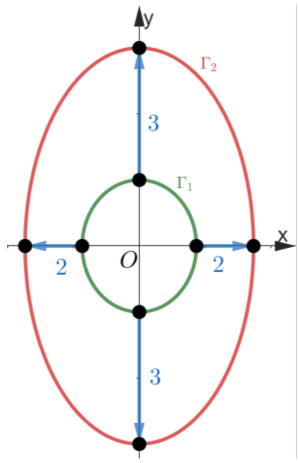
$$\Rightarrow \frac{x^2}{12} + \frac{y^2}{36} = 1.$$

Teacher:



Hence, the equation of an ellipse Γ_2 is $\frac{x^2}{12} + \frac{y^2}{36} = 1.$

老師：



設 (x, y) 為橢圓 Γ_2 上的任意一點，由題意可得點(x, y) 是由 Γ_1上的點 $\left(\frac{x}{2}, \frac{y}{3}\right)$ 伸縮得來。

學生： 因此將 $\left(\frac{x}{2}, \frac{y}{3}\right)$ 代入 $\Gamma_1 : \frac{x^2}{3} + \frac{y^2}{4} = 1$，即

$$\Gamma_2 : \frac{\left(\frac{x}{2}\right)^2}{3} + \frac{\left(\frac{y}{3}\right)^2}{4} = 1。$$

老師： 整理得橢圓方程式。

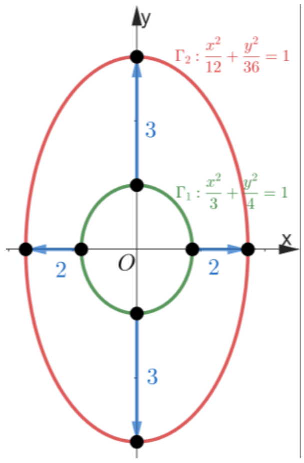
學生： 得

$$\Gamma_2 : \frac{\left(\frac{x}{2}\right)^2}{3} + \frac{\left(\frac{y}{3}\right)^2}{4} = 1$$

$$\Rightarrow \frac{x^2}{4} \cdot \frac{1}{3} + \frac{y^2}{9} \cdot \frac{1}{4} = 1$$

$$\Rightarrow \frac{x^2}{12} + \frac{y^2}{36} = 1。$$

老師：



故橢圓 Γ_2 的方程式為 Γ_2 is $\frac{x^2}{12} + \frac{y^2}{36} = 1$。

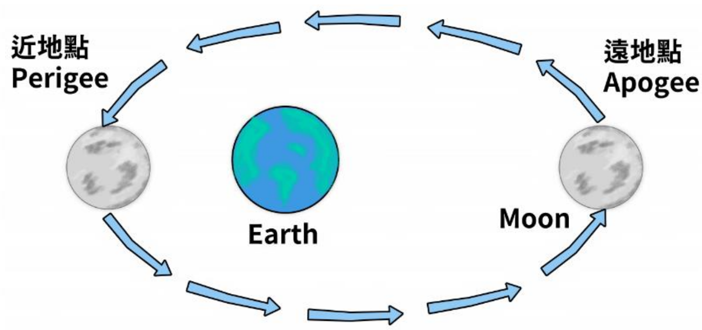
應用問題 / 學測指考題

例題一

說明：近地／遠地點應用問題。

（英文）The moon orbits the Earth in an elliptical path with the center of the Earth at one focus, as shown in the figure. The major and minor axes of the orbit have lengths of 768,800 kilometers and 767,641 kilometers, respectively. Find the greatest and least distances (the apogee and perigee) from the Earth's center to the moon's center.

（中文）月亮的軌道是以地球為焦點的橢圓。其軌道的長軸長為 768,800 公里；短軸長為 767,641 公里。求月球於近地點及遠地點和地球的距離。



Teacher: Begin by solving for a and b. Given that $2a = 768{,}800$ and $2b = 767{,}641$, find a and b.

Student: $a = 384{,}400$ and $b = 383{,}820.5$.

Teacher: Now, using these values and the definition of an ellipse, we can solve for c.

Student: $c = \sqrt{a^2 - b^2} \approx 21{,}099$.

Teacher: Perfect! So, the greatest distance between the center of the Earth and the center of the moon is…

Student: $a + c \approx 405{,}499$ kilometers.

Teacher: and the least distance is…

Student: $a - c \approx 363{,}301$ kilometers.

老師： 由題目知長軸長 $2a = 768{,}800$、短軸長 $2b = 767{,}641$，所以我們可以求 a 及 b 之值。

學生： $a = 384{,}400$ 及 $b = 383{,}820.5$。

老師： 我們用 a 跟 b 及橢圓的定義求 c。

學生： $c=\sqrt{a^2-b^2}\approx 21{,}099$。

老師： 很好，所以月亮從遠地點到地球的距離為？

學生： $a+c\approx 405{,}499$ 公里。

老師： 月亮從近地點到地球的距離為？

學生： $a-c\approx 363{,}301$ 公里。

例題二

說明：關於橢圓旋轉的應用題。

（英文）In the coordinate plane, let Γ be a vertical ellipse centered at the origin. Rotating ellipse Γ by an angle θ $(0<\theta<\pi)$ counterclockwise around the origin produces the new ellipse $\Gamma':40x^2+4\sqrt{5}xy+41y^2=180$. The point $\left(-\frac{5}{3},\frac{2\sqrt{5}}{3}\right)$ lying on ellipse Γ' is one of the two points that is the greatest distance from the center.

(1) Find the length of the major axis of ellipse Γ'.

(2) Find the line equation where minor axis of ellipse Γ' lies and the length of the minor axis of ellipse Γ'.

(3) Given that P is a point on the ellipse Γ. Rotating point P results in point P' lying on the x-axis, and the x-coordinate of point P' is greater than 0, find the coordinates of point P.

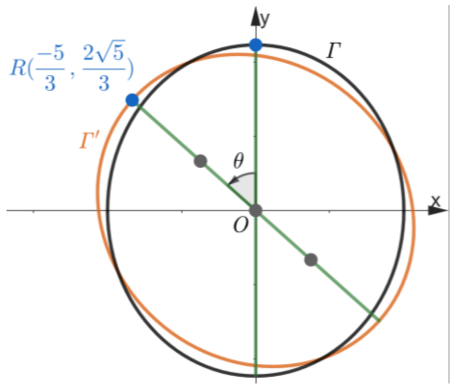
（中文）坐標平面上，設 Γ 為中心在原點且長軸落在 y 軸上的橢圓。已知對原點逆時針旋轉 θ 角（其中$0<\theta<\pi$）的線性變換將 Γ 變換到新橢圓 $\Gamma':40x^2+4\sqrt{5}xy+41y^2=180$，點 $\left(-\frac{5}{3},\frac{2\sqrt{5}}{3}\right)$ 為 Γ' 上離原點最遠的兩點之一。

(1) 橢圓 Γ' 的長軸長為何？（化為最簡根式）

(2) 試求 Γ' 短軸所在的直線方程式與短軸長。

(3) 已知在Γ上的一點 P 經由此旋轉後得到的點 P' 落在 x 軸上，且P' 點的 x 坐標大於 0。試求 P 點的坐標。

（112 年分科數甲）

Teacher: 

Teacher: **Question 1.** As shown in the figure, we use the distance formula to find the major axis of length $2a = 2\overline{OR}$.

Student: $\overline{OR} = \sqrt{\left(-\frac{5}{3}\right)^2 + \left(\frac{2\sqrt{5}}{3}\right)^2} = \sqrt{5},\ 2\overline{OR} = 2a = 2\sqrt{5}.$

Teacher: **Question 2.** To find the line equation, there are two elements we need to know, which are…?

Student: The slope of the line and any point on the line.

Teacher: Right, and we know that the minor axis passes through…

Student: The origin.

Teacher: And, the minor axis is perpendicular to…

Student: The major axis. The product of two slopes is -1.

Teacher: Perfect! Then, how can we find the slope of the major axis.

Student: The slope of the major axis is exactly the slope of $\overline{OR}$, which is

$$m_{\text{major}} = m_{\text{OR}} = \frac{\frac{2\sqrt{5}}{3}}{\frac{-5}{3}} = -\frac{2}{\sqrt{5}}.$$

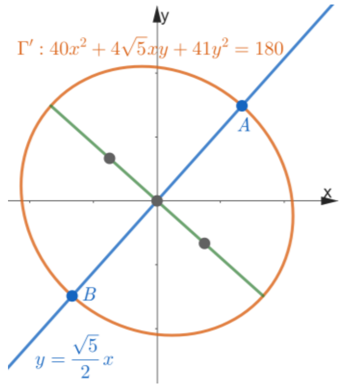
Since we know that $m_{\text{major}} \times m_{\text{minor}} = -1$, this gives

$\mathrm{m_{minor}} = -\frac{1}{\mathrm{m_{major}}} = -\frac{1}{-\frac{2}{\sqrt{5}}} = \frac{\sqrt{5}}{2}.$

Teacher: Find the line equation that passes through the origin with the slope $\frac{\sqrt{5}}{2}$.

Student: The equation of the minor axis is $y - 0 = \frac{\sqrt{5}}{2}(x - 0) \Rightarrow y = \frac{\sqrt{5}}{2}x.$

Teacher:



As shown in the figure, we use the system of equations,

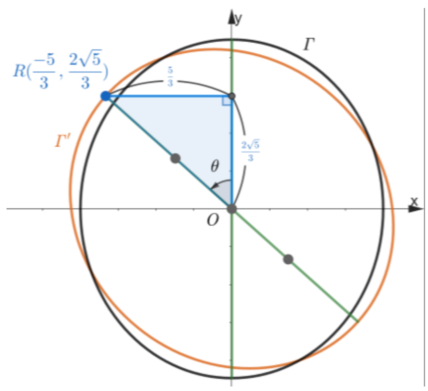
Γ' : $40x^2+4\sqrt{5}xy+41y^2=180$ and $y=\frac{\sqrt{5}}{2}x$, to find the intersects A and B.

Student: Solve $\begin{cases} 40x^2+4\sqrt{5}xy+41y^2=180 \\ y=\frac{\sqrt{5}}{2}x \end{cases} \Rightarrow (x,y)=\pm\left(\frac{4}{3},\frac{4\sqrt{5}}{6}\right)$.

Teacher: By applying the distance formula, we get the length of minor axis is

$\overline{AB}=2\sqrt{\left(\frac{4}{3}\right)^2+\left(\frac{4\sqrt{5}}{6}\right)^2}=4$.

Teacher:



As shown in the figure, there is a right triangle, and we can find the matrix of rotation.

Student: Yes, we can find $\tan\theta=\frac{\frac{5}{3}}{\frac{2\sqrt{5}}{3}}=\frac{\sqrt{5}}{2}$, $\cos\theta=\frac{2}{3}$, and $\sin\theta=\frac{\sqrt{5}}{3}$.

Teacher: To figure out point P, we have to find point P' first.

Student: We can use the system of the equation:

$\begin{cases} 40x^2+4\sqrt{5}xy+41y^2=180 \\ y=0 \end{cases} \Rightarrow x=\pm\frac{3}{\sqrt{2}}$ (P' is the positive one.)

So, the coordinates of point $P'\left(\frac{3}{\sqrt{2}},0\right)$.

Teacher: By substituting the values for the rotation matrix

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

, what can we get?

Student: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{\sqrt{5}}{3} \\ -\frac{\sqrt{5}}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{3}{\sqrt{2}} \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ -\frac{\sqrt{10}}{2} \end{bmatrix} \Rightarrow P\left(\sqrt{2}, -\frac{\sqrt{10}}{2}\right)$.

Teacher: Wonderful! Now, we've solved the question. Also, we can use a rotation matrix to find the original equation of the ellipse, which can prove the answers.

Plug in $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{2}{3}x - \frac{\sqrt{5}}{3}y \\ \frac{\sqrt{5}}{3}x + \frac{2}{3}y \end{bmatrix}$ for the equation of ellipse $\Gamma' : 40x^2 + 4\sqrt{5}xy + 41y^2 = 180$.

After simplifying, we get the equation of the original ellipse Γ.

Student: $40\left(\frac{2}{3}x - \frac{\sqrt{5}}{3}y\right)^2 + 4\sqrt{5}\left(\frac{2}{3}x - \frac{\sqrt{5}}{3}y\right)\left(\frac{\sqrt{5}}{3}x + \frac{2}{3}y\right) + 41\left(\frac{\sqrt{5}}{3}x + \frac{2}{3}y\right)^2 = 180$

$$\Rightarrow 40\left(\frac{4}{9}x^2 - \frac{4\sqrt{5}}{9}xy + \frac{5}{9}y^2\right) + 4\sqrt{5}\left(\frac{2\sqrt{5}}{9}x^2 - \frac{1}{9}xy - \frac{2\sqrt{5}}{9}y^2\right)$$

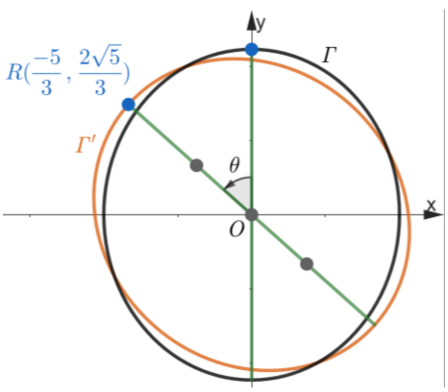
$$+41\left(\frac{5}{9}x^2 + \frac{4\sqrt{5}}{9}xy + \frac{4}{9}y^2\right) = 180$$

$$\Rightarrow \frac{160 + 40 + 205}{9}x^2 + \frac{-160\sqrt{5} - 4\sqrt{5} + 164\sqrt{4}}{9}xy + \frac{200 - 40 + 164}{9}y^2$$

$$= 180$$

$$\Rightarrow \frac{405}{9}x^2 + \frac{324}{9}y^2 = 180 \Rightarrow \frac{x^2}{4} + \frac{y^2}{5} = 1$$

老師：



老師： **問題 1**，如圖所示，利用距離公式可知，長軸長為 $2a = 2\overline{OR}$。

學生： $\overline{OR}=\sqrt{\left(-\frac{5}{3}\right)^2+\left(\frac{2\sqrt{5}}{3}\right)^2}=\sqrt{5}$，$2\overline{OR}=2a=2\sqrt{5}$

老師： **問題 2**，要求直線方程式，要先找到哪兩個條件？

學生： 斜率跟直線上任一點。

老師： 沒錯，由題目可知，短軸通過哪個點？

學生： 原點。

老師： 而且短軸與什麼垂直？

學生： 長軸，所以他們的斜率相乘為 -1。

老師： 很好！那我們要如何找到長軸的斜率？

學生： 長軸的斜率就是 $\overline{OR}$ 的斜率，

$$m_{\text{major}}=m_{\text{OR}}=\frac{\frac{2\sqrt{5}}{3}}{\frac{-5}{3}}=-\frac{2}{\sqrt{5}}$$

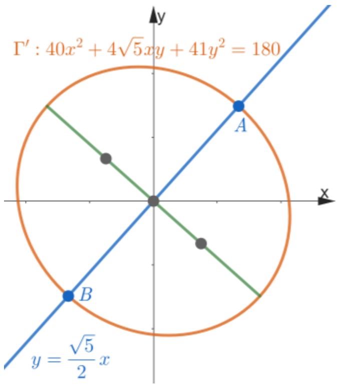
又因 $m_{\text{major}}\times m_{\text{minor}}=-1$，所以

$$m_{\text{minor}}=-\frac{1}{m_{\text{major}}}=-\frac{1}{-\frac{2}{\sqrt{5}}}=\frac{\sqrt{5}}{2}$$

老師： 求直線方程式通過原點且斜率為 $\frac{\sqrt{5}}{2}$。

學生： 短軸的直線方程式為 $y-0=\frac{\sqrt{5}}{2}(x-0)\Rightarrow y=\frac{\sqrt{5}}{2}x$

老師：



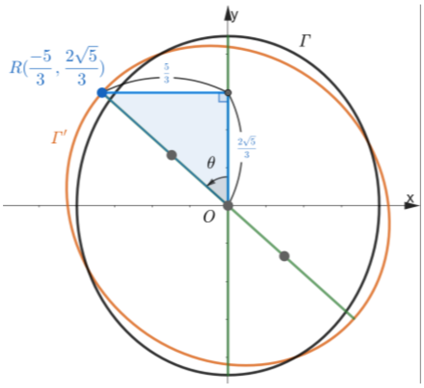
如圖所示，要求交點 A 與 B，我們可以用聯立方程組

Γ' : $40x^2+4\sqrt{5}xy+41y^2=180$ 與 $y=\frac{\sqrt{5}}{2}x$。

學生： 解 $\begin{cases}40x^2+4\sqrt{5}xy+41y^2=180\\ y=\frac{\sqrt{5}}{2}x\end{cases}$，得 $(x,y)=\pm\left(\frac{4}{3},\frac{4\sqrt{5}}{6}\right)$。

老師：　由距離公式我們得到短軸長為 $\overline{AB}=2\sqrt{\left(\frac{4}{3}\right)^2+\left(\frac{4\sqrt{5}}{6}\right)^2}=4$。

老師：



如圖所示，我們可以從直角三角形得知橢圓的旋轉角。

學生：　可以得到 $\tan\theta=\frac{\frac{5}{3}}{\frac{2\sqrt{5}}{3}}=\frac{\sqrt{5}}{2}$、$\cos\theta=\frac{2}{3}$ 與 $\sin\theta=\frac{\sqrt{5}}{3}$。

老師：　要求點 P，我們要先求點 P'。

學生：　利用聯立方程組：

$$\begin{cases}40x^2+4\sqrt{5}xy+41y^2=180\\ y=0\end{cases}\Rightarrow x=\pm\frac{3}{\sqrt{2}}\ (\text{點 } P'\text{取正。})$$

所以，其點坐標為 $P'\left(\frac{3}{\sqrt{2}},0\right)$。

老師：　將上述的值代入旋轉矩陣，

$$\begin{bmatrix}x'\\y'\end{bmatrix}=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}\Rightarrow\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x'\\y'\end{bmatrix}$$

，我們會得到什麼？

學生：　$$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\frac{2}{3} & \frac{\sqrt{5}}{3}\\ -\frac{\sqrt{5}}{3} & \frac{2}{3}\end{bmatrix}\begin{bmatrix}\frac{3}{\sqrt{2}}\\0\end{bmatrix}=\begin{bmatrix}\sqrt{2}\\ -\frac{\sqrt{10}}{2}\end{bmatrix}\Rightarrow P\left(\sqrt{2},-\frac{\sqrt{10}}{2}\right)$$ 。

老師：　非常好！現在我們已解出所有問題。然而，我們也可以用旋轉矩陣求出原始橢圓方程式來驗證答案。

將 $\begin{bmatrix}x'\\y'\end{bmatrix}=\begin{bmatrix}\frac{2}{3}x-\frac{\sqrt{5}}{3}y\\ \frac{\sqrt{5}}{3}x+\frac{2}{3}y\end{bmatrix}$ 代入新橢圓方程式：$\Gamma':40x^2+4\sqrt{5}xy+41y^2=180$

化簡後，我們可以得到旋轉前的橢圓方程式 Γ。

學生： $40\left(\frac{2}{3}x-\frac{\sqrt{5}}{3}y\right)^2+4\sqrt{5}\left(\frac{2}{3}x-\frac{\sqrt{5}}{3}y\right)\left(\frac{\sqrt{5}}{3}x+\frac{2}{3}y\right)+41\left(\frac{\sqrt{5}}{3}x+\frac{2}{3}y\right)^2=180$

$\Rightarrow 40\left(\frac{4}{9}x^2-\frac{4\sqrt{5}}{9}xy+\frac{5}{9}y^2\right)+4\sqrt{5}\left(\frac{2\sqrt{5}}{9}x^2-\frac{1}{9}xy-\frac{2\sqrt{5}}{9}y^2\right)$

$+41\left(\frac{5}{9}x^2+\frac{4\sqrt{5}}{9}xy+\frac{4}{9}y^2\right)=180$

$\Rightarrow \frac{160+40+205}{9}x^2+\frac{-160\sqrt{5}-4\sqrt{5}+164\sqrt{4}}{9}xy+\frac{200-40+164}{9}y^2$

$=180$

$\Rightarrow \frac{405}{9}x^2+\frac{324}{9}y^2=180 \Rightarrow \frac{x^2}{4}+\frac{y^2}{5}=1$

單元三　雙曲線
Hyperbolas

臺北市陽明高級中學　吳柏萱老師

■ 前言 Introduction

我們將從雙曲線的定義開始，依序介紹雙曲線的圖形與方程式。進一步認識雙曲線圖形的漸近線，並解決生活中的應用問題。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
hyperbola	**雙曲線**	**conjugate axis**	**共軛軸**
foci (單數：focus)	**焦點**	standard form	標準式
concentric circles	同心圓	orientation	方向
branch	分支	**asymptote**	**漸近線**
center	中心	approach	接近
vertices (單數：vertex)	頂點	infinity	無窮大
transverse axis	**貫軸**		

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ Using ________, we obtain ________.

例句：**Using** a similar procedure, **we obtain** the equation $-\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$

利用相似的方法，我們可以得到方程式為$-\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$。

❷ A summary of ________ is given below.

例句：**A summary of** the standard form of the hyperbola **is given below**.

總結雙曲線的標準式如下。

❸ ________ are centered at ________.

例句：The equations of the hyperbola **are centered at** the origin.

雙曲線的中心在原點。

❹ From ________, it follows that ________.

例句：**From** this standard form, **it follows that** the center is the origin.

從標準式得知，中心為原點。

■ 問題講解 Explanation of Problems

🙞 說明 🙜

[The Definition of a Hyperbola]

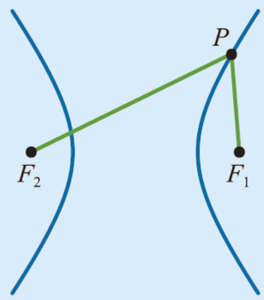
The definition of a hyperbola is similar to that of an ellipse. In the previous chapter, we learned that the definition of an ellipse is that the sum of the distances between the foci and a point on the ellipse is constant. If we replace "sum" with "difference", then it becomes the definition of a hyperbola, which is that the absolute value of the difference of the distances between the foci and a point on the hyperbola is constant.

The definition of a Hyperbola

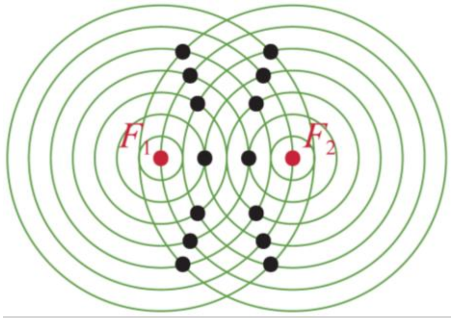
A hyperbola is the set of all points $P(x,y)$ in a plane. The absolute value of the difference of the distances from two distinct fixed points, F_1 and F_2, is constant $2a$, where $0<2a<\overline{F_1F_2}$. That is

$$\left|PF_1-PF_2\right|=2a,$$

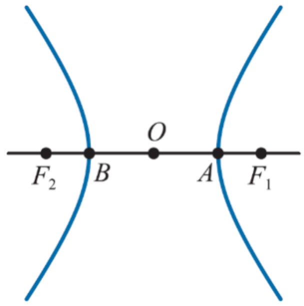
then F_1 and F_2 are called **foci** (singular focus).



Consider two points, F_1 and F_2, as the centers of **concentric circles** with radii 1, 2, 3, 4, 5, 6, and 7, as shown in the figure. All the black dots satisfy the condition that the absolute value of the difference of distances from each black dot to F_1 and from the same black dot to F_1 is 2. We can construct more concentric circles to create additional black dots, enabling us to sketch a smoother curve, which is a hyperbola.



As shown in the figure, the graph of a hyperbola has two disconnected parts (**branches**). The hyperbola is symmetrical both vertically (up and down) and horizontally (left and right); its midpoint of the foci is called the center.



In this figure, F_1 and F_2 are the foci of a hyperbola. The absolute value of the difference of the distances from any point on the hyperbola to the two foci is $2a$.

Key features of a hyperbola:

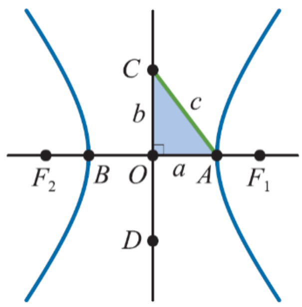
(1) **A Center**: The midpoint of $\overline{F_1F_2}$ is called the center. Let $\overline{F_1F_2}$ be $2c$, then $\overline{OF_1}=\overline{OF_2}=c$.

(2) **Vertices**: The two points A and B are the intersections that the line F_1F_2 through the foci intersects the hyperbola.

(3) **A Transverse Axis**: The line segment AB connecting the vertices is the transverse axis, as illustrated below. Since point A lies on the hyperbola, $AF_2-AF_1=2a$. Also,

$$\overline{AF_2}-\overline{AF_1}=\left(\overline{AO}+\overline{OF_2}\right)-\overline{AF_1}$$
$$=\overline{AO}+\left(\overline{OF_1}-\overline{AF_1}\right)$$ (O is the midpoint of the foci, so $\overline{OF_2}=\overline{OF_1}$.)
$$=\overline{AO}+\overline{OA}=2\overline{OA}$$

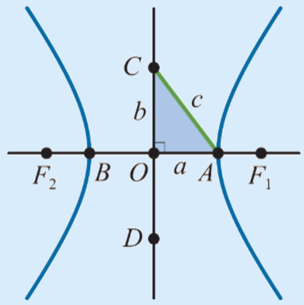
(4) **A Conjugate Axis:** The conjugate axis of a hyperbola is the line segment CD with the midpoint at the center O, and is the perpendicular bisector of the transverse axis The length of CD is $2b$, which satisfies the equation $c^2=a^2+b^2$, as shown in the figure.



Properties of the Hyperbola

Let F_1 and F_2 be the foci of the hyperbola, and the absolute value of the difference of the distances from any point on the hyperbola to the two foci is constant $2a$.

Let O be the center, $\overline{AB}$ be the transverse axis, and $\overline{CD}$ be the conjugate axis, as shown in the figure.



(4) The center O is the midpoint of $\overline{F_1F_2}$, the transverse axis $\overline{AB}$, and the conjugate axis $\overline{CD}$.

(5) The transverse axis of the length $\overline{AB}$ is $2a$.

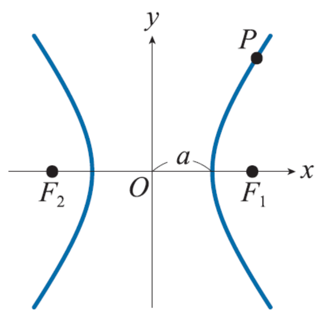
(6) When the conjugate axis of the length $\overline{CD}$ is $2b$ and $\overline{F_1F_2}$ is $2c$, the constants a, b, and c satisfy the equation $c^2 = a^2 + b^2$.

[The Equations of a Hyperbola Centered at the Origin]

Based on the definition and properties of a hyperbola, we can derive its standard equation centered at the origin with the foci on the x-axis or y-axis.

(1) A Horizontal Transverse Axis

Given a hyperbola centered at the origin with a horizontal transverse axis, and transverse and conjugate axes of lengths $2a$ and $2b$, respectively. Determining c using the equation $c^2 = a^2 + b^2$, this gives the values for the foci and provides one and only one equation of the hyperbola. To derive the equation of the hyperbola centered at the origin, we can consider the hyperbola with the foci $F_1(c,0)$ and $F_2(-c,0)$, as shown in the figure. Moreover, $c^2 = a^2 + b^2$.



Let $P(x,y)$ be any point on the hyperbola for which the absolute value of the difference of the distances from two foci is $2a$. That is,

$$\sqrt{(x-c)^2+y^2}-\sqrt{(x+c)^2+y^2}=\pm 2a$$

which, after expanding and regrouping, reduces to $(c^2-a^2)x^2-a^2y^2=a^2(c^2-a^2)$

From the equation

$$c^2=a^2+b^2$$
$$\Rightarrow b^2=c^2-a^2$$

which implies that the equation of the hyperbola is

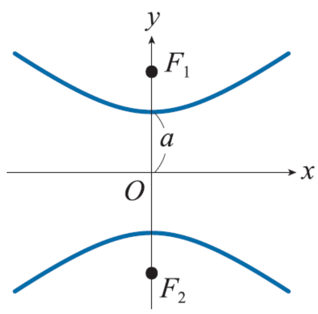
$$b^2x^2-a^2y^2=a^2b^2$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

On the contrary, the point (x,y) satisfying the equation lies on the hyperbola.

(2) A Vertical Transverse Axis

Given a hyperbola centered at the origin with a vertical transverse axis, and transverse and conjugate axes of lengths $2a$ and $2b$, respectively. To derive the equation of the hyperbola centered at the origin, we can consider the hyperbola with the foci $F_1(0,c)$ and $F_2(0,-c)$, as shown in the figure. Using a similar procedure, we would obtain the equation

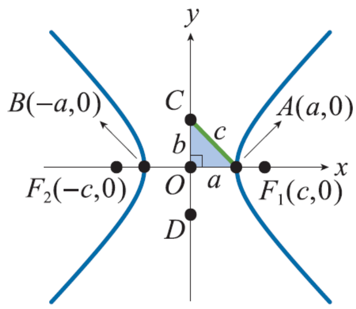
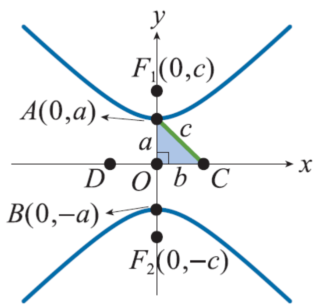


$$-\frac{x^2}{b^2}+\frac{y^2}{a^2}=1.$$

A summary of these results is given below.

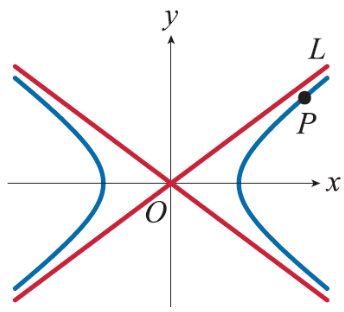
The Standard Equation of a Hyperbola Centered at the Origin

The **standard form** of the equation of a hyperbola centered at the origin is

Equations	**Orientations**	Graphs
$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	The transverse axis is horizontal.	
$-\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$	The transverse axis is vertical.	

The vertices are a units from the center, and the foci are c units from the center. Moreover, $c^2=a^2+b^2$.

(3) The Asymptotes of a Hyperbola



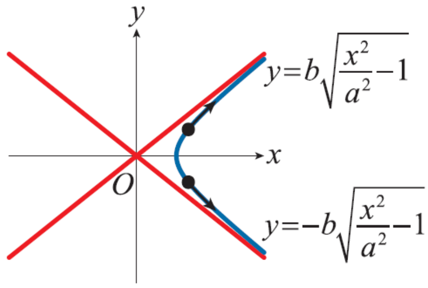
The graph of a hyperbola consists of two disconnected branches, each of which extends indefinitely outward. As the branches of the hyperbola extend outward, they **approach** but never actually touch certain straight lines called **asymptotes**. As a point P lying on the hyperbola moves away from the center along one of its branches, the distance between P and the line L approaches zero. Therefore, the line L is one of the asymptotes of the hyperbola.

To derive the equations of the two asymptotes, consider the hyperbola centered at the origin with a horizontal transverse axis, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, as illustrated below.

Rewrite the equation as $y=\pm b\sqrt{\frac{x^2}{a^2}-1}$. As x becomes larger and larger, approaching **infinity**, then

$$y=b\sqrt{\frac{x^2}{a^2}-1}\approx b\left|\frac{x}{a}\right|=\frac{b}{a}x,$$

$$y=-b\sqrt{\frac{x^2}{a^2}-1}\approx -b\left|\frac{x}{a}\right|=-\frac{b}{a}x.$$

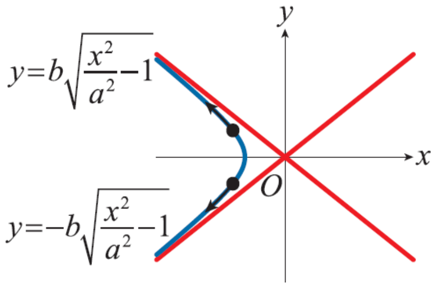


As shown in the figure, the blue curve approaches the red line.

On the other hand, when x approaches negative infinity, then

$$y=b\sqrt{\frac{x^2}{a^2}-1}\approx b\left|\frac{x}{a}\right|=-\frac{b}{a}x,$$

$$y=-b\sqrt{\frac{x^2}{a^2}-1}\approx -b\left|\frac{x}{a}\right|=\frac{b}{a}x.$$



As shown in the figure, the blue curve approaches the red line.

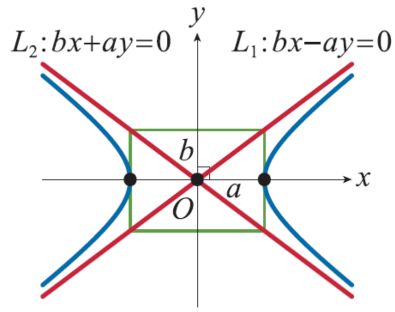
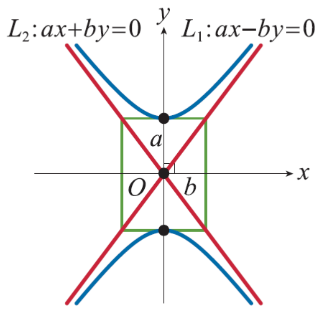
Hence, we know that the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ has two asymptotes $y=\pm\frac{b}{a}x$, which is $bx+ay=0$ and $bx-ay=0$.

We would obtain the same result for a hyperbola with a vertical transverse axis. the hyperbola $-\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ has two asymptotes $y=\pm\frac{a}{b}x$, which is $ax-by=0$ and $ax+by=0$.

Every hyperbola has two asymptotes that intersect at the center of the hyperbola. The asymptotes pass through the vertices of a rectangle of dimensions $2a$ by $2b$, with its center at the origin.

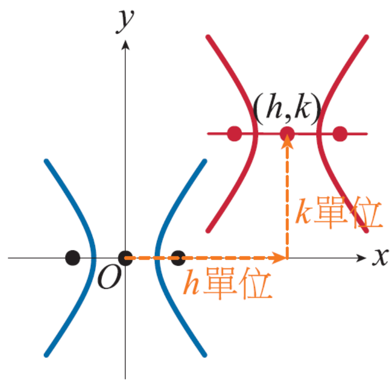
The Asymptotes of a Hyperbola Centered at the Origin

The equations of the asymptotes of a hyperbola are

Equations	Asymptotes	Orientations	Graphs
$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$bx+ay=0$ and $bx-ay=0$	A horizontal transverse axis	
$-\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$	$ax-by=0$ and $ax+by=0$	A vertical transverse axis	

[The Standard Equation of a Hyperbola]

If the center of the hyperbola is any point (h,k), and the transverse axis is either horizontal or vertical, we can shift the graph of the hyperbola centered at the origin with a horizontal or vertical transverse axis. Consider shifting a hyperbola centered at the origin with a horizontal transverse axis, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, by moving h units to the right and by k units up.



That is, replace x with $x-h$ and y with $y-k$, which means the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is transformed to $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$. Meanwhile, the center of the new hyperbola is (h,k), its foci are $(h+c,k)$ and $(h-c,k)$, and the equation of asymptotes are $b(x-h)-a(y-k)=0$ and $b(x-h)+a(y-k)=0$.

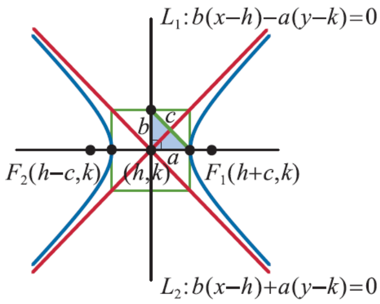
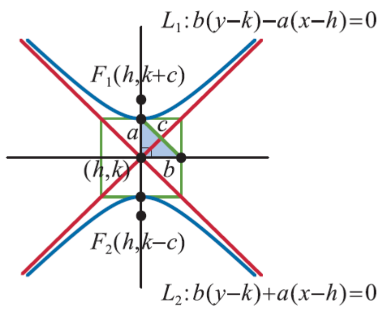
Similarly, by shifting the hyperbola centered at the origin with a vertical transverse axis, $-\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$, by moving h units to the right and k units up, we would obtain the new equation:

$$-\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1.$$

A summary of the standard form of the hyperbola is given below.

The Standard Equation of a Hyperbola

The standard form of the equation of a hyperbola is

Equations	Orientations	Graphs
$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$	The transverse axis is horizontal.	
$-\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$	The transverse axis is vertical.	

The vertices are a units from the center, and the foci are c units from the center. Moreover, $c^2=a^2+b^2$.

運算問題的講解

例題一

說明：已知雙曲線的一些元素時，求出其方程式。

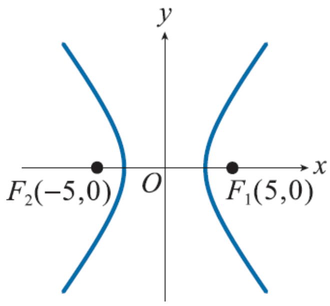
（英文）Write an equation for the given hyperbola that satisfies the following conditions.

Foci: $F_1(5,0)$ and $F_2(-5,0)$; the length of the transverse axis: 6.

（中文）求滿足下列各條件的雙曲線方程式。

焦點為 $F_1(5,0)$ 與 $F_2(-5,0)$，貫軸長為 6。

Teacher:



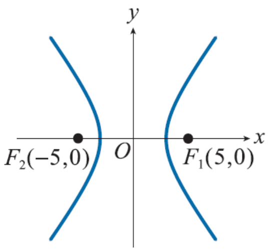
As shown in the figure, it's a horizontal hyperbola centered at the origin. Given that $2a=6$ and $2c=\overline{F_1F_2}=10$, we get $a=3$ and $c=5$. How can we find the value of b?

Student: Using the definition of the hyperbola $c^2=a^2+b^2$, we get $b=4$.

Teacher: By applying the standard equation of a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, what can we get?

Student: The equation of the hyperbola is $\frac{x^2}{9}-\frac{y^2}{16}=1$

老師：



如圖所示，可得雙曲線的貫軸在 x 軸上，且其中心為原點。而由題目可知，長軸長為$2a=6$，$2c=\overline{F_1F_2}=10$，所以$a=3$及$c=5$。我們要如何得知 b 之值？

學生： 由橢圓的定義 $c^2=a^2+b^2$ ，得 $b=4$ 。

老師： 橢圓的方程式形如 $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ ，所以我們可以得到？

學生： 其方程式為： $\frac{x^2}{9}-\frac{y^2}{16}=1$ 。

例題二

說明：本題練習由雙曲線的方程式求出其圖形的各元素。

（英文）Find the vertices and foci of the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$.

（中文）求雙曲線 $\frac{x^2}{16}-\frac{y^2}{9}=1$ 的頂點與焦點坐標。

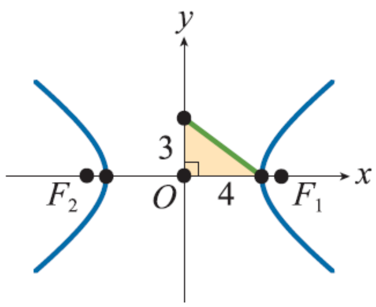
Teacher: Rewrite the equation as $\frac{x^2}{4^2}-\frac{y^2}{3^2}=1$. We can get the value of a and b.

Student: We obtain $a=4$ and $b=3$, and it's a horizontal hyperbola centered at the origin.

Teacher: Using the definition of the hyperbola $c^2=a^2+b^2$, what can we get?

Student: We obtain $c=5$.

Teacher:



As shown in the figure, we can find the vertices and foci. What are they?

Student: Vertices: $(4, 0)$ and $(-4, 0)$.

Foci: $(5, 0)$ and $(-5, 0)$.

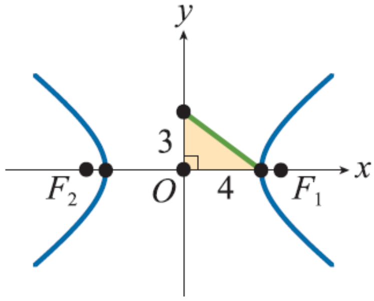
老師： 將方程式改寫成 $\frac{x^2}{4^2}-\frac{y^2}{3^2}=1$，我們可以得到 a 跟 b 的值。

學生： 我們得到 $a=4$、$b=3$，且雙曲線的中心為原點，貫軸在 x 軸上。

老師： 從雙曲線的定義 $c^2=a^2+b^2$，我們可以得到？

學生： 我們得知 $c=5$。

老師：



從圖可知及上述條件，我們可以得到頂點及焦點坐標。

學生：頂點：(4, 0) 及 (−4, 0).

焦點：(5, 0) 及 (−5, 0)。

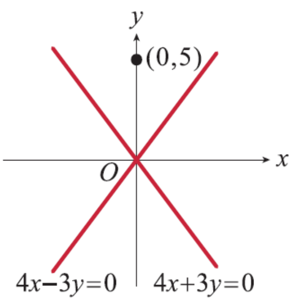
例題三

說明：本題練習知道雙曲線的漸近線方程式與其中一個焦點時，求出雙曲線的方程式。

（英文）Find the equation of the hyperbola whose asymptotes are $4x+3y=0$ and $4x-3y=0$, with one of the foci at (0, 5).

（中文）求漸近線方程式為 $4x+3y=0$ 與 $4x-3y=0$，且其中一個焦點為(0 , 5) 的雙曲線方程式。

Teacher:



In the figure shown, what can we get?

Student: It's a vertical hyperbola centered at the origin and $c=5$.

Teacher: Use the standard equation $-\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$. The asymptotes are $ax-by=0$ and $ax+by=0$. What can we get?

Student: Comparing the coefficients, we can let $a=4k$ and $b=3k$. So, we can obtain the equation of the hyperbola, which is $-\dfrac{x^2}{(3k)^2}+\dfrac{y^2}{(4k)^2}=1$.

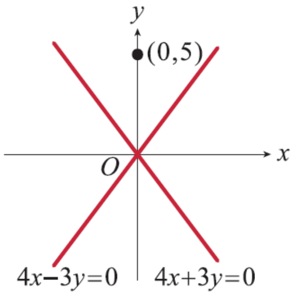
Teacher: Using the definition of the hyperbola $c^2=a^2+b^2$, what can we get?

Student: $c^2=(4k)^2+(3k)^2=25k^2=25$. This gives $k^2=1$.

Teacher: By plugging in $k^2=1$ into the equation, we can get the solution.

Student: The equation of the hyperbola is $-\dfrac{x^2}{9}+\dfrac{y^2}{16}=1$.

老師：



從圖及上述條件，我們可以得到？

學生：雙曲線中心為 $(0,0)$，且貫軸在 y 軸上，$c=5$。

老師：雙曲線的方程式形如 $-\dfrac{x^2}{b^2}+\dfrac{y^2}{a^2}=1$，且其漸近線方程式為 $ax-by=0$ 與 $ax+by=0$。我們可以得到？

學生：比較係數可設 $a=4k$ 及 $b=3k$，即雙曲線的方程式為 $-\dfrac{x^2}{(3k)^2}+\dfrac{y^2}{(4k)^2}=1$。

老師：再利用雙曲線的定義，我們可以得到？

學生：$c^2=(4k)^2+(3k)^2=25k^2=25$，解得 $k^2=1$。

老師：將 $k^2=1$ 代入方程式就可以得到答案。

學生：故雙曲線的方程式為 $-\dfrac{x^2}{9}+\dfrac{y^2}{16}=1$。

例題四

說明：本題由雙曲線的方程式求出其圖形的各元素。

（英文）Find the vertices, foci and asymptotes of the hyperbola $4x^2-9y^2-8x+36y+4=0$.

（中文）求雙曲線 $4x^2-9y^2-8x+36y+4=0$ 的頂點、焦點坐標與漸近線方程式。

Teacher: After completing the squares, what can we get?

Student: $4(x^2-2x+1)-9(y^2-4y+4)=-36$

$\Rightarrow 4(x-1)^2-9(y-2)^2=-36$

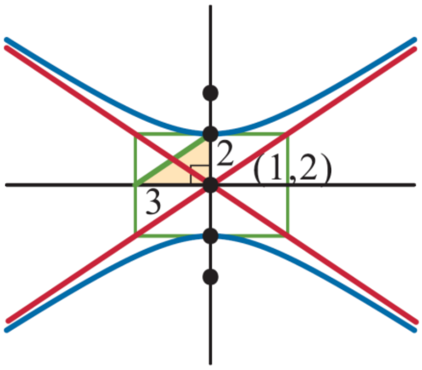
$\Rightarrow -\frac{(x-1)^2}{3^2}+\frac{(y-2)^2}{2^2}=1$

Teacher: By using the standard equation $-\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$, what can we get?

Student: We get $a=2$ and $b=3$, and it is a vertical hyperbola with the center at $(1,2)$.

Using the definition of the hyperbola $c^2=a^2+b^2$, we obtain $c=\sqrt{13}$.

Teacher:



In the figure shown, we can find the vertices, foci and asymptotes. What are they?

Student: Vertices: $(1,4)$ and $(1,0)$.

Foci: $\left(1,2+\sqrt{13}\right)$ and $\left(1,2-\sqrt{13}\right)$.

Asymptotes: $3(y-2)-2(x-1)=0$ and $3(y-2)+2(x-1)=0$, which is

$2x-3y+4=0$ and $2x+3y-8=0$.

老師： 利用配方法，然後我們會得到什麼？

學生： $4(x^2-2x+1)-9(y^2-4y+4)=-36$

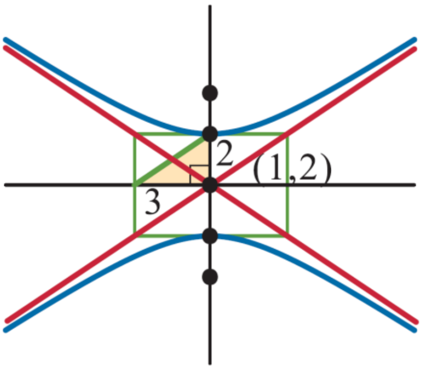
$$\Rightarrow -\frac{(x-1)^2}{3^2}+\frac{(y-2)^2}{2^2}=1$$

老師：雙曲線的方程式形如 $-\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$，我們得到？

學生：可得知 $a=2$ 及 $b=3$，且雙曲線的中心在 $(1,2)$，貫軸平行 y 軸。

再利用 $c^2=a^2+b^2$，解得 $c=\sqrt{13}$.

老師：



從圖可知及上述條件，我們可以得到頂點、焦點坐標與漸近線方程式。

學生：頂點：$(1,4)$及$(1,0)$。

焦點：$(1,2+\sqrt{13})$及$(1,2-\sqrt{13})$。

漸近線方程式：$3(y-2)-2(x-1)=0$與$3(y-2)+2(x-1)=0$，即$2x-3y+4=0$與$2x+3y-8=0$。

應用問題 / 學測指考題

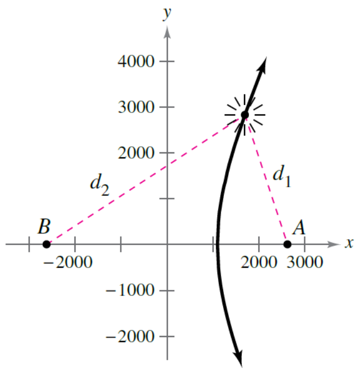
例題一

說明：雙曲線應用於聲音傳導。

（英文）Two microphones, 1 mile apart, record an explosion. Microphone A receives the sound 2 seconds before microphone B. Where did the explosion might occur? (Assume sound travels at 1100 feet per second.)

（中文）相距 1 英里的兩個麥克風記錄了一次爆炸的聲音。麥克風 A 比麥克風 B 早 2 秒收到聲音。爆炸可能發生在哪裡？假設聲音以每秒 1100 英尺的速度傳播。

Teacher:



We know that the explosion took place $1100\times 2 = 2200$ feet farther from B than from A. The locus of all points that are 2200 feet closer to A than to B is one branch of the hyperbola. As shown in the figure, let the hyperbola be…

Student: Let the hyperbola be centered at the origin with a vertical transverse axis. Its form is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ where $2a=2200 \Rightarrow a=1100$.

Teacher: Because $2c = 1$ mile $= 5280$ feet, we obtain $c = 2640$.

Student: Using the definition of the hyperbola $c^2=a^2+b^2$, it follows that

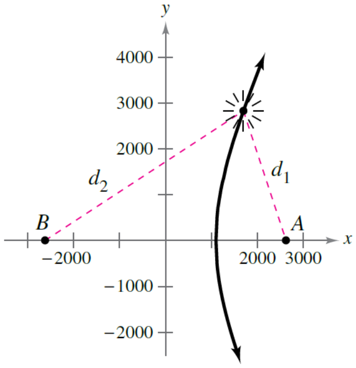
$$\begin{aligned} b^2 &= c^2-a^2 \\ &= 2640^2-1100^2 . \\ &= 5759600 \end{aligned}$$

Teacher: So, the explosion happened on…

Student: Somewhere on the right branch of the hyperbola. Its equation is

$$\frac{x^2}{1210000}-\frac{y^2}{5759600}=1.$$

老師：



由題目可知，爆炸發生的地點距離 B 比距離 A 還要遠 $1100 \times 2 = 2200$ 英尺。而距離 A 比距離 B 近 2200 的所有點形成的軌跡為雙曲線的一支。如圖所示，我們可以設雙曲線為...

學生：設雙曲線之中心為 $(0,0)$，且貫軸在 y 軸上。雙曲線的方程式形如 $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$，又 $2a=2200 \Rightarrow a=1100$。

老師：因為 $2c = 1$ 英里 $= 5280$ 英尺，可得 $c = 2640$。

學生：再利用雙曲線的定義 $c^2=a^2+b^2$，解得

$$b^2=c^2-a^2$$
$$=2640^2-1100^2 。$$
$$=5759600$$

老師：所以，爆炸發生在...

學生：爆炸發生在雙曲線的右葉某處，其方程式為

$$\frac{x^2}{1210000}-\frac{y^2}{5759600}=1 。$$

例題二

說明：關於圓錐曲線的應用題。

（英文）Which of the following conics have a focus that is also the focus of a parabola $y^2 = 2x$?

(1) $y = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4}$　(2) $\frac{x^2}{4} + \frac{y^2}{3} = 1$　(3) $x^2 + \frac{4y^2}{3} = 1$

(4) $8x^2 - 8y^2 = 1$　(5) $4x^2 - 4y^2 = 1$

（中文）試問下列哪些選項中的二次曲線，其焦點（之一）是拋物線 $y^2 = 2x$ 的焦點？

(1) $y = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4}$　(2) $\frac{x^2}{4} + \frac{y^2}{3} = 1$　(3) $x^2 + \frac{4y^2}{3} = 1$

(4) $8x^2 - 8y^2 = 1$　(5) $4x^2 - 4y^2 = 1$

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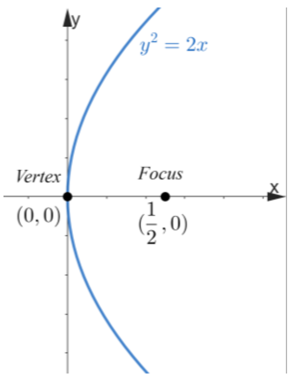
Teacher: Rewrite the equation $y^2 = 2x$ in standard form, and compare it with $(y-k)^2 = 4c(x-h)$.

Student: It shows that the vertex of the parabola $(y-0)^2 = 4 \times \frac{1}{2}(x-0)$ is at the origin with $c = \frac{1}{2}$.

Teacher:



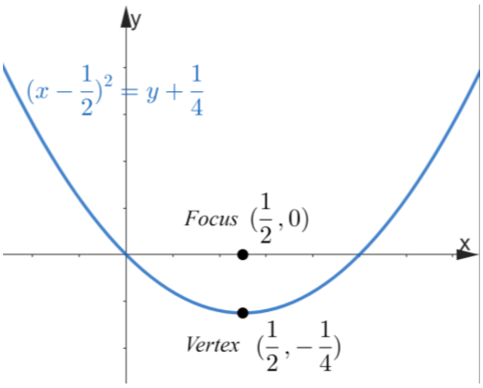
So, the parabola opens to the right, because $c = \frac{1}{2}$ is positive, as shown in the figure.

Student: The focus of the parabola is $\left(\frac{1}{2}, 0\right)$.

Teacher: (1) Rewrite $y=\left(x-\frac{1}{2}\right)^2-\frac{1}{4}$ in standard form, and compare it with $(x-h)^2=4c(y-k)$.

Student: It shows that the vertex of parabola $\left(x-\frac{1}{2}\right)^2=4\times\frac{1}{4}\left(y+\frac{1}{4}\right)$ is at $\left(\frac{1}{2},-\frac{1}{4}\right)$ with $c=\frac{1}{4}$.

Teacher:



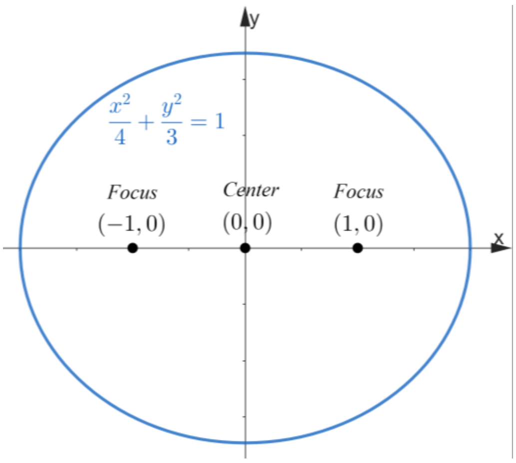
So, the parabola opens upward, because $c=\frac{1}{4}$ is positive, as shown in the figure.

Student: The focus of the parabola is $\left(\frac{1}{2},0\right)$.

Teacher: (2) Rewrite the equation $\frac{x^2}{4}+\frac{y^2}{3}=1$ in standard form $\frac{x^2}{2^2}+\frac{y^2}{\sqrt{3}^2}=1$.

Student: From this standard form, it follows that the center is the origin. The major axis is vertical, $a=2$, $b=\sqrt{3}$, and $c=\sqrt{2^2-\sqrt{3}^2}=1$.

Teacher:



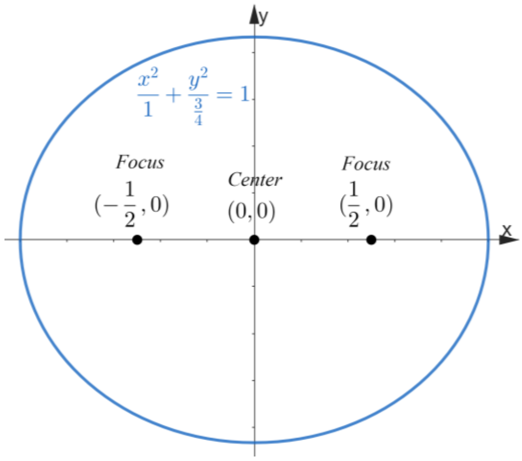
In the figure shown, the foci are...

Student: The foci are $(1,0)$ and $(-1,0)$.

Teacher: Rewrite (3) the equation $x^2+\frac{4y^2}{3}=1$ in standard form $\frac{x^2}{1^2}+\frac{y^2}{\left(\frac{\sqrt{3}}{2}\right)^2}=1$.

Student: From this standard form, it follows that the center is the origin. The major axis is vertical, $a=1$, $b=\frac{\sqrt{3}}{2}$, and $c=\sqrt{1^2-\left(\frac{\sqrt{3}}{2}\right)^2}=\frac{1}{2}$.

Teacher:



In the figure shown, the foci are...

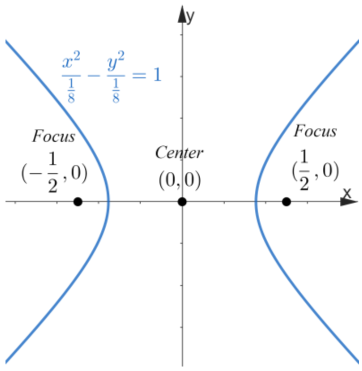
Student: The foci are $\left(\frac{1}{2},0\right)$ and $\left(-\frac{1}{2},0\right)$.

Teacher: (4) Rewrite the equation $8x^2-8y^2=1$ in standard form $\frac{x^2}{\left(\frac{1}{2\sqrt{2}}\right)^2}-\frac{y^2}{\left(\frac{1}{2\sqrt{2}}\right)^2}=1$.

Student: From this standard form, it follows that the center is the origin. The x^2-term is positive, so the transverse axis is horizontal.

Also, we obtain $a=b=\frac{1}{2\sqrt{2}}$ and $c=\sqrt{\left(\frac{1}{2\sqrt{2}}\right)^2+\left(\frac{1}{2\sqrt{2}}\right)^2}=\frac{1}{2}$.

Teacher:



In the figure shown, the foci are...

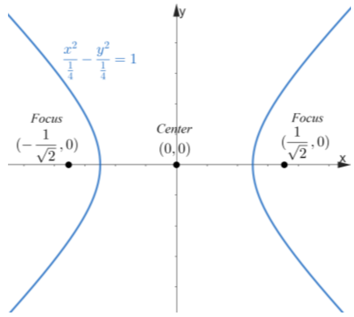
Student: The foci are $\left(\frac{1}{2},0\right)$ and $\left(-\frac{1}{2},0\right)$.

Teacher: (5) Rewrite the equation $4x^2-4y^2=1$ in standard form $\frac{x^2}{\left(\frac{1}{2}\right)^2}-\frac{y^2}{\left(\frac{1}{2}\right)^2}=1$.

Student: From this standard form, it follows that the center is the origin. The x^2-term is positive, so the transverse axis is horizontal.

Also, we obtain $a = b = \frac{1}{2}$ and $c = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}}$.

Teacher:



As shown in the figure, the foci are...

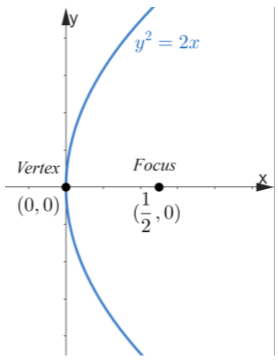
Student: The foci are $\left(\frac{1}{\sqrt{2}}, 0\right)$ and $\left(-\frac{1}{\sqrt{2}}, 0\right)$.

Teacher: So, the correct selections are (1)(3)(4).

老師： 改寫 $y^2 = 2x$ 為標準式$(y-k)^2 = 4c(x-h)$並比較係數。

學生： 由拋物線的標準式$(y-0)^2 = 4 \times \frac{1}{2}(x-0)$ 的頂點為原點，且 $c = \frac{1}{2}$。

老師：



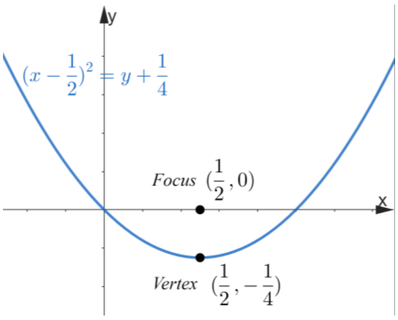
因為 $c = \frac{1}{2}$ 為正，所以拋物線開口朝右，如圖所示。

學生： 所以焦點為 $\left(\frac{1}{2}, 0\right)$。

老師： (1) 改寫 $y = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4}$ 為標準式 $(x-h)^2 = 4c(y-k)$ 並比較係數。

學生： 由拋物線的標準式$\left(x - \frac{1}{2}\right)^2 = 4 \times \frac{1}{4}\left(y + \frac{1}{4}\right)$ 的頂點為 $\left(\frac{1}{2}, -\frac{1}{4}\right)$，且 $c = \frac{1}{4}$。

老師：



因為 $c=\frac{1}{4}$ 為正，所以拋物線開口朝上，如圖所示。

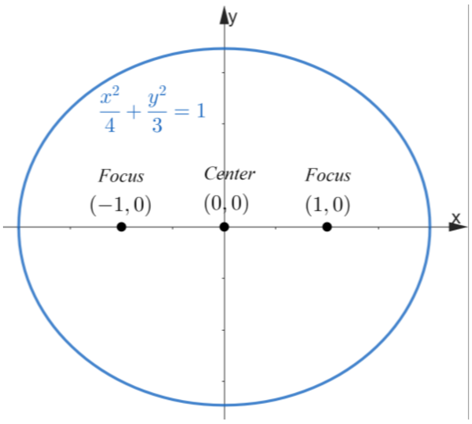
學生：所以焦點為 $\left(\frac{1}{2}, 0\right)$。

老師：(2) 改寫 $\frac{x^2}{4}+\frac{y^2}{3}=1$ 為標準式 $\frac{x^2}{2^2}+\frac{y^2}{\sqrt{3}^2}=1$。

學生：由標準式得其中心為原點，長軸在 x 軸上，$a=2$、$b=\sqrt{3}$

且$c=\sqrt{2^2-\sqrt{3}^2}=1$。

老師：



如圖所示，焦點為…

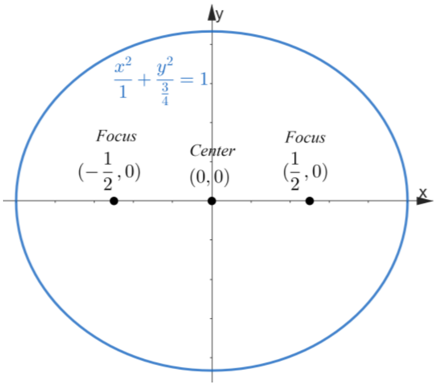
學生：焦點為 $(1,0)$ 及 $(-1,0)$。

老師：(3) 改寫 $x^2+\frac{4y^2}{3}=1$為標準式 $\frac{x^2}{1^2}+\frac{y^2}{\left(\frac{\sqrt{3}}{2}\right)^2}=1$。

學生：由標準式得其中心為原點，長軸在 x 軸上，$a=1$、$b=\frac{\sqrt{3}}{2}$

且 $c=\sqrt{1^2-\left(\frac{\sqrt{3}}{2}\right)^2}=\frac{1}{2}$。

老師：



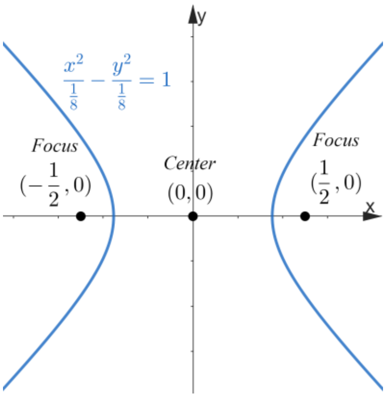
如圖所示，焦點為…

學生：焦點為 $\left(\frac{1}{2},0\right)$ 及 $\left(-\frac{1}{2},0\right)$。

老師：(4) 改寫 $8x^2-8y^2=1$ 為標準式 $\frac{x^2}{\left(\frac{1}{2\sqrt{2}}\right)^2}-\frac{y^2}{\left(\frac{1}{2\sqrt{2}}\right)^2}=1$。

學生：由標準式得其中心為原點。又因 x^2 項為正，得知貫軸為 x 軸，並得 $a=b=\frac{1}{2\sqrt{2}}$ 及 $c=\sqrt{\left(\frac{1}{2\sqrt{2}}\right)^2+\left(\frac{1}{2\sqrt{2}}\right)^2}=\frac{1}{2}$。

老師：



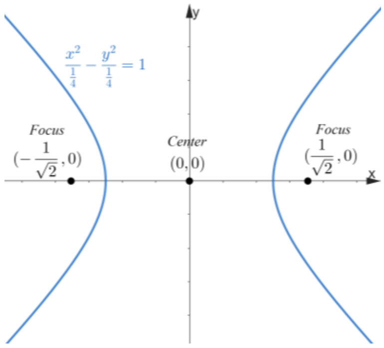
如圖所示，焦點為…

學生：焦點為 $\left(\frac{1}{2},0\right)$ 及 $\left(-\frac{1}{2},0\right)$。

老師：(5) 改寫 $4x^2-4y^2=1$ 為標準式 $\frac{x^2}{\left(\frac{1}{2}\right)^2}-\frac{y^2}{\left(\frac{1}{2}\right)^2}=1$。

學生：由標準式得其中心為原點。又因 x^2 項為正，得知貫軸為 x 軸，並得 $a=b=\frac{1}{2}$ 及 $c=\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}=\frac{1}{\sqrt{2}}$。

老師：



如圖所示，焦點為…

學生： 焦點為$\left(\frac{1}{\sqrt{2}},0\right)$ 及 $\left(-\frac{1}{\sqrt{2}},0\right)$。

老師： 故此題應選 **(1)(3)(4)**。

單元四　複數
Complex Number

國立臺灣師範大學附屬高級中學　蕭煜修老師

■ 前言 Introduction

我們在國中曾經學過利用公式法求解一元二次方程式，在求解方程式的過程中如果根號中出現負數，當時我們稱這樣的情況為「無解」(無實數解)。到了高中，我們依然會遇見一元二次方程式或是更高次的方程式(如 $x^3+1=0$、$x^4-4=0$ 等)求解的問題，當負數出現在根號中時，我們引進一個新的數 $\sqrt{-1}$，並將其記作 i，其中 $i^2=-1$。舉例來說，如果有一個方程式的根為 $3+\sqrt{-5}$，我們會改以 $3+\sqrt{-5}=3+\sqrt{5}i$ 來表示。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
quadratic equation	二次方程式	polynomial	多項式
polynomial equation	多項式方程式	discriminant	判別式
complex conjugate	**共軛複數**	system of simultaneous equations	聯立方程(組)
imaginary number	**虛數**	denominator	分母
complex number	**複數**	repeated root	重根

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ If ________, there are ________.

例句：**If** the expression under the square root is negative, **there are** no real solutions available.
若在平方根內的數為負數，則（該方程式）沒有實數解。

❷ A combination of ________ and ________ is known as ________.

例句：**A combination of** real numbers **and** imaginary numbers **is known as** a complex number.
一個由實數和虛數相加而成的數被稱為複數。

❸ Let's start by ________.

例句：**Let's start by** tackling this basic transformation problem together.
讓我們一起從解決這個基本的轉換問題開始。

❹ ________ makes use of ________.

例句：The division of complex numbers **makes use of** the formula of the reciprocal of a complex number.
複數的除法運用了複數的倒數公式。

❺ After ________, we just need to ________.

例句：**After** substituting $a+bi$ back into the equation and performing the calculations, **we just need to** compare the real and imaginary parts on both sides of the equation to find the result.
在將 $a+bi$ 代回方程式中並完成計算後，我們只需要比較方程式兩邊的實部和虛部，就可以找到結果了。

■ 問題講解 Explanation of Problems

☙ 說明 ❧

Complex numbers:

The quadratic equation $ax^2+bx+c=0$ has solutions given by $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.

This equation has a discriminant: $D=b^2-4ac$:

Case1	If $b^2-4ac>0$, there are two distinct real roots.
Case2	If $b^2-4ac=0$, there are two equal real roots.
Case3	If $b^2-4ac<0$, there are no real roots. (Two imaginary roots)

If the expression under the square root is negative, there are no real solutions available.
For example:

Solve the equation: $x^2-2x+4=0 \Rightarrow x=\dfrac{2\pm\sqrt{4-16}}{2}=1\pm\sqrt{-3}$

We got two roots $1+\sqrt{-3}$ and $1-\sqrt{-3}$.
Now consider the root $1+\sqrt{-3}$, how can we represent "$\sqrt{-3}$" in a proper way?

We are going to extend the number system to include $\sqrt{-1}$. Since there is no real number that is squared to be -1, the number $\sqrt{-1}$ is the **imaginary unit**, and is represented by the letter "i".

For example, the sum of real numbers and imaginary numbers are:

$$1-\sqrt{-3}=1-\sqrt{3\times-1}=1-\sqrt{3}\times\sqrt{-1}=1-\sqrt{3}\times i=1-\sqrt{3}i$$

$1-\sqrt{3}i$ is known as a **complex number** (複數).

<**key**> The "unit" imaginary number is: 1. $i=\sqrt{-1},\ i^2=-1$ 2. When $a>0,\ \sqrt{-a}=\sqrt{a}i$

<**key**> The set of all complex numbers is written as $\mathbb{C}$.

Complex number

For the complex number $z=a+bi \quad a,b\in\mathbb{R}$.

Re(z)=a is the real part of z. Im(z)=b is the imaginary part of z.

If two complex numbers have the same real part and the same imaginary part, we say these two complex numbers are equal. For example, $a,b,c,d\in\mathbb{R}$, $a+bi=c+di \Leftrightarrow a=c, b=d$.

Arithmetic operations with complex number

We've learned the arithmetic operations with real numbers. When we expand the number system to include complex numbers, similar arithmetic rules still apply. Here are the definitions of complex addition, subtraction, and multiplication: (We'll talk about the division later.)

Suppose a,b,c,d are real numbers:

(1) Addition: $(a+bi)+(c+di)=(a+c)+(b+d)i$

(2) Subtraction: $(a+bi)-(c+di)=(a-c)+(b-d)i$

(3) Multiplication: $(a+bi)(c+di)=(ac-bd)+(ad+bc)i$

<key>

When two complex numbers are multiplied by each other, the multiplication process should be similar to the multiplication of two binomials.

$$\begin{aligned}(a+bi)(c+di) &= a(c+di)+bi(c+di)\\ &= ac+adi+bci+bdi\\ &= ac+adi+bci-bd\\ &= (ac-bd)+(ad+bc)i\end{aligned}$$

Properties of complex number arithmetic:

Suppose z_1, z_2, z_3 are complex numbers, the following statements hold true:

(1) Commutative property of addition and multiplication: $z_1+z_2=z_2+z_1$, $z_1z_2=z_2z_1$

(2) Associative property of addition and multiplication:

$$z_1+(z_2+z_3)=(z_1+z_2)+z_3,\ z_1(z_2z_3)=(z_1z_2)z_3$$

(3) Distributive property of multiplication: $z_1(z_2+z_3)=z_1z_2+z_1z_3$

(4) Identity of addition and multiplication: $z_1+0=0+z_1=z_1$, $z_1\times 1=1\times z_1=z_1$

Just like operations with real numbers, we can define exponentiation for complex numbers.

Let n be a positive integer:

$$z^n = \underbrace{z \times z \times \ldots \times z}_{n \text{ items}},\ z^0 = 1\ (z \neq 0).$$

<**key**>The power of "i": (for $k \in \mathbb{Z}$)

(1) $i^{4k+1} =$ ___i___ (2) $i^{4k+2} =$ ___-1___ (3) $i^{4k+3} =$ ___$-i$___ (4) $i^{4k+4} =$ ___1___

complex conjugates

A complex conjugate of a complex number is another complex number where the real part remains the same as the original complex number, while the imaginary part is the negative of the original's imaginary part. For example, $3+4i$ and $3-4i$ are two complex numbers with opposite imaginary parts.

The complex conjugate of $a+bi\ (a,b \in \mathbb{R})$ is $a-bi$, and vice versa. We can denote the complex conjugate with the symbol "$\overline{}$", for example, the complex conjugate of $a+bi$ is $\overline{a+bi} = a-bi$. If we multiply $a+bi$ by its complex conjugate $a-bi$, we have:

$(a+bi)(a-bi) = a^2-(bi)^2 = a^2 - i^2b^2 = a^2+b^2$

It is a nonnegative real number. We can use this result to find the reciprocal and division of complex numbers.

Division of complex numbers

The division of complex numbers makes use of the formula of the reciprocal of a complex number.

For a complex number $z_1 = a+bi$ ($ab \neq 0$), we have the reciprocal of z_1 as following:

$$\frac{1}{z_1} = \frac{1}{a+bi} = \frac{1}{a+bi} \times \frac{a-bi}{a-bi} = \left(\frac{a}{a^2+b^2}\right) + \left(\frac{-b}{a^2+b^2}\right)i$$

For the two complex numbers $z_1 = a+bi, z_2 = c+di$ ($ab \neq 0$), we have the division as following:

$$\frac{z_2}{z_1} = (c+di) \times \frac{1}{(a+bi)} = \frac{(c+di)(a-bi)}{(a+bi)(a-bi)} = \left(\frac{ac+bd}{a^2+b^2}\right) + \left(\frac{ad-bc}{a^2+b^2}\right)i$$

運算問題的講解

例題一

說明：根據 i 的定義轉換數值。

（英文）According to the rules of i, express the following numbers using i.

(1) $\sqrt{-10}$　(2) $\sqrt{-\frac{1}{2}}$　(3) $\sqrt{-12}$

（中文）根據 i 的定義，將下列的數用 i 表示。

(1) $\sqrt{-10}$　(2) $\sqrt{-\frac{1}{2}}$　(3) $\sqrt{-12}$

Teacher: Let's start by tackling this basic transformation problem together. Based on the definition of i, how can we express the following numbers using i? How should we begin?

Student: To find the solution, we have to extract the square root of -1. See what inside the square root multiplied by -1 is, then replace $\sqrt{-1}$ with i, and we're done!

Teacher: Very good. Now let's give everyone 1 minute to solve these three problems. Later, we'll come back to check everyone's answer.

Student: (Responding)

Teacher: Okay. Times up. Let's check the answers you got. The first one is? The second one is? And the last one is?

Student: $\sqrt{10}i$ (square root of ten i), $\sqrt{\frac{1}{2}}i$ (square root of one over two i), $2\sqrt{3}i$ (two times the square root of three i).

Teacher: Good job. Especially for the third one, we need to remember to simplify when giving the answer. If there is a perfect square under the square root, remember to extract it. Let's move on to the next example.

老師：首先，我們先一起來看看這個基本的轉換問題。請根據虛數 i 的定義，把下列的數字用 i 表示。我們可以怎麼開始呢？

學生：要求出這題的答案，首先我們要把根號中的 -1 拆分出來，看看根號中的數是多少與 -1 相乘，再把 $\sqrt{-1}$ 換成以 i 表示，這樣我們就可以完成了！

老師：非常好，接下來我們給大家一分鐘完成這三個問題，等等我們一起來確認大家的答案。

學生：（計算問題）

老師：好！時間到！我們來確認你所求出的答案吧！第一題是？第二題是？第三題是？

學生：$\sqrt{10}i$（根號十 i）、$\sqrt{\frac{1}{2}}i$（根號二分之一 i）、$2\sqrt{3}i$（二根號三 i）。

老師：做得好，尤其是第三個，我們在回答答案時要記得化簡，如果有完全平方數在根號中要記得提出來，接著讓我們繼續下一個例子吧！

例題二

說明：利用複數相等的性質求解未知數。

（英文）Suppose $a,b\in\mathbb{R}$, if $(a-2)+5i=5+(b+3)i$, find the ordered pair (a,b).

（中文）設 a,b 為實數，若 $(a-2)+5i=5+(b+3)i$，求數對 (a,b)。

Teacher: In this math problem, we're going to use the properties of complex numbers and methods of solving simultaneous equations. Let's all read together.

Student: Suppose…

Teacher: What simultaneous equations can we derive from the conditions given in the problem?

Student: $(a-2)=5$ and $5=(b+3)$. The real parts and the imaginary parts are equal.

Teacher: Well-done! So, we have $(a,b)=(7,2)$. Sometimes the simultaneous equations set by the problem are more complex, but we can still use the same method to find the solution. Next, let's look at the last problem. It has four arithmetic operations.

老師：在這一題當中，我們會使用複數相等的性質以及解二元一次聯立方程組的技巧。請大家一起把題目讀過一遍吧！

學生：（閱讀題目）

老師：透過題目所給的條件，我們可以取出甚麼樣的聯立方程組？

學生：$(a-2)=5$ 以及 $5=(b+3)$。（本題的條件其實為兩個一元一次方程組）實數部分與虛數部分要相等。

老師：很好。我們可以得到 $(a,b)=(7,2)$。有時候題目所設定的聯立方程組比較複雜，不過我們依然可以用相同的手法求出結果。接著我們一起看看最後一個包含四則運算的問題吧！

例題三

說明：利用複數的四則運算求值。

（英文）Suppose $z_1 = 3+4i,\ z_2 = -1+2i$, find the following values:

(1) $z_1 + z_2$　(2) $z_1 - z_2$　(3) $z_1 z_2$　(4) $\dfrac{z_1}{z_2}$

（中文）設 $z_1 = 3+4i,\ z_2 = -1+2i$ 試求下列各小題的值：

(1) $z_1 + z_2$　(2) $z_1 - z_2$　(3) $z_1 z_2$　(4) $\dfrac{z_1}{z_2}$

Teacher: This problem gives us $z_1 = 3+4i,\ z_2 = -1+2i$, and asks us to utilize the rules of complex number arithmetic to compute the values of the following four expressions. Shall we start with the first one?

Student: The first one is $z_1 + z_2$. We have to add the real parts of the complex numbers together, add the imaginary parts together, and the same applies for subtraction.

Teacher: Good, so what are the solutions to these problems?

Student: $z_1 + z_2$ equals $2+6i$ and $z_1 - z_2$ equals $4+2i$.

Teacher: Well-done! How about the next two problems?

Student: For $z_1 z_2$, we can use the “FOIL” method. (註) (The distributive law/property)

$$z_1 z_2 = (3+4i)(-1+2i) = -3+6i-4i+8i^2 = -3+2i-8 = -11+2i.$$

Teacher: And the last one?

Student: For $\dfrac{z_1}{z_2}$, we can make use of the formula of reciprocal of a complex number.

Teacher: Yes, for $\dfrac{z_1}{z_2} = \dfrac{3+4i}{-1+2i} = \dfrac{3+4i}{-1+2i} \times \dfrac{-1-2i}{-1-2i} = \dfrac{-3-6i-4i-8i^2}{1-4i^2} = \dfrac{5-10i}{5} = 1-2i$.

We need to simplify the denominator by multiplying by the complex conjugate. According to the steps above to obtain our final answer. These are the four arithmetic operations frequently used with complex numbers, and I hope everyone can understand and use them effectively.

Student: No problem teacher!

※ 註：“FOIL” method，是國外的數學教材中在教同學乘法分配律的方法。在兩項乘以兩

項的分配律中，因為總共有四種不同的可能，分別為 First（第一項）、outer（外側項）、inner（內側項）、last（最末項），取這四個單字的字首組成 FOIL，方便同學記憶與學習，老師可以視情況決定是否要補充介紹。

老師：　這題中題目給定了 $z_1 = 3+4i,\ z_2 = -1+2i$，並要我們利用複數的四則運算規則求出下列四者的值。請問第一題要怎麼操作？

學生：　第一題是 $z_1 + z_2$ 。我們需要把這兩個複數的虛部與實部分別相加，而當計算減法的時候則改為相減。

老師：　很好，所以這兩題的答案是？

學生：　$z_1 + z_2$ 是 $2+6i$ ，$z_1 - z_2$ 是 $4+2i$ 。

老師：　做得好，那麼剩下的兩題呢？

學生：　$z_1 z_2$ 我們可以使用乘法的分配律：

$$z_1 z_2 = (3+4i)(-1+2i) = -3+6i-4i+8i^2 = -3+2i-8 = -11+2i$$

老師：　最後一題呢？

學生：　$\dfrac{z_1}{z_2}$ 我們可以使用複數的倒數進行計算。

老師：　是的，$\dfrac{z_1}{z_2} = \dfrac{3+4i}{-1+2i} = \dfrac{3+4i}{-1+2i} \times \dfrac{-1-2i}{-1-2i} = \dfrac{-3-6i-4i-8i^2}{1-4i^2} = \dfrac{5-10i}{5} = 1-2i$

我們不允許虛數出現在分母，因此要經過上面的化簡才是我們最後的答案。這就是四種我們常用的複數的四則運算，希望大家都可以十分熟練。

學生：　沒問題的老師！

應用問題 / 學測指考題

例題一

說明：利用複數的相等與複數的四則運算求解未知數。(102 學測數學)

(英文) Let $a,b\in\mathbb{R}$ and $(a+bi)(2+6i)=-80$, then $(a,b)=?$

(中文) 設 a,b 為實數且 $(a+bi)(2+6i)=-80$，其中 $i^2=-1$，則數對 $(a,b)=?$

(92 年學測數學選填題 B)

Teacher: This is a problem from a college entrance exam. We need to utilize the principles of complex numbers equality and arithmetic operations to solve for the unknowns. Let's read through it together and talk about the methods we can use.

Student: Let …. We should multiply the complex numbers and compare the real part and imaginary part.

Teacher: Very good. The challenge with this question, unlike the previous ones, arises from the need to solve a system of simultaneous equations to determine the values of *a* and *b*. Our set of simultaneous equations is as follows:

Student: $2a+6ai+2bi+6bi^2=-80 \Rightarrow \begin{cases} 2a-6b=-80 \\ 6ai+2bi=0 \end{cases}$

Teacher: Yes, we can solve the simultaneous equations to get $(a,b)=(-4,12)$. Once we have those, we're finished.

老師：這一題是大考的入學試題，我們要使用複數的相等與複數的四則運算來解出未知數，我們一起把題目的條件閱讀一遍並討論看看我們可以從哪裡下手吧！

學生：設…（閱讀題目）我們要先完成複數的乘法，之後比較等號兩邊複數的實數部分與虛數部分要分別相等。

老師：非常好，這題比前面的問題困難的點在於，這一題我們真的需要解一個聯立方程組（而不是先前的兩個一元一次方程式）。透過這些條件，我們可以找到的聯立方程組為：

學生：$2a+6ai+2bi+6bi^2=-80 \Rightarrow \begin{cases} 2a-6b=-80 \\ 6ai+2bi=0 \end{cases}$

老師：是的，我們可以把上面的聯立方程組解出來得到 $(a,b)=(-4,12)$，這樣我們就完成這一題了。

例題二

說明：利用複數的四則運算求解方程式。

（英文）Let z be a complex number, and let the real part of z be a positive number. If $z^2 = 15 - 8i$, then $z = ?$

（中文）設 z 為複數，且 z 的實數部分為正數，若 $z^2 = 15 - 8i$ ，則 $z = ?$

Teacher: The last problem needs us to use the properties of complex numbers to find the values of the equation. Does anyone have any suggestions on how we should tackle this?

Student: Since z is a complex number, we can start by assuming z to be $a + bi$, and then substitute it back into the equation to calculate.

Teacher: Great! After substituting back into the equation and performing the calculations, we just need to compare the real and imaginary parts on both sides of the equation to find the result.

Student: So $(a+bi)^2 = a^2 - b^2 + 2abi = 15 - 8i$. We have the simultaneous equations:

$\begin{cases} a^2 - b^2 = 15 \ldots (*) \\ 2ab = -8 \ldots (**) \end{cases}$. Then substitute $a = \dfrac{-4}{b}$ into $a^2 - b^2 = 15$.

We have: $(b^2 - 1)(b^2 + 16) = 0 \Rightarrow b = \pm 1$. Could there be two possibilities for z?

Teacher: That's a good point! In this problem, the condition tells us that the real part of z is a positive number. So, the only possible solution is $a = 4,\ b = -1$. I encourage everyone to give it a try and see if you can solve this question on your own.

老師：最後的這一題，我們要應用複數的性質解方程式的值，有沒有同學可以說說看我們該如何下手呢？

學生：因為 z 是複數，我們可以先假設 $z = a + bi$ ，再把這個結果代回方程組計算。

老師：很好，代回方程組完成計算後，我們只要比較等號兩邊複數的實部與虛部就可以求出結果了。

學生：因為 $(a+bi)^2 = a^2 - b^2 + 2abi = 15 - 8i$ 。我們得到下列聯立方乘組：

$\begin{cases} a^2 - b^2 = 15 \ldots (*) \\ 2ab = -8 \ldots (**) \end{cases}$ 。我們可以把 $a = \dfrac{-4}{b}$ 代入 $a^2 - b^2 = 15$ 。

得到 $(b^2-1)(b^2+16)=0 \Rightarrow b=\pm1$…這一題有兩個可能的答案嗎？

老師： 問的好！在這題當中，條件給定了複數 z 的實數部分為正，因此這題唯一的可能答案為 $a=4,\ b=-1$。希望所有的同學都可以再自己試試看是否可以獨立完成這一題喔！

單元五　多項式方程式
Polynomial Equations

國立臺灣師範大學附屬高級中學　林佳葦老師

■ 前言 Introduction

利用上一節學過的複數，我們來了解一元二次方程式的解以及根與係數的關係，進一步介紹代數基本定理與實係數方程式的虛根成對定理。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
polynomial	多項式	quadratic equation	一元二次方程式
equation	方程式	repeated root	重根
imaginary root	**虛根**	**conjugate imaginary root**	**共軛虛根**
complex conjugate	**共軛複數**	the quadratic formula	一元二次方程式公式解
discriminant	判別式	corollary	推論
complete the square	配方	**Fundamental Theorem of Algebra**	**代數基本定理**
cubic equation	三次方程式		

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ ________, and vice versa.

例句：When real roots exist, the discriminant is $b^2-4ac\geq 0$, **and vice versa**.
當實根存在時，則判別式 $b^2-4ac\geq 0$，反之亦然。

❷ _______ taking into account _______.

例句：**Taking into account** repeated roots, we'll find an equation of degree *n* has *n* complex roots
考慮到重根，我們會發現到 *n* 次方程式有 *n* 個複數根。

❸ There are _______ such that _______.

例句：**There are** rational numbers a,b,c **such that** $f(1), f(2), f(3), f(4)$ sequentially forms an arithmetic sequence.
存在有理數 a,b,c 使得 $f(1), f(2), f(3), f(4)$ 依序形成等差數列。

■ 問題講解 Explanation of Problems

☙ 說明 ❧

How to Solve the Quadratic Equations

With complex numbers, all the quadratic equations of one variable have roots. When the root of a quadratic equation is an imaginary number, the equation is said to have an imaginary root.

For example, we solve a quadratic equation $x^2+2x+2=0$.

Complete the square:

$$x^2+2x+1=-1$$
$$\Rightarrow (x+1)^2=-1$$

Take the square root of both sides:

$$\Rightarrow x+1=\pm i$$
$$\Rightarrow x=-1\pm i$$

Note that the two imaginary roots of this equation are conjugate complex numbers with each other. The two imaginary roots are called **conjugate imaginary roots**.

To solve a general quadratic equation $ax^2+bx+c=0$ where *a, b,* and *c* are real numbers and $a\neq 0$, you can use the method of completing the square.

$$a(x+\frac{b}{2a})^2-\frac{b^2}{4a}+c=0$$

Move the terms and simplify:

$$a(x+\frac{b}{2a})^2=\frac{b^2-4ac}{4a}$$
$$\Rightarrow (x+\frac{b}{2a})^2=\frac{b^2-4ac}{4a^2}$$

Take the square root of both sides:

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$
$$\Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

This important result is known as **the quadratic formula.**

It is not difficult to see that the positive, negative, or 0 value of b^2-4ac affects whether the two roots are different real roots, conjugate imaginary roots, or repeated roots. Therefore, we call b^2-4ac the **discriminant**.

$b^2-4ac>0$	$b^2-4ac=0$	$b^2-4ac<0$
Two distinct real roots.	Two equal real roots. (repeated roots).	Two conjugate imaginary roots.

Note that when real roots exist, the discriminant is $b^2-4ac\geq 0$, and vice versa.

Vieta's Formula for Quadratic Equations

The relationship between the roots and the coefficients of the polynomial equations was discovered by a French mathematician François Viète. We will only present Vieta's formula for quadratic equations.

For example,

$$\begin{aligned}&(x-2)(x+3)=0\\ \Rightarrow\ &x^2+(-2+3)x+(-2)\times 3=0\\ \Rightarrow\ &x^2+1x-6=0\end{aligned}$$

Two roots of $(x-2)(x+3)=0$ are $2,-3$. You can see that the sum of the two roots is the opposite of the coefficient of x, and the constant term is the product of the two roots.

For a general quadratic equation $ax^2+bx+c=0$ where *a*, *b*, and *c* are both real numbers and $a\neq 0$, if the roots of $ax^2+bx+c=0$ are α,β, then we can factor the equation into $a(x-\alpha)(x-\beta)=0$. Comparing the coefficients of the two equations, we can get $-a(\alpha+\beta)=b$ and $a(\alpha\beta)=c$. Then we can get

$$\alpha+\beta=\frac{-b}{a},\alpha\beta=\frac{c}{a}.$$

Fundamental Theorem of Algebra

By inventing complex numbers, we know that a quadratic equation must have two complex roots. Does a cubic equation have exactly three complex roots? For any equation of degree 10, how many roots will there be? Gauss and Jean-Robert Argand proved the fundamental theorem of algebra, thereby answering those questions.

Fundamental Theorem of Algebra Every single-variable polynomial equation must have at least one complex root.

Now let’s return to our question, “Does the cubic equation have exactly three complex roots?”. According to the fundamental theorem of algebra, the cubic equation $f(x)=0$ must have at least one complex root α. So, $f(x)=g(x)(x-\alpha)$, where $g(x)$ is a quadratic polynomial. Similarly, according to the fundamental theorem of algebra, $g(x)=0$ must have two complex root β,γ. So, $f(x)=(x-\alpha)(x-\beta)(x-\gamma)$.

The same corollary shows that an equation of degree n has n complex roots if we take into account repeated roots.

A single-variable, degree polynomial equation with complex coefficients of degree *n* has *n* complex roots (taking into account repeated roots).

A Pair of Imaginary Roots in A Polynomial Equation with Real Coefficients

We now introduce another important theorem. In the case of polynomial equations with real coefficients, complex roots always occur in conjugate pairs.

For example, if $2+3i$ is a root of a polynomial equation with real coefficients, then its complex conjugate $2-3i$ must be another root.

> Suppose $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ is a real coefficient polynomial. If $z=a+bi$ is an imaginary root of $f(x)=0$ (a,b are real numbers and $b\neq 0$), then $\overline{z}=a-bi$ is also a root of $f(x)=0$.

Let's prove the theorem.

Since z is a root of $f(x)=0$, we have

$$a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z^n+a_0=0$$

$$\Rightarrow \overline{a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0}=0$$

$$\Rightarrow \overline{a_nz^n}+\overline{a_{n-1}z^{n-1}}+\cdots+\overline{a_1z}+\overline{a_0}=0$$

$$\Rightarrow a_n\left(\overline{z}\right)^n+a_{n-1}\left(\overline{z}\right)^{n-1}+\cdots+a_1\left(\overline{z}\right)+a_0=0$$

So, $\overline{z}$ is also a root of $f(x)=0$.

Thus, the complex roots of a polynomial equation with real coefficients occur as a conjugate pair.

運算問題的講解

例題一

說明： 由一元二次方程式的係數觀察兩根的關係。

（英文）Suppose k is a real number and the equation $x^2+2x+k=0$. Let's discuss the relationship between the two roots of this equation based on the range of k.

（中文）設 k 為實數，方程式 $x^2+2x+k=0$，試就 k 的範圍討論此方程式兩根的關係。

Teacher: Let's calculate the discriminant first. What's the discriminant of this equation?

Student: From the equation $x^2+2x+k=0$, we have $a=1, b=2, c=k$. The discriminant is $b^2-4ac=2^2-4\cdot1\cdot k=4-4k$.

Teacher: Yes. If the discriminant is positive, what can we get?

Student: If $4-4k$ is positive, we get $k<1$. The equation $x^2+2x+k=0$ has two distinct roots.

Teacher: Good. If the discriminant is zero, what can we get?

Student: If $4-4k$ is zero, we can get $k=1$. The equation $x^2+2x+k=0$ has two equal roots.

Teacher: Great. If the discriminant is negative, what can we get?

Student: If $4-4k$ is negative, we get $k>1$. The equation $x^2+2x+k=0$ has two conjugate imaginary roots.

Teacher: Well-done!

老師：我們先計算一下判別式。這個方程式的判別式是什麼？

學生：從這個方程式 $x^2+2x+k=0$，可得 $a=1, b=2, c=k$。
判別式 $b^2-4ac=2^2-4\cdot1\cdot k=4-4k$。

老師：是的。如果判別式為正，我們能得到什麼？

學生：若 $4-4k$ 為正，則 $k<1$。方程式 $x^2+2x+k=0$ 有兩相異實根。

老師：很好。如果判別式為零，我們能得到什麼？

學生：若 $4-4k$ 為零，則 $k=1$。方程式 $x^2+2x+k=0$ 有兩相等實根。

老師：很好。如果判別式為負，我們能得到什麼？

學生： 若 $4-4k$ 為負，則 $k>1$。方程式 $x^2+2x+k=0$ 有兩共軛虛根。
老師： 做得好。

例題二

說明：利用實係數方程式的虛根成對定理性質回推方程式。

（英文）Suppose a,b are real numbers, and the polynomial equation $x^3+ax^2+bx-4=0$ has one root $-1+i$. Find the value of a,b.

（中文）設 a,b 為實數，且多項式方程式 $x^3+ax^2+bx-4=0$ 有一根為 $-1+i$，試求 a,b 的值。

Teacher: First, we observe the coefficients of this polynomial $x^3+ax^2+bx-4=0$. What do they have in common?
Student: They are all real numbers. So, we know another root is $-1-i$.
Teacher: Really? But why?
Student: In the case of polynomial equations with real coefficients, complex roots always occur in conjugate pairs.
Teacher: Very good. So, we now have two roots in this equation. How can we use those roots?
Student: We can get that

$(x-(-1+i))(x-(-1-i))=x^2+2x+2$ is a factor of x^3+ax^2+bx-4.

Teacher: Good job! Next, comparing the linear coefficient and the constant term, we can see that the polynomial can be factored into $x^3+ax^2+bx-4=(x^2+2x+2)(x-2)$. Expand the right-hand equation and compare the coefficients. What can we get?
Student: We get $a=0,b=-2$.
Teacher: Excellent!

老師： 首先我們先觀察這個多項式 $x^3+ax^2+bx-4=0$ 的係數。他們有什麼共同點？
學生： 它們都是實數。所以我們知道另一個根是 $-1-i$。
老師： 真的嗎？但為什麼？
學生： 因為具有實數係數的多項式方程式，複數根總是以共軛的形式出現。
老師： 非常好。現在我們有了這個方程式的兩個根，我們要如何使用這些根呢？

學生：　我們可得 $(x-(-1+i))(x-(-1-i))=x^2+2x+2$ 是 x^3+ax^2+bx-4的因式。

老師：　做得好。接下來，比較一次項係數和常數項，我們可以看到多項式可以分解為 $x^3+ax^2+bx-4=(x^2+2x+2)(x-2)$。

將右邊的方程式展開並比較係數，我們可以得到什麼？

學生：　我們可得 $a=0,b=-2$。

老師：　很棒。

應用問題 / 學測指考題

例題一

說明：乘法公式和方程式的根的應用。

（英文）Consider the polynomial $f(x)=3x^4+11x^2-4$, and choose the correct option. (Multiple choices)

(1) The y-coordinate of the intersection point of the graph and the y-axis is less than 0.

(2) $f(x)=0$ has 4 real roots.

(3) $f(x)=0$ has at least one rational root.

(4) $f(x)=0$ has a root between 0 and 1.

(5) $f(x)=0$ has a root between 1 and 2.

（中文）考慮多項式 $f(x)=3x^4+11x^2-4$，試選出正確的選項。（多選）

(1) $y=f(x)$ 的圖形和 y 軸交點的 y 坐標小於 0。

(2) $f(x)=0$ 有 4 個實根。

(3) $f(x)=0$ 至少有一個有理根。

(4) $f(x)=0$ 有一根介於 0 與 1 之間。

(5) $f(x)=0$ 有一根介於 1 與 2 之間。

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Teacher: This is a multiple-choice question. Let's see the first option. What's the *y*-coordinate of the intersection point of the graph $y = f(x)$ and the *y*-axis? How do we get the *y*-coordinate?

Student: We let $x = 0$, and the y-coordinate is $f(0) = -4$. So, it is less than 0.

Teacher: Good. For the second and the third option, we need to check the roots of $f(x) = 0$. Can you try to solve the equation?

Student: Ok, I'll try. $3x^4 + 11x^2 - 4 = 0$ implies $(x^2 + 4)(3x^2 - 1) = 0$. Then we get $x^2 = -4$ or $x^2 = \frac{1}{3}$. We have $x = \pm 2i$ or $x = \pm \frac{1}{\sqrt{3}}$.

Teacher: Great. We have two real roots and two imaginary roots but no rational roots. How about the last two options?

Student: We can easily get that $f(x) = 0$ has a root between 0 and 1.

Teacher: Good job. So, we have to choose (1)(4).

老師：這是一道多重選擇題。我們看第一個選項。圖形 $y = f(x)$ 與 y 軸交點的 y 坐標是多少？如何取得 y 坐標？

學生：我們取 $x = 0$，得到 y 坐標是 $f(0) = -4$，所以小於 0 。

老師：很好。對於第二個和第三個選項，我們需要檢查 $f(x) = 0$ 的根。你能嘗試解這個方程式嗎？

學生：好，我來試試。$3x^4 + 11x^2 - 4 = 0$ 推得 $(x^2 + 4)(3x^2 - 1) = 0$。則我們得 $x^2 = -4$ 或 $x^2 = \frac{1}{3}$。我們有 $x = \pm 2i$ 或 $x = \pm \frac{1}{\sqrt{3}}$。

老師：很不錯。我們得到了兩個實根和兩個虛根，但沒有得到有理根。最後兩個選項怎麼樣呢？

學生：我們很輕易得知 $f(x) = 0$ 有一根介於 0 與 1 之間。

老師：做的好。所以我們選(1)(4)。

例題二

說明：實係數多項式方程式的綜合應用。

（英文）Let the polynomial function $f(x)=x^3+ax^2+bx+c$, where a,b,c are all rational numbers. Try to choose the correct options. (Multiple choices)

(1) The graph of the function $y=f(x)$ and the parabola $y=x^2+100$ may not intersect.

(2) If $f(0)f(1)<0<f(0)f(2)$, then the equation $f(x)=0$ must have three distinct real roots.

(3) If $1+3i$ is a complex root of the equation $f(x)=0$, then the equation $f(x)=0$ has a rational root.

(4) There are rational numbers a,b,c such that $f(1), f(2), f(3), f(4)$ sequentially forms an arithmetic sequence.

(5) There are rational numbers a,b,c such that $f(1), f(2), f(3), f(4)$ sequentially forms a geometric sequence.

（中文）設多項式函數 $f(x)=x^3+ax^2+bx+c$，其中 a,b,c 均為有理數。試選出正確的選項。（多選）

(1) 函數 $y=f(x)$ 與拋物線 $y=x^2+100$ 的圖形可能沒有交點

(2) 若 $f(0)f(1)<0<f(0)f(2)$，則方程式 $f(x)=0$ 必有三個相異實根

(3) 若 $1+3i$ 是方程式 $f(x)=0$ 的複數根，則方程式 $f(x)=0$ 有一個有理根

(4) 存在有理數 a,b,c 使得 $f(1), f(2), f(3), f(4)$ 依序形成等差數列

(5) 存在有理數 a,b,c 使得 $f(1), f(2), f(3), f(4)$ 依序形成等比數列。

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Teacher: The degree of the polynomial function $f(x)=x^3+ax^2+bx+c$ is 3 and the coefficients of $f(x)$ are rational numbers. Let's see the first option. Do the graph of the function $y=f(x)$ and the parabola $y=x^2+100$ have intersection points?

Student: We can solve a pair of simultaneous equations below to find out.

$$\begin{cases} y=f(x) & ① \\ y=x^2+100 & ② \end{cases}$$

Teacher: Alright. Let's do this. Substituting for *y* in the equation ②, what can you get?

Student: $f(x)=x^2+100$.

Teacher: Good. Now rearranging the equation gives $f(x)-x^2-100=0$.

$f(x)-x^2-100=0$ is a cubic real coefficient equation. So, what is the property of its root?

Student: A cubic real coefficient equation is with at least one real root.

Teacher: Yes. What will happen to the intersection points of these two graphs?

Student: The two figures must have at least one intersection point.

Teacher: Good job. Then the first option is false. For the second option, what can we get from $f(0)f(1)<0$?

Student: From $f(0)f(1)<0$, we know that $f(x)=0$ has at least a root between 0 and 1.

Teacher: How about $0<f(0)f(2)$?

Student: We know that $f(0)$ and $f(2)$ has the same signs. $f(0)f(1)<0$ implies $f(0)$ and $f(1)$ has distinct signs. We get $f(1)$ and $f(2)$ has distinct signs, so $f(x)=0$ has at least a root between 1 and 2.

Teacher: Excellent! Since the degree of $f(x)$ is 3, then $f(x)=0$ has 3 distinct real roots. The second option is true. For the third option, $1+3i$ is a complex root of the equation $f(x)=0$. Since all coefficients of $f(x)$ are rational numbers, then what can we get?

Student: Its complex conjugate $1-3i$ must be another root.

Teacher: Yes. We can deduce that $f(x)$ has two factors
$(x-(1+3i))(x-(1-3i))=(x^2-2x+10)$.
Factor $f(x)$. What can we get?

Student: $f(x)=(x^2-2x+10)(x+\frac{c}{10})$.

Teacher: Yes. Since c is a rational number, $x=-\frac{c}{10}$ is rational. The third option is true. For the fourth option, if the sequence $f(1), f(2), f(3), f(4)$ is an arithmetic sequence, are $(1, f(1))$, $(2, f(2))$, $(3, f(3))$, $(4, f(4))$ on the same straight line?

Student: Yes. If $f(1), f(2), f(3), f(4)$ has equal differences between two consecutive terms, then $(1, f(1))$, $(2, f(2))$, $(3, f(3))$, $(4, f(4))$ are on the same straight line. However, the cubic function $y=f(x)$ cannot intersect the straight line at four different points, so there are no such a, b and c.

Teacher: Good points! Then is the sequence $f(1), f(2), f(3), f(4)$ a geometric sequence?

Student: If the sequence $f(1), f(2), f(3), f(4)$ is a geometric sequence, $(1, f(1))$, $(2, f(2))$, $(3, f(3))$, $(4, f(4))$ are on the graph of an exponential function. Plus, the cubic function $y = f(x)$ may intersect the exponential function at four different points, so there may be such a, b and c.

Teacher: Excellent! Now we try to create a function that matches the description of option 5. First, we assume that the common ration is 2. What relations can we get?

Student: We get $\frac{f(2)}{f(1)} = \frac{f(3)}{f(2)} = \frac{f(4)}{f(3)} = 2$

Teacher: Good.
Let $f(1) = t$ and $f(x) = (x-1)(x-2)(x-3) + B(x-1)(x-2) + C(x-1) + D$.
Since $f(1) = t, f(2) = 2t, f(3) = 4t, f(4) = 8t$, then calculate t and the coefficients B, C, and D.

Student: We get $D = t$, $C = t$, $B = \frac{1}{2}t$, $t = 6$. So $D = 6$, $C = 6$, $B = 3$.
So we get $f(x) = (x-1)(x-2)(x-3) + 3(x-1)(x-2) + 6(x-1) + 6$. Simplify the function, we get $f(x) = x^3 - 3x^2 + 8x$.

Teacher: Good job. Now we created a function that matches the description of option 5. So, we have to choose (2)(3)(5).

老師： 多項式函數 $f(x) = x^3 + ax^2 + bx + c$ 的次數為 3， $f(x)$ 的係數為有理數。讓我們來看看第一個選項。函數 $y = f(x)$ 圖形和拋物線 $y = x^2 + 100$ 有交點嗎？

學生： 我們可以解出下面的一對聯立方程式來找出答案。

$$\begin{cases} y = f(x) & ① \\ y = x^2 + 100 & ② \end{cases}$$

老師： 好吧。我們開始吧。將 y 代入方程式 ② 中，可以得到什麼？

學生： $f(x) = x^2 + 100$ 。

老師： 好的。現在重新排列方程式給出 $f(x) - x^2 - 100 = 0$ 。 $f(x) - x^2 - 100 = 0$ 是三次實係數方程式。那麼它的根有什麼性質呢？

學生： 三次實係數方程式至少有一個實根。

老師： 是的。這兩個圖形的交點會怎麼樣呢？

學生： 兩個圖形必至少有一個交點。

老師：很好。那麼第一個選項就是錯的。對於第二個選項，從 $f(0)f(1)<0$，我們能得到什麼？

學生：從 $f(0)f(1)<0$，我們知道 $f(x)=0$ 至少有一個介於 0 和 1 之間的根。

老師：那 $0<f(0)f(2)$ 呢？

學生：我們知道 $f(0)$ $f(2)$ 同號。$f(0)f(1)<0$ 推得 $f(0)$ 和 $f(1)$ 異號。我們得到 $f(1)$ 和 $f(2)$ 異號，所以 $f(x)=0$ 至少有一個介於 1 和 2 之間的根。

老師：太棒了。由於 $f(x)$ 的次數為 3，因此 $f(x)=0$ 有 3 個不同的實根。第二個選項是正確的。對於第三個選項，$1+3i$ 是方程式 $f(x)=0$ 的複數根，由於 $f(x)$ 的所有係數都是有理數，那麼我們能得到什麼呢？

學生：它的共軛複數 $1-3i$ 一定是另一個根。

老師：是的。我們推斷這有兩個因式 $(x-(1+3i))(x-(1-3i))=(x^2-2x+10)$。因式分解 $f(x)$，我們能得到什麼？

學生：$f(x)=(x^2-2x+10)(x+\frac{c}{10})$。

老師：是的，既然 c 是有理數，那麼 $x=-\frac{c}{10}$ 是有理數。第三個選項是正確的。至於第四個選項，如果數列 $f(1), f(2), f(3), f(4)$ 是等差數列，$(1, f(1))$、$(2, f(2))$、$(3, f(3))$、$(4, f(4))$ 是否在同一條直線上？

學生：是的。如果 $f(1), f(2), f(3), f(4)$ 連續兩項之間的差異相等，則 $(1, f(1))$、$(2, f(2))$、$(3, f(3))$、$(4, f(4))$ 在同一條直線上。然而，三次函數不能與直線相交於四個不同點，因此不存在這樣的 a、b、c。

老師：做得好。那麼這個數列是幾何數列呢？

學生：若數列 $f(1), f(2), f(3), f(4)$ 是等比數列，則 $(1, f(1))$、$(2, f(2))$、$(3, f(3))$、$(4, f(4))$ 位於指數函數圖形上。此外，三次函數可能與此指數函數在四個不同的點相交，因此可能存在這樣的 a、b、c。

老師：做的好。現在我們嘗試建立一個符合選項 5 所描述的函數。首先我們假設公比為 2，我們可以得到什麼關係式呢？

學生：我們可得 $\frac{f(2)}{f(1)}=\frac{f(3)}{f(2)}=\frac{f(4)}{f(3)}=2$。

老師：很好。令 $f(1)=t$ 和 $f(x)=(x-1)(x-2)(x-3)+B(x-1)(x-2)+C(x-1)+D$。因為 $f(1)=t, f(2)=2t, f(3)=4t, f(4)=8t$，接著計算 t 和係數 B、C 和 D。

學生：可得 $D=t$，$C=t$，$B=\frac{1}{2}t$，$t=6$。所以 $D=6$，$C=6$，$B=3$。所以可得

$f(x)=(x-1)(x-2)(x-3)+3(x-1)(x-2)+6(x-1)+6$。

化簡得 $f(x)=x^3-3x^2+8x$。

老師：做得好。現在我們製造了符合選項 5 的多項式函數了。

所以我們選 (2)(3)(5)。

單元六　複數的極式與幾何意義
The Polar Form of Complex Numbers and Their Geometric Interpretation

國立臺灣師範大學附屬高級中學　林佳葦老師

■ 前言 Introduction

本節將藉由極坐標的概念來介紹複數的極式，並說明極式如何應用在複數的乘除運算。最後將介紹如何利用複數的極式求一個複數的 n 次方根。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
complex plane / Argand diagram	複數平面	coterminal angle	同界角
polar form / modulus-argument form	極式	cube root	三次方根
principal argument	主幅角	generalized angle	廣義角
modulus (plural: **moduli**)	模/向徑		
de Moivre's theorem	棣美弗定理		
*n*th roots of one / ***n*th roots of unity**	1 的 n 次方根		

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ ________ generalizes ________.

例句：For $n>3$, de Moivre's theorem **generalize**s this to show that to raise a complex number to the *n*th power, the absolute value is raised to the *n*th power and the argument is multiplied by *n*.

對於 $n>3$，棣美弗定理對上述進行了推廣，表明如果要推廣複數的 n 次方，則須將複數的絕對值變成 n 次方，並將其幅角乘以 n。

❷ We stipulate ________.

例句：**We stipulate** $z^0=1$.

我們規定 $z^0=1$。

❸ ________, which is indeed the case.

例句：These roots seem to be evenly distributed on the unit circle, **which is indeed the case.**

這些根看起來均勻分佈在單位圓上，事實確實如此。

❹ Furthermore, ________.

例句：**Furthermore,** since the angle between any two consecutive roots is $\frac{2\pi}{n}$, the *n*th roots of unity are evenly spaced around the unit circle.

此外，由於任兩個連續根之間的角度為 $\frac{2\pi}{n}$，因此 1 的 n 次複數根圍繞單位圓均勻分佈。

❺ In general, ________.

例句：**In general**, any non-zero complex number a also has n *n*th roots.

一般來說，任何非零複數 a 也有 n 個 n 次方根。

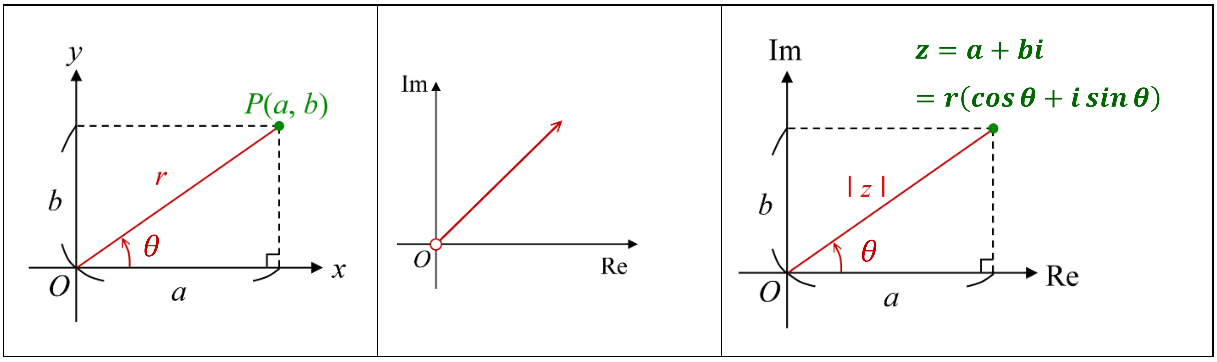
■ 問題講解 Explanation of Problems

說明

The polar form of complex numbers

We have already learned about the relationship between the rectangular coordinates and the polar coordinates. If the polar coordinates of point $P(a,b)$ are $[r,\theta]$, then $a=r\cos\theta$ and $b=r\sin\theta$.

Therefore, the point $z=a+bi$ on the complex plane can be transformed into the form: $z=a+bi=r\cos\theta+(r\sin\theta)i=r(\cos\theta+i\sin\theta)$. We call $z=r(\cos\theta+i\sin\theta)$ the **polar or modulus-argument form**. In this form, $r=\sqrt{a^2+b^2}$ is the absolute value of *z* or the modulus of *z*, denoted by $r=|z|=\sqrt{a^2+b^2}$. Moreover, θ is the generalized angle whose initial side is on the real axis and whose final side is on the ray OP. θ is called the **argument** of *z*. We write $\arg z=\theta$. If θ is on the interval $[0,2\pi)$, then θ is called the **principal argument**.



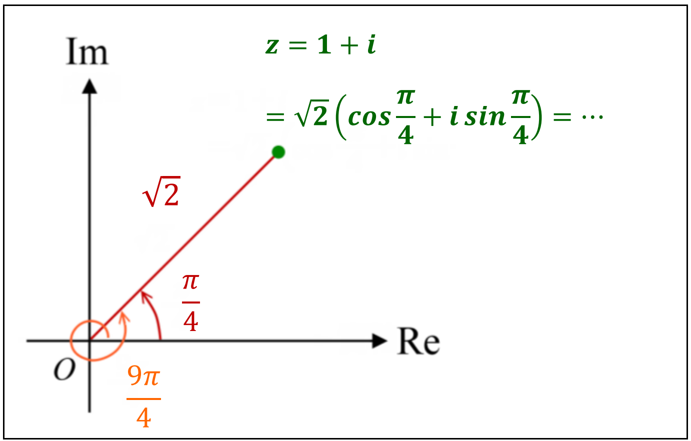
For example, the polar form of $z=1+i$ is:

$$z=\sqrt{2}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})$$
$$=\sqrt{2}(\cos\frac{9\pi}{4}+i\sin\frac{9\pi}{4})$$
$$=\cdots$$

where $\sqrt{2}$ is the modulus of z, and $\frac{\pi}{4}$ is the principal argument of z.

Usually, when z is written in polar form, we let θ be a principal argument.

We stipulate that the polar form of the complex number 0 is $0(\cos\theta + i\sin\theta)$, its absolute value is 0, and the argument θ can be any real number.



From the definition of the polar form, we can immediately get that when the two polar forms are equal, their absolute values are equal, and the arguments are coterminal angles.

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, if $z_1 = z_2$, then $r_1 = r_2$ and $\theta_1 = \theta_2 + 2k\pi, k \in \mathbb{Z}$.

The geometric meaning of complex number multiplication

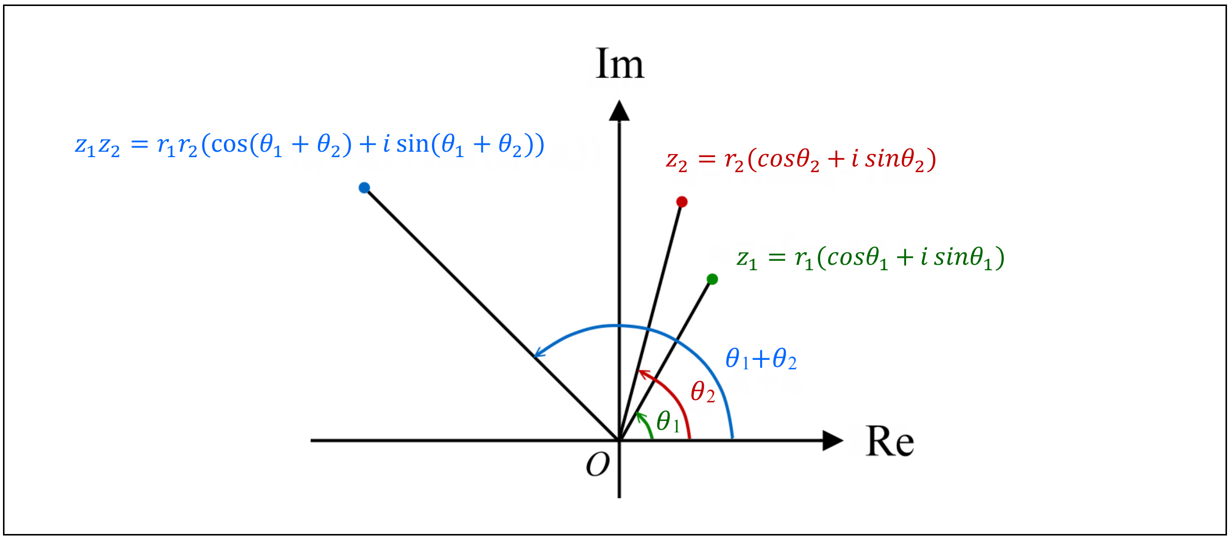
Let $z_1 = r_1(cos\theta_1 + i\ sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, we get:

$$\begin{aligned} z_1z_2 &= r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1r_2((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)) \\ &= r_1r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)) \end{aligned}$$

(the compound-angle formula)

This is the complex number with modulus r_1r_2 and argument $(\theta_1+\theta_2)$, so we have the great result that $|z| = |z_1||z_2|$ and $\arg z = \arg z_1 + \arg z_2$ ($\pm 2\pi$ if necessary, to give the principal argument).

So, to multiply complex numbers in polar form you multiply their moduli and add their arguments.



If $z_2 \neq 0$, then:

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$= \frac{r_1}{r_2}\cdot\frac{(\cos\theta_1 + i\sin\theta_1)}{(\cos\theta_2 + i\sin\theta_2)}\cdot\frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)}$$

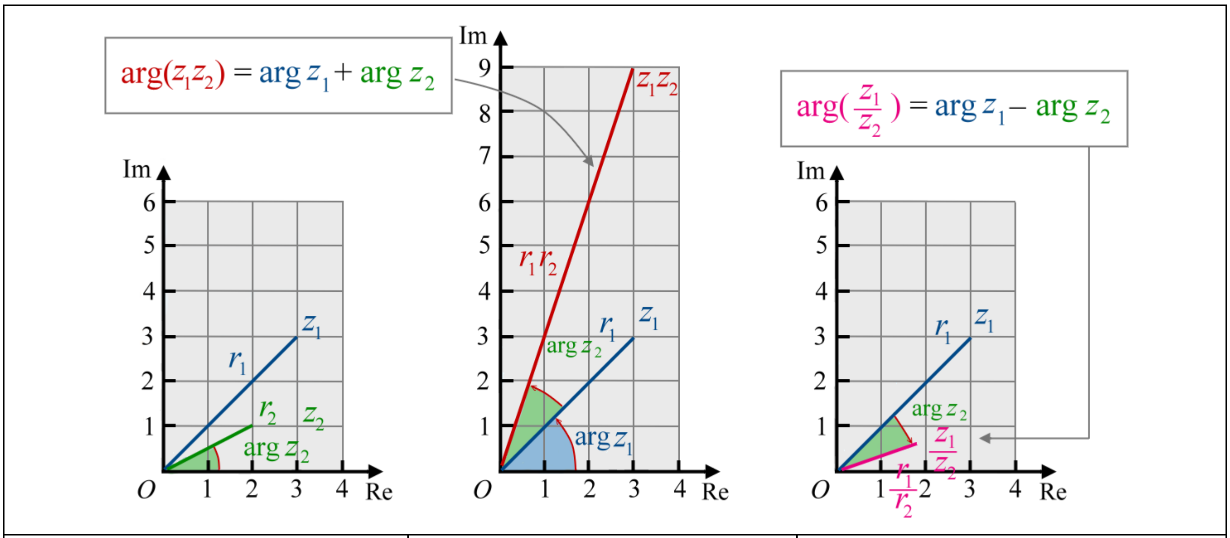
$$= \frac{r_1}{r_2}\cdot\frac{(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{1}$$

$$= \frac{r_1}{r_2}(\cos\theta_1 + i\sin\theta_1)(\cos(-\theta_2) + i\sin(-\theta_2))$$

$$= \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$$

So, to divide complex numbers in polar form you divide their moduli and subtract their arguments.

This gives the following simple geometrical interpretation of multiplication and division.



(i) z_1 and z_2 | (ii) Multiply z_1 by z_2 | (iii) divide z_1 by z_2

De Moivre's Theorem

Let $z = r(\cos\theta + i\sin\theta)$, we calculate:

$$\begin{aligned} z^2 &= zz \\ &= r(\cos\theta + i\sin\theta)r(\cos\theta + i\sin\theta) \\ &= r^2(\cos 2\theta + i\sin 2\theta), \end{aligned}$$

$$\begin{aligned} z^3 &= z^2z \\ &= r^2(\cos 2\theta + i\sin 2\theta)r(\cos\theta + i\sin\theta) \\ &= r^3(\cos 3\theta + i\sin 3\theta), \\ &\vdots \end{aligned}$$

This shows that by squaring a complex number, the absolute value is squared, and the argument is multiplied by 2. Also, by cubing a complex number, the absolute value is cubed, and the argument is multiplied by 3. For $n > 3$, de Moivre's theorem generalizes this to show that to raise a complex number to the nth power, the absolute value is raised to the nth power and the argument is multiplied by n. We can guess that if $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then: $z^n = r^n(\cos n\theta + i\sin n\theta)$.

We will prove the formula by mathematical induction.

(i) For $n = 1$, $z^1 = r^1(\cos(1\theta) + i\sin(1\theta))$ is true.

(ii) We assume the formula is true for $n = k$, so we have:

$$z^k = r^k(\cos(k\theta) + i\sin(k\theta))$$

For $n=k+1$, we get:

$$\begin{aligned} z^{k+1} &= z^k z \\ &= r^k(\cos(k\theta)+i\sin(k\theta))r(\cos\theta+i\sin\theta) \\ &= r^{k+1}(\cos(k\theta+\theta)+i\sin(k\theta+\theta)) \\ &= r^{k+1}(\cos(k+1)\theta+i\sin(k+1)\theta) \end{aligned}$$

Thus, the formula is true for: $n=k+1$

By (i), (ii) and the principle of mathematical induction, the formula is true for all $n\geq 1, n\in\mathbb{N}$. This is the de Moivre's theorem.

De Moivre's theorem
If $z=r(\cos\theta+i\sin\theta)$ and n is a positive integer, then: $z^n=r^n(\cos n\theta+i\sin n\theta)$.

If $z\neq 0$ and n is a positive integer, we stipulate that the $-n$ power of z is $z^{-n}=\left(\frac{1}{z}\right)^n$,

then de Moivre's theorem is true for the integer $-n$, that is:

$$z^{-n}=r^{-n}(\cos(-n\theta)+i\sin(-n\theta)).$$

This is because:

$$\begin{aligned} z^{-n} &= \left(\frac{1}{r(\cos\theta+i\sin\theta)}\right)^n \\ &= \left(\frac{1}{r}(\cos\theta-i\sin\theta)\right)^n \\ &= \frac{1}{r^n}(\cos(-\theta)+i\sin(-\theta))^n \\ &= r^{-n}(\cos(-n\theta)+i\sin(-n\theta)) \end{aligned}$$

In addition, when $z=r(\cos\theta+i\sin\theta)$ is a non-zero complex number, since $z^0=r^0(\cos 0\theta+i\sin 0\theta)=1$, we stipulate $z^0=1$, de Moivre's theorem is also true for the $n=0$. So, the theorem holds for all non-zero complex numbers z and for all integers n.

If a nonzero complex number z with its polar form $z=r(\cos\theta+i\sin\theta)$ and n is an integer, then: $z^n=r^n(\cos n\theta+i\sin n\theta)$.

De Moivre's theorem gives a formula for computing higher powers of complex numbers.

Now we shall discuss ***n*th roots of one**, also often called ***n*th roots of unity**.

***N*th roots of unity(one)**

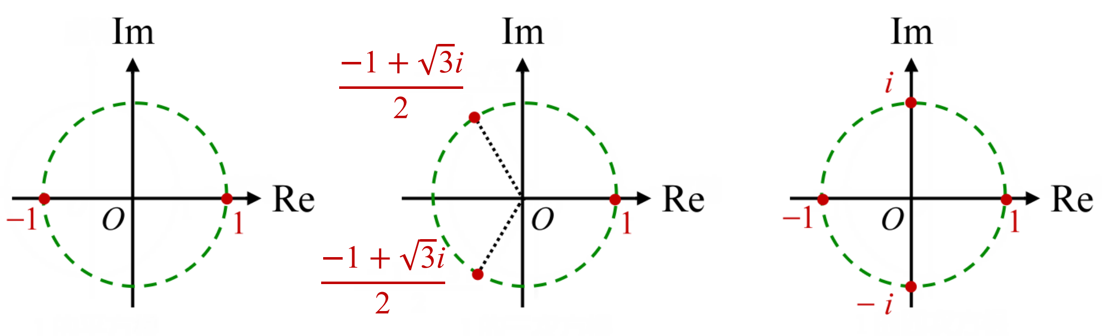
Suppose n is a positive integer, the root of the equation $z^n - 1 = 0$ is called the ***n*th root of unity**. According to the fundamental theorem of algebra, it is known that an nth-degree equation has n complex roots.

(i) For $n = 2$, $z^2 - 1 = 0 \Rightarrow (z+1)(z-1) = 0 \Rightarrow z = \pm 1$. So, the square roots of unity are: $1, -1$

(ii) For $n = 3$, $z^3 - 1 = 0 \Rightarrow (z-1)(z^2+z+1) = 0 \Rightarrow z = 1, \dfrac{-1+\sqrt{3}i}{2}, \dfrac{-1-\sqrt{3}i}{2}$. So, the cube roots of unity are: $1, \dfrac{-1+\sqrt{3}i}{2}, \dfrac{-1-\sqrt{3}i}{2}$

(iii) For $n = 4$, $z^4 - 1 = 0 \Rightarrow (z^2-1)(z^2+1) = 0 \Rightarrow (z-1)(z+1)(z^2+1) = 0 \Rightarrow z = 1, -1, i, -i$. So, the 4th roots of unity are: $1, -1, i, -i$

We plot these roots on the complex plane like this:



the square root of unity	the cubic root of unity	the fourth root of unity

These roots seem to be evenly distributed on the unit circle, which is indeed the case.

We return to the **nth root of unity**.

Suppose n is a positive integer, we are going to find the roots of the equation $z^n - 1 = 0$.

Assume that the complex number $z = r(\cos\theta + i\sin\theta)$ is a **nth root of unity**. Then, by de Moivre's theorem, we have: $z^n = r^n(\cos n\theta + i\sin n\theta)$. Since $z^n - 1 = 0$, we know:

$$r^n(\cos n\theta + i\sin n\theta) = 1$$
$$= 1(\cos 0 + i\sin 0).$$

We get:

$$\begin{cases} r^n = 1 \\ n\theta = 0 + 2k\pi, k \in \mathbb{Z} \end{cases}$$

$$\Rightarrow \begin{cases} r = 1 \\ \theta = \dfrac{2k\pi}{n}, k \in \mathbb{Z}. \end{cases}$$

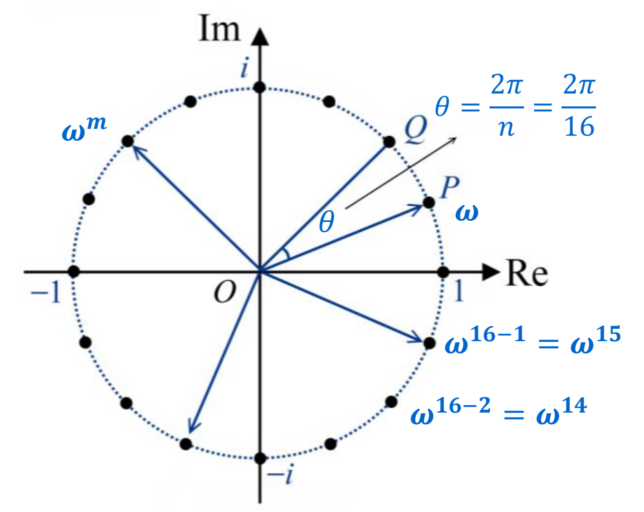
Now, the values $k = 0,1,2,\cdots,n-1$ give distinct values of θ and for any other value of k, we can add or subtract an integer multiple of n to reduce to one of these values of θ.

Therefore, the nth roots of unity are the complex numbers:

$$\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, k = 0,1,2,\cdots,n-1$$

Since all the complex roots of unity have absolute value 1, these points lie on the unit circle. Furthermore, since the angle between any two consecutive roots is $\frac{2\pi}{n}$, the complex roots of unity are evenly spaced around the unit circle.

If $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, then these n roots can also be written as: $1, \omega, \omega^2, \omega^3, \cdots, \omega^{n-1}$ and these roots form the vertices of a regular n-gon with the origin $O(0,0)$ as the center on the complex plane.



16th roots of unity

Next, we discuss *n*th roots of complex numbers.

Nth roots of complex numbers

In general, any non-zero complex number a also has *n* *n*th roots. We can find these roots in the same way.

Assume that complex number $z = r(\cos\theta + i\sin\theta)$ is an *n*th root of a. In other words, z is the root of an equation $z^n = a$. Let the polar form of a be $a = |a|(\cos\phi + i\sin\phi)$, we get:

$$r^n(\cos n\theta + i\sin n\theta) = |a|(\cos\phi + i\sin\phi)$$

Then:

$$\begin{cases} r^n = |a| \\ n\theta = \phi + 2k\pi, k \in \mathbb{Z} \end{cases}$$

$$\Rightarrow \begin{cases} r = \sqrt[n]{|a|} \\ \theta = \dfrac{\phi + 2k\pi}{n}, k \in \mathbb{Z} \end{cases}$$

We choose: $k = 0, 1, 2, \cdots, n-1$

Therefore, the roots $z_0, z_1, \cdots, z_{n-1}$ of the equation $z^n = a$ can be expressed in the following form:

$$z_k = \sqrt[n]{|a|}\left(\cos\frac{\phi + 2k\pi}{n} + i\sin\frac{\phi + 2k\pi}{n}\right), k = 0, 1, 2, \cdots, n-1$$

Let's plot all the roots of $z^n = a$ on a complex plane. Since the angle between any two consecutive roots is $\frac{2\pi}{n}$,the roots are evenly spaced around the circle with its radius. In other words, as long as the first root z_0 is determined, $z_0, z_1, \cdots, z_{n-1}$ form a regular n-sided polygon with the origin $O(0,0)$ as the center and $z_0, z_1, \cdots, z_{n-1}$ as the vertices.

運算問題的講解

例題一

說明：將複數化為極式。

（英文）Write the following complex numbers in polar form:

(i) $1+\sqrt{3}i$

(ii) $-2i$

（中文）將下列複數化為極式：

(i) $1+\sqrt{3}i$

(ii) $-2i$

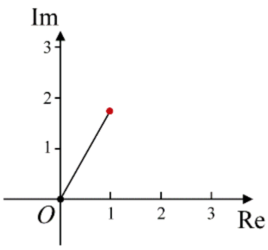
Teacher: According to the definition of the polar form, we need to find two things now. What two things do we need to get?

Student: The modulus and the argument.

Teacher: Yes. What is the modulus of $1+\sqrt{3}i$?

Student: The modulus of $1+\sqrt{3}i$ is: $\left|1+\sqrt{3}i\right|=\sqrt{1^2+\sqrt{3}^2}=2$

Teacher: Good. Draw $1+\sqrt{3}i$ on the complex plane. Since $1+\sqrt{3}i$ lies in the first quadrant, what is the argument of $1+\sqrt{3}i$?



Student: From the relationship between sides and angles of a 30-60-90 right triangle, I get the argument of $1+\sqrt{3}i$ is 60 degrees.

Teacher: Good job. That is, $\arg\left(1+\sqrt{3}i\right)=\frac{\pi}{3}$ radians which is also the principal argument of $1+\sqrt{3}i$. From the discussion, please write the polar form of $1+\sqrt{3}i$.

Student: Ok. $1+\sqrt{3}i=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

Teacher: Great. Now let's follow the same method to address the second example. What is the modulus of $-2i$?

Student: The modulus of $-2i$ is 2.

Teacher: Yes. What is the argument of $-2i$?

Student: Since $-2i$ lies to the left of the origin on the x-axis, the principal argument of $-2i$ is π. So, I get the polar form of $-2i$ is $2(\cos\pi + i\sin\pi)$.

Teacher: Excellent!

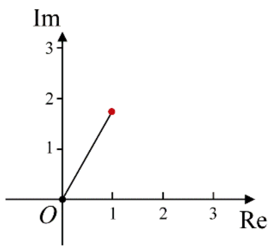
老師：根據複數極式的定義，我們現在需要找到兩件事。我們需要得到哪兩件事？

學生：模和幅角。

老師：是的。$1+\sqrt{3}i$ 的模為何？

學生：$1+\sqrt{3}i$ 的模為 $|1+\sqrt{3}i| = \sqrt{1^2 + \sqrt{3}^2} = 2$。

老師：很好。將 $1+\sqrt{3}i$ 畫在複數平面上。因為 $1+\sqrt{3}i$ 在第一象限，$1+\sqrt{3}i$ 的幅角為何？



學生：根據 30-60-90 直角三角形的邊角的關係，我得到 $1+\sqrt{3}i$ 的幅角是 60 度。

老師：做得好。換句話說，$\arg(1+\sqrt{3}i) = \frac{\pi}{3}$ 弧度，同時也是其主幅角。從上述討論，請寫出 $1+\sqrt{3}i$ 的極式。

學生：好的。$1+\sqrt{3}i = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

老師：很好。現在我們按照同樣的方法來寫第二個範例。$-2i$ 的模是多少？

學生：$-2i$ 的模是 2。

老師：是的。$-2i$ 的幅角為何？

學生：由於 $-2i$ 位於 x 軸原點的左側，因此 $-2i$ 的主幅角是 π。所以我得到了 $-2i$ 的極坐標為 $2(\cos\pi + i\sin\pi)$。

老師：太棒了。

例題二

說明：複數極式的乘除法。

（英文）Let $z_1=2\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$ and $z_2=3\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$, find:

(i) z_1z_2

(ii) $\frac{z_1}{z_2}$

（中文）令 $z_1=2\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$ 和 $z_2=3\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$，試求下列各式之值：

(i) z_1z_2

(ii) $\frac{z_1}{z_2}$

Teacher: Recall that multiplying complex numbers in polar form means multiplying their moduli and adding their arguments. What is the product of z_1 and z_2?

Student: The product z_1z_2 is:

$$z_1z_2=2\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)\cdot 3\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=6\left(\cos\left(\frac{\pi}{12}+\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{12}+\frac{\pi}{4}\right)\right)$$

$$=6\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=3+3\sqrt{3}i$$

Teacher: Very good. And what is the division of $\frac{z_1}{z_2}$?

Student: The division of $\frac{z_1}{z_2}$ is:

$$\frac{z_1}{z_2}=\frac{2\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)}{3\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)}=\frac{2}{3}\left(\cos\left(\frac{\pi}{12}-\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{12}-\frac{\pi}{4}\right)\right)$$

$$=\frac{2}{3}\left(\cos\frac{-\pi}{6}+i\sin\frac{-\pi}{6}\right)=\frac{2}{3}\left(\frac{\sqrt{3}}{2}+i\left(-\frac{1}{2}\right)\right)=\frac{\sqrt{3}}{3}-\frac{1}{3}i$$

Teacher: Excellent!

老師：回想一下，要將複數以極式相乘，需要乘以它們的模並將它們的幅角相加。
z_1 和 z_2 的乘積是什麼呢？

學生：z_1, z_2 的乘積為

$$z_1z_2=2\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)\cdot 3\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=6\left(\cos\left(\frac{\pi}{12}+\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{12}+\frac{\pi}{4}\right)\right)$$

$$=6\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=3+3\sqrt{3}i$$ 。

老師： 很好。那 $\frac{z_1}{z_2}$ 是什麼呢？

學生： $$\frac{z_1}{z_2}=\frac{2\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)}{3\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)}=\frac{2}{3}\left(\cos\left(\frac{\pi}{12}-\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{12}-\frac{\pi}{4}\right)\right)$$

$$=\frac{2}{3}\left(\cos\frac{-\pi}{6}+i\sin\frac{-\pi}{6}\right)=\frac{2}{3}\left(\frac{\sqrt{3}}{2}+i\left(-\frac{1}{2}\right)\right)=\frac{\sqrt{3}}{3}-\frac{1}{3}i$$

老師： 太棒了。

ꕤ 應用問題 / 學測指考題 ꕤ

例題一

說明：複數的極式、棣美弗定理及三角函數綜合應用。

（英文）On the complex plane, the complex number z is in the first quadrant and satisfies $|z|=1$ and $\left|\frac{-3+4i}{5}-z^3\right|=\left|\frac{-3+4i}{5}-z\right|$, where $i=\sqrt{-1}$. If the real part of z is a and the imaginary part is b, then $a=$______，$b=$______.

（中文）在複數平面上，複數 z 在第一象限且滿足 $|z|=1$ 以及 $\left|\frac{-3+4i}{5}-z^3\right|=\left|\frac{-3+4i}{5}-z\right|$，其中 $i=\sqrt{-1}$。若 z 的實部為 a、虛部為 b，則 $a=$______，$b=$______。

（111 分科數甲）

Teacher: Since $|z|=1$, what can we assume z is?

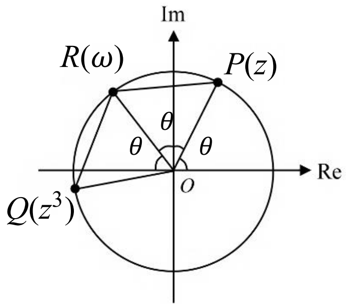
Student: We can assume $z=\cos\theta+i\sin\theta$, and $z^3=\cos 3\theta+i\sin 3\theta$ by de Moivre's theorem.

Teacher: Good job. We observe the complex numbers $\frac{-3+4i}{5}$, z, and z^3. What is the common character among the three numbers?

Student: If we graph them on the complex plane, they are all on the same unit circle, where the center of this circle is the origin.

Teacher: Very good. If we let $\omega=\frac{-3+4i}{5}$, we know that the distance between ω and z^3 equals the distance between ω and z because: $\left|\frac{-3+4i}{5}-z^3\right|=\left|\frac{-3+4i}{5}-z\right|$

Let $P(z), Q(z^3), R(\omega)$ and draw these three points on the complex plane as follows:



Then what is the principal argument of ω?

Student: It is clear that the principal argument of ω is 2θ.

Teacher: Good. That is, $\omega=\cos 2\theta+i\sin 2\theta$. We get:

$$\begin{cases}\cos 2\theta=-\dfrac{3}{5}\\ \sin 2\theta=\dfrac{4}{5}\end{cases}$$

How do we get z?

Student: From the half angle formula and z is in the first quadrant, we get:

$$\cos\theta=\sqrt{\frac{1+\cos 2\theta}{2}}$$

$$=\sqrt{\frac{1+\left(\frac{-3}{5}\right)}{2}}=\frac{\sqrt{5}}{5}\text{, and: }\sin\theta=\sqrt{\frac{1-\cos 2\theta}{2}}=\sqrt{\frac{1-\left(-\frac{3}{5}\right)}{2}}=\frac{2\sqrt{5}}{5}$$

Teacher: Bravo!

老師： 既然 $|z|=1$，我們可以假設 z 是什麼？

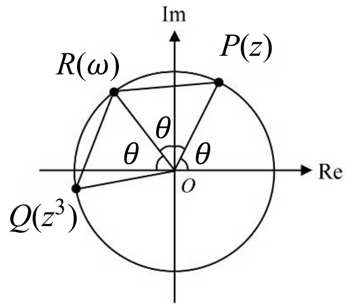
學生： 我們可以假設 $z=\cos\theta+i\sin\theta$，並且根據棣美弗定理，$z^3=\cos 3\theta+i\sin 3\theta$。

老師： 做得好。我們觀察複數 $\dfrac{-3+4i}{5}$、z 和 z^3。這三個數字的共同特徵是什麼？

學生： 如果我們將它們繪製在複數平面上，它們都在同一個單位圓上，其中圓心是原點。

老師： 非常好。如果我們令 $\omega=\dfrac{-3+4i}{5}$，因為 $\left|\dfrac{-3+4i}{5}-z^3\right|=\left|\dfrac{-3+4i}{5}-z\right|$，我們知道 ω 和 z^3 之間的距離等於 ω 和 z 之間的距離。

設 $P(z),Q(z^3),R(\omega)$，我們在複數平面上繪製這三個點，如下所示：



那麼 ω 的主幅角是什麼？

學生： 顯然的，ω 的主幅角是 2θ。

老師： 很好。也就是說，$\omega=\cos 2\theta+i\sin 2\theta$。可得

$$\begin{cases}\cos 2\theta=-\dfrac{3}{5}\\ \sin 2\theta=\dfrac{4}{5}\end{cases}$$

z 要如何得到呢？

學生： 由半角公式且 z 在第一象限，我們得到

$$\cos\theta=\sqrt{\frac{1+\cos 2\theta}{2}}=\sqrt{\frac{1+\left(\frac{-3}{5}\right)}{2}}=\frac{\sqrt{5}}{5}$$，and

$$\sin\theta=\sqrt{\frac{1-\cos 2\theta}{2}}=\sqrt{\frac{1-\left(-\frac{3}{5}\right)}{2}}=\frac{2\sqrt{5}}{5}$$。

老師： 太棒了！

例題二

說明：複數的乘除法及棣美弗定理的應用。

（英文）On the complex plane, let $\bar{z}$ represent the complex conjugate of the complex number z, and $i=\sqrt{-1}$. Try to choose the correct options. (Multiple choices)

(1) If $z=2i$, then: $z^3=4i\bar{z}$

(2) If a nonzero complex number α satisfies $\alpha^3=4i\bar{\alpha}$, then $|\alpha|=2$

(3) If a nonzero complex number α satisfies $\alpha^3=4i\bar{\alpha}$ and let $\beta=i\alpha$, then $\beta^3=4i\bar{\beta}$

(4) Among all non-zero complex numbers z satisfying $z^3=4i\bar{z}$, the smallest possible principal argument is: $\frac{\pi}{6}$

(5) There are exactly 3 different non-zero complex numbers z such that: $z^3=4i\bar{z}$

（中文）複數平面上，設 $\bar{z}$ 代表複數 z 的共軛複數，且 $i=\sqrt{-1}$。試選出正確的選項。（多選）

(1) 若 $z=2i$，則 $z^3=4i\bar{z}$

(2) 若非零複數 α 滿足 $\alpha^3=4i\bar{\alpha}$，則 $|\alpha|=2$

(3) 若非零複數 α 滿足 $\alpha^3=4i\bar{\alpha}$ 且令 $\beta=i\alpha$，則 $\beta^3=4i\bar{\beta}$

(4) 滿足 $z^3=4i\bar{z}$ 的所有非零複數 z 中，其主輻角的最小可能值為 $\frac{\pi}{6}$

(5) 恰有 3 個相異非零複數 z 滿足 $z^3=4i\bar{z}$。

（112 分科數甲）

Teacher: If $z=2i$, then what is z^3?

Student: $z^3=(2i)^3=-8i$

Teacher: Simplify $4i\bar{z}$

Student: $4i\bar{z}=4i\cdot(-2i)=8$

Teacher: Is $z^3=4i\bar{z}$?

Student: No. The first option is false.

Teacher: Good. Look at the second option. If $\alpha^3 = 4i\overline{\alpha}$, what is $|\alpha|$? To give you a hint, add absolute values to the left and right sides of the equal sign.

Student: Ok. I get:

$$|\alpha^3| = |4i\overline{\alpha}| \Rightarrow |\alpha|^3 = |4i||\overline{\alpha}| \Rightarrow |\alpha|^3 = 4|\overline{\alpha}|$$. Since $|\alpha| = |\overline{\alpha}|$, $|\alpha|^3 = 4|\alpha| \Rightarrow |\alpha|^2 = 4$

So $|\alpha| = 2$. The second option is true.

Teacher: Good job. Next, look at the third option. If $\beta = i\alpha$, what is $\overline{\beta}$?

Student: $\overline{\beta} = \overline{i\alpha} = -i\overline{\alpha}$

Teacher: Express β^3 by α

Student: $\beta^3 = (i\alpha)^3 = -i\alpha^3$

Teacher: Since $\alpha^3 = 4i\overline{\alpha}$, then: $\beta^3 = -i\alpha^3 = -i(4i\overline{\alpha}) = 4\overline{\alpha}$. Express $4i\overline{\beta}$ by α.

Student: $4i\overline{\beta} = 4i(-i) = 4\overline{\alpha}$. So, the third option is true.

Teacher: As for the fourth option, to solve $z^3 = 4i\overline{z}$, we let $z = 2(\cos\theta + i\sin\theta)$. So, what do we get?

Student: $z^3 = 4i\overline{z} \Rightarrow \left(2(\cos\theta + i\sin\theta)\right)^3 = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\left(2(\cos\theta - i\sin\theta)\right)$

$$\Rightarrow 8(\cos 3\theta + i\sin 3\theta) = 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\left(\cos(-\theta) + i\sin(-\theta)\right)$$

$$\Rightarrow \cos 3\theta + i\sin 3\theta = \cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)$$

Teacher: So 3θ and $\frac{\pi}{2} - \theta$ are coterminal angles. Then $3\theta = \frac{\pi}{2} - \theta + 2k\pi$, $k \in \mathbb{Z}$ implies $\theta = \frac{\pi}{8} + \frac{k}{2}\pi$, $k \in \mathbb{Z}$. So, what is the smallest possible principal argument?

Student: It is $\frac{\pi}{8}$ when $k = 0$. So, the fourth option is false.

Teacher: Very good. For the last option, we can use the result of the fourth option. What are the solutions that satisfy $z^3 = 4i\overline{z}$?

Student: The solutions that satisfy $z^3 = 4i\bar{z}$ are: $z = 2\left(\cos\left(\frac{\pi}{8}+\frac{k}{2}\pi\right)+i\sin\left(\frac{\pi}{8}+\frac{k}{2}\pi\right)\right)$, $k = 0,1,2,3$. There are four solutions. So, the fifth option is false.

Teacher: Excellent! So, we choose (2)(3).

老師： 如果 $z = 2i$，那麼 z^3 是什麼？

學生： $z^3 = (2i)^3 = -8i$。

老師： 化簡 $4i\bar{z}$。

學生： $4i\bar{z} = 4i\cdot(-2i) = 8$。

老師： $z^3 = 4i\bar{z}$ 嗎?

學生： 不是，第一個選項是錯誤的。

老師： 好的。我們看第二個選項。如果 $\alpha^3 = 4i\bar{\alpha}$，$|\alpha|$ 是什麼？給你一個提示，在等號的左側和右側加上絕對值。

學生： 好的。得到 $|\alpha^3| = |4i\bar{\alpha}| \Rightarrow |\alpha|^3 = |4i||\bar{\alpha}| \Rightarrow |\alpha|^3 = 4|\bar{\alpha}|$。因為 $|\alpha| = |\bar{\alpha}|$，則 $|\alpha|^3 = 4|\alpha| \Rightarrow |\alpha|^2 = 4$。所以 $|\alpha| = 2$。第二個選項是正確的。

老師： 做得好。接下來我們看第三個選項。如果 $\beta = i\alpha$，$\bar{\beta}$ 是什麼？

學生： $\bar{\beta} = \overline{i\alpha} = -i\bar{\alpha}$。

老師： 既然 $\alpha^3 = 4i\bar{\alpha}$，則 $\beta^3 = -i\alpha^3 = -i(4i\bar{\alpha}) = 4\bar{\alpha}$。用 α 來表示 $4i\bar{\beta}$。

學生： $4i\bar{\beta} = 4i(-i) = 4\bar{\alpha}$。所以第三個選項是正確的。

老師： 對於第四個選項，為了解 $z^3 = 4i\bar{z}$，我們令 $z = 2(\cos\theta + i\sin\theta)$。那我們能得到什麼呢？

學生： $z^3 = 4i\bar{z} \Rightarrow \left(2(\cos\theta + i\sin\theta)\right)^3 = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\left(2(\cos\theta - i\sin\theta)\right)$

$\Rightarrow 8(\cos 3\theta + i\sin 3\theta) = 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\left(\cos(-\theta) + i\sin(-\theta)\right)$

$\Rightarrow \cos 3\theta + i\sin 3\theta = \cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)$。

老師： 所以 3θ 和 $\frac{\pi}{2} - \theta$ 是同界角。然後 $3\theta = \frac{\pi}{2} - \theta + 2k\pi$，$k \in \mathbb{Z}$ 推得 $\theta = \frac{\pi}{8} + \frac{k}{2}\pi$，$k \in \mathbb{Z}$。那麼最小可能的主幅角是什麼？

學生： 當 $k=0$ 時，是 $\frac{\pi}{8}$ 。所以第四個選項是錯的。

老師： 非常好。對於最後一個選項，我們可以使用第四個選項的結果。滿足 $z^3=4i\bar{z}$ 的解是什麼？

學生： 滿足 $z^3=4i\bar{z}$ 的解是 $z=2\left(\cos\left(\frac{\pi}{8}+\frac{k}{2}\pi\right)+i\sin\left(\frac{\pi}{8}+\frac{k}{2}\pi\right)\right)$，$k=0,1,2,3$。有四個解。所以第五個選項是錯的。

老師： 太棒了。所以選(2)(3)。

單元七　離散型隨機變數 Discrete Random Variables

臺北市中正高中　鄧宇凱老師

■ 前言 Introduction

本單元將介紹離散型隨機變數，進而定義機率質量函數，且由隨機變數的觀點，定義期望值、標準差與變異數，並了解其意涵。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
trial	試驗	**random phenomenon**	隨機現象
discrete	離散型	**random variable**	隨機變數
event	事件	**sample space**	樣本空間
probability mass function	機率質量函數	**probability distribution**	機率分布
probability histogram	機率質量函數圖	**expected value**	期望值
variance	變異數	**standard deviation**	標準差

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ _______ is/are classified as _______.

例句：The unpredictable outcomes of each trial can **be classified as** random phenomena.

每次試驗的結果無法預知，可歸類為隨機現象。

❷ _______ correspond to _______.

例句：Each sample point in the sample space **corresponds to** a real number.

樣本空間中每一個樣本點對應到一個實數。

❸ _______ list _______ one by one.

例句：Let's **list** the possible values **one by one.**

我們把可能的值一一列出來。

❹ _________ stand for________.

例句：The height of the vertical line **stands for** the probability value.

鉛直線的高度代表機率值。

■ 問題講解 Explanation of Problems

說明

[Discrete random variables]

Flip a fair coin three times successively to observe which side comes up. The sample space is:

$S=\{(+,+,+),(+,+,-),(+,-,+),(-,+,+),(+,-,-),(-,+,-),(-,-,+),(-,-,-)\}$. Let $\boldsymbol{X}$ be the number of heads that appear in the three flips. $(+,+,+)$ corresponds to $X=3$. $(+,+,-),(+,-,+)$, and $(-,+,+)$ correspond to $X=2$. $(+,-,-),(-,+,-)$, and $(-,-,+)$ correspond to $X=1$. On the other hand, $(-,-,-)$ corresponds to $X=0$. Each sample point corresponds to a real number, so X is a random variable.

Definition: The meaning of random phenomena
If the outcomes of each trial are unpredictable, this situation can be classified as a random phenomenon.

Definition: Random variables
A random variable is a function that assigns a value to each outcome in a sample space.

Definition: Discrete random variables
A random variable is discrete if all possible values can be listed one by one.

Definition: Probability mass functions
For a random variable X of the discrete type, the probability $P(X=x)$ is denoted by $p(x)$, and this function $p(x)$ is called a probability mass function.

In general, a probability mass function has the following properties:

Assuming that the possible values of the discrete random variable are $x_1, x_2, ..., x_n$, then its probability mass function must satisfy the following conditions:
(1) $p(x_i)>0$, $i=1,2,3,\cdots,n$
(2) $\sum_{i=1}^{n} p(x_i)=p(x_1)+p(x_2)+p(x_3)+\cdots+p(x_n)=1$

Example 1 will show these properties.

[Expected values]

Consider tossing an even coin three times. Record the total number of heads, and then repeat the trial several more times. Let X represent the number of heads from each trial. We know that $X = 0, 1, 2,$ or 3. We can also define a value which can be regarded as the "representative" of X.

If the probability mass function of the discrete random variable X is listed as follows:

x	x_1	x_2	$\cdots$	x_i	$\cdots$	x_n
$p(x)$	$p(x_1)$	$p(x_2)$	$\cdots$	$p(x_i)$	$\cdots$	$p(x_n)$

Then, the expected value of X is:

$$\mu = E(X) = x_1 p(x_1) + x_2 p(x_2) + \ldots + x_n p(x_n) = \sum_{i=1}^{n} x_i p(x_i)$$

[Variance and Standard deviation]

The expected value μ is a representative number of a random variable, and the values of the random variable will be scattered around it. However, the expected value does not tell us whether the data values cluster together around μ or are spread far apart. If there is a larger number, it means that the data values are spread farther apart from μ. Conversely, a smaller number tells us that the data values are concentrated near μ. In this case, this variation, or the standard deviation, can provide information about the spread of X.

The probability mass function of the discrete random variable X is listed as follows:

x	x_1	x_2	$\cdots$	x_i	$\cdots$	x_n
$p(x)$	$p(x_1)$	$p(x_2)$	$\cdots$	$p(x_i)$	$\cdots$	$p(x_n)$

The expected value of X is μ, and the variance of X is denoted as σ^2 or $Var(X)$.

It is defined as: $\sigma^2 = \mathrm{Var}(X) = \sum_{i=1}^{n} (x_i - \mu)^2 p(x_i)$.

The standard deviation is: $\sigma = \sqrt{\sigma^2} = \sqrt{\mathrm{Var}(X)}$.

The variance can also be expressed in another way, and the proof is as follows:

$$\sigma^2 = \mathrm{Var}(X) = \sum_{i=1}^{n}(x_i-\mu)^2 p(x_i) = \sum_{i=1}^{n}(x_i^2 - 2x_i\mu + \mu^2)p(x_i)$$

$$= \sum_{i=1}^{n} x_i^2 p(x_i) - 2\mu\sum_{i=1}^{n} x_i p(x_i) + \mu^2 \sum_{i=1}^{n} p(x_i)$$

$$= \sum_{i=1}^{n} x_i^2 p(x_i) - 2\mu^2 + \mu^2 = \sum_{i=1}^{n} x_i^2 p(x_i) - \mu^2 = E(X^2) - (E(X))^2$$

運算問題的講解

例題一

說明：寫出隨機變數 X 的機率質量函數並用圖形表示。

（英文）Flip a coin three times in a row. Let X be the number of heads that appear in the three flips. Find the probability mass function $p(x)$ and sketch the graph of $p(x)$. The finished graph is known as the "probability histogram."

（中文）連續擲一枚均勻硬幣 3 次，令 X 代表擲出之正面總數。試求 X 之機率質量函數 $p(x)$ 並用圖表示出來 $p(x)$。（機率質量函數圖）

Teacher: As I explained previously, list the possible values of X.

Student: The possible values of X are 0, 1, 2, and 3.

Teacher: What does the probability mass function $p(x)$ stand for?

Student: $p(x)$ stands for the probability of $X = x$.
It is denoted as $P(X = x)$ where $x = 0, 1, 2, 3$.

Teacher: Find the values of $p(0), p(1), p(2)$, and $p(3)$.

Student: There are 8 possible outcomes in these trials. And they are equally likely.
The sample space is:

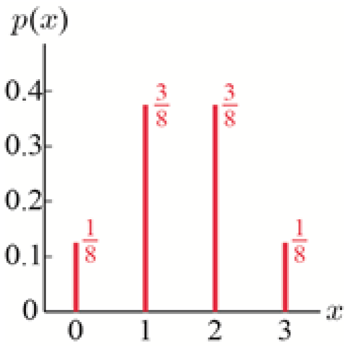
$$S = \{(+,+,+),(+,+,-),(+,-,+),(-,+,+),(+,-,-),(-,+,-),(-,-,+),(-,-,-)\}.$$

It shows that $p(0) = P(X=0) = \frac{1}{8}$, which is the probability of zero heads occurring in the three flips. Based on the sample space, we can get:

$p(1) = P(X=1) = \frac{3}{8}$, $p(2) = P(X=2) = \frac{3}{8}$, and $p(3) = P(X=3) = \frac{1}{8}$.

Teacher: Right! Let's sketch the probability histogram of $p(x)$. First mark the possible values 0, 1, 2, and 3 on the x-axis and then draw vertical lines above 0, 1, 2, and 3. The height of each vertical line should represent the probability value of the corresponding point.

Student: We mark $x=0,1,2,3$ on the x-axis and draw vertical lines above 0, 1, 2, and 3 respectively. The heights of each vertical line are $\frac{1}{8},\frac{3}{8},\frac{3}{8}$, and $\frac{1}{8}$ corresponding to x = 0, 1, 2, and 3, respectively. We have finished drawing the probability histogram of $p(x)$ as below:



老師： 如之前的說明，列出 X 可能的值。

學生： $p(x)$ 可能的值為 0、1、2 和 3。

老師： 那機率質量函數 $p(x)$ 代表為何？

學生： $p(x)$ 代表當 $X=x$ 發生的機率。我們記為 $P(X=x)$，其中 x 等於 0、1、2 和 3。

老師： 試找出 $p(0)$ 、 $p(1)$ 、 $p(2)$ 和 $p(3)$ 的值。

學生： 每次試驗都 8 種可能的結果，他們出現的機會皆均等，所以樣本空間為 $S=\{$(正、正、正)，(正、正、反)，(正、反、正)，(反、正、正)，(正、反、反)，(反、正、反)，(反、反、正)，(反、反、反)$\}$。

$p(0)=P(X=0)=\frac{1}{8}$ 表示連續擲 3 次皆沒出現正面的機率，並由樣本空間得知

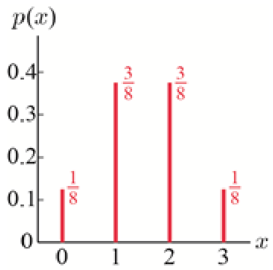
$p(1)=P(X=1)=\frac{3}{8}$， $p(2)=P(X=2)=\frac{3}{8}$ 和 $p(3)=P(X=3)=\frac{1}{8}$ 。

老師： 是的，我們用圖形表示機率質量函數，先畫出 x 軸，把 X 的可能值 0、1、2 和 3 在 x 軸上標示出來，並在可能值上方畫鉛直線，鉛直線的高度代表所對應的機率值，

學生： 我們在 x 軸上標示出來 X 的可能值 0、1、2 和 3，其上所鉛直線的高度分別

$\frac{1}{8}, \frac{3}{8}, \frac{3}{8},$ 和 $\frac{3}{8}$ 為代表該點所對應的機率值，

就完成機率質量函數圖如下：



例題二

說明：求隨機變數 X 的期望值與變異數。

（英文）Roll a fair dice three times in a row, and let X represent the number of outcomes where you roll a "6." Find the expected value and variance of X.

（中文）連續擲一顆公正骰子 3 次，令 X 代表擲出 6 點的次數，試求 X 的期望值及變異數。

Teacher: First, we have to figure out the possible values of X and construct a table of the probability mass function of X.

Student: The possible values of X are 0, 1, 2 and 3. We have to find the possibilities of $P(X=0)$, $P(X=1)$, $P(X=2)$ and $P(X=3)$.

Teacher: These three rolls are independent events, which means they are not affected by previous events. When rolling a dice, the probability of getting a 6 is $\frac{1}{6}$ and the probability of not getting a 6 is $\frac{5}{6}$. Now, what are the values of $P(X=0)$ and $P(X=3)$?

Student: When $X=0$, it means 6 didn't occur in any of the three rolls. And $X=3$ means 6 occurred in all three rolls. Since they are independent events, we can calculate the respective probabilities of $P(X=0)$ and $P(X=3)$ by multiplying the three probabilities corresponding to these three independent events. So, we get:

$P(X=0)=(\frac{5}{6})^3=\frac{125}{216}$ and $P(X=3)=(\frac{1}{6})^3=\frac{1}{216}$.

Teacher: Next, try to find the values of $P(X=1)$ and $P(X=2)$.

Student: When $X=1$, it means that a 6 was thrown in one of the three rolls and the other two outcomes are numbers other than 6. If we present all the cases, they can be indicated as (6, non-6, non-6), (non-6, 6, non-6), and (non-6, non-6, 6). As a result, we know that there are 3 possible cases.

We thus can get $P(X=1)=3\times(\frac{1}{6})\times(\frac{5}{6})^2=\frac{75}{216}$. Similarly, $X=2$ means that two of the three rolls had a 6 and only one outcome is a number other than 6. We can also get: $P(X=2)=3\times(\frac{1}{6})^2\times(\frac{5}{6})=\frac{15}{216}$.

Teacher: Fill in the blanks in the following table. And find the expected value and variance of *X*.

x	0	1	2	3
$p(x)$				

Student:

x	0	1	2	3
$p(x)$	$\frac{125}{216}$	$\frac{75}{216}$	$\frac{15}{216}$	$\frac{1}{216}$

We first complete the table above.

Then we calculate the expected value of *X*.

$$\mu=E(X)=0\times\frac{125}{216}+1\times\frac{75}{216}+2\times\frac{15}{216}+3\times\frac{1}{216}=\frac{1}{2}$$

Finally, according to the definition of variance, the variance of *X* is:

$$\sigma^2=(0-\frac{1}{2})^2\times\frac{125}{216}+(1-\frac{1}{2})^2\times\frac{75}{216}+(2-\frac{1}{2})^2\times\frac{15}{216}+(3-\frac{1}{2})^2\times\frac{1}{216}=\frac{5}{12}$$

老師： 首先，我們先找出 X 可能的值，並完成機率質量函數的列表。

學生： X 可能的值為 0, 1, 2 和 3，我們分別計算所對應的機率 $P(X=0)$、$P(X=1)$、$P(X=2)$ 和 $P(X=3)$。

老師： 明顯可知，連續擲一顆骰子 3 次互為獨立事件，代表不受前次的事件影響。擲一顆公正骰子出現 6 點的機率為 $\frac{1}{6}$，非 6 點的機率為 $\frac{5}{6}$。

$P(X=0)$ 和 $P(X=3)$ 的值為何？

學生： $X=0$，代表當連續擲 3 次，6 點皆沒出現；$X=3$代表當連續擲 3 次，6 點連續出現 3 次，因為他們是獨立事件，我只需將三事件的所對應的機率相乘即可得機率分別為：$P(X=0)=(\frac{5}{6})^3=\frac{125}{216}$ 和 $P(X=3)=(\frac{1}{6})^3=\frac{1}{216}$。

老師： 接著，試找出$P(X=1)$ 和 $P(X=2)$的值。

學生： $X=1$，代表擲一顆骰子 3 次中 6 點出現一次，其他兩次為非 6 點，因此有三種可能的情況。我們可得$P(X=1)$ 的機率 $P(X=1)=3\times(\frac{1}{6})\times(\frac{5}{6})^2=\frac{75}{216}$。

相似的方法，$X=2$，代表擲一顆骰子 3 次中 6 點出現二次，一次出現為非 6 點，我們可得$P(X=2)$ 的機率 $P(X=2)=3\times(\frac{1}{6})^2\times(\frac{5}{6})=\frac{15}{216}$。

老師： 將數值填入下列表格，求 X 的期望值和變異數。

學生： 我們先完成上列表格，再計算出 X 的期望值，

$\mu=E(X)=0\times\frac{125}{216}+1\times\frac{75}{216}+2\times\frac{15}{216}+3\times\frac{1}{216}=\frac{1}{2}$，最後利用變異數的定義求出 X 的變異數，

$\sigma^2=(0-\frac{1}{2})^2\times\frac{125}{216}+(1-\frac{1}{2})^2\times\frac{75}{216}+(2-\frac{1}{2})^2\times\frac{15}{216}+(3-\frac{1}{2})^2\times\frac{1}{216}=\frac{5}{12}$。

應用問題 / 學測指考題

例題一

說明：此問題在了解隨機變數與期望值的定義。

（英文）Let the random variable X represent the point(s) that occur(s) when throwing an unfair dice, and let $P(X=k)$ represent the probability of the random variable $X=k$. The probability distribution of X is as follows: (note that x and y are unknown constants).

We also know that the expected value of X is equal to 3.

(1) Find the values of x and y.

(2) Throw the unfair dice twice. What is the probability that the sum of the outcomes is 3?

（中文）設隨機變數 X 表示投擲一個不公正骰子出現的點數，$P(X=k)$ 表示隨機變數 X 取值為 k 的機率。已知 X 的機率分布如下表：（x、y 為未知常數）

k	1	2	3	4	5	6
$P(X=k)$	x	y	y	x	y	y

又知 X 的期望值等於 3。

(1) 試求 x、y 之值。

(2) 投擲此骰子兩次，試求點數和為 3 的機率。

（105 年指考數乙）

Teacher: First, we already have the sum of $P(X=k)$ equals 1, as shown in the table above. Now, write down the first equation.

Student: After adding up all the possible values of $P(X=k)$, the first equation is as follows: $2x+4y=1$.

Teacher: Second, use the definition of expected value that you already learned. Write down the second equation.

By multiplying each of the possible values $X=k$ by its corresponding probability, and adding up the results, we get the expected value of X. The second equation satisfies $x+2y+3y+4x+5y+6y=3$. It follows that $5x+16y=3$.

Solve these two simultaneous equations for x and y.

Student: The solutions are $x=\frac{1}{3}$ and $y=\frac{1}{12}$.

Teacher: Next, find the probability that the sum is 3 after throwing the dice twice.

Student: There are two possible cases: $(1, 2)$ or $(2, 1)$. The x coordinate stands for the outcome at the first throw. And the y coordinate stands for the outcome at the second throw. So, we get the probability: $\frac{1}{3}\times\frac{1}{12}+\frac{1}{12}\times\frac{1}{3}=\frac{1}{18}$.

老師： 我們之前所說的，下列表格呈現的$P(X=k)$的機率和為 1，因此可得第一個方程式。

學生： 將$P(X=k)$可能的值加總後，第一個方程式為$2x+4y=1$。

老師： 接著根據期望值定義，寫出第二個方程式。

學生： $X=k$乘上它所對應的機率再加總，可得到期望值。
第二個方程式滿足$x+2y+3y+4x+5y+6y=3$，可得$5x+16y=3$。

老師： 解 x 和 y 的聯立方程組。

學生： 其解為$x=\frac{1}{3}$ 和 $y=\frac{1}{12}$。

老師： 繼續求投擲此骰子兩次，點數和為 3 的機率。

學生： 有兩種可能，$(1, 2)$ 和 $(2, 1)$，其中x軸坐標代表擲第一次的點數，y軸坐標代表擲第二次的點數。因此可得機率為 $\frac{1}{3}\times\frac{1}{12}+\frac{1}{12}\times\frac{1}{3}=\frac{1}{18}$。

例題二

說明：這題利用期望值連結日常生活問題。

（英文）In a certain exam, there is a multiple-choice question with only one correct answer. This question has four possible answers: A, B, C, D. If you answer this question correctly, you will get 6 points. If you answer the question incorrectly, x points will be deducted. Assume that a student decides to guess one of the answers randomly. If the student's expected score on this question is 0, what should the value of x be?

（中文）某次考試中，有一試題採單選題，此題有 A、B、C、D 四個選項，其中只有一個答案是正確的，若答對此題，可得 6 分，而答錯倒扣 x 分，假設有考生決定「靠運氣」瞎猜其中一答案，若要讓該考生在此題得分的期望值為 0，則 x 的值應為多少？

Teacher: First, let the random variable X stand for the points you get. We can construct a table according to the description above. In this table, the first row shows the possible values of X and the second row lists each of the probability values corresponding to its respective value of X.

Student: There are two cases: First, if you are lucky enough to answer this question correctly, you will get 6 points with a probability of $P(X=6)=\frac{1}{4}$. On the other hand, if you answer this question incorrectly, x points will be deducted with a probability of $P(X=-x)=\frac{3}{4}$. The table of this is as follows:

X	6	$-x$
P	$\frac{1}{4}$	$\frac{3}{4}$

Teacher: Based on the definition of expected value, let the expected value $E(X)$ equal 0 as described in the question. What equation does $E(X)=0$ yield? And solve for x.

Student: It yields the equation: $E(X)=6\times\frac{1}{4}+(-x)\times\frac{3}{4}=0$. We can find $x=2$.

老師： 我們先讓隨機變數 X 代表所得分數，根據題目所描述，建立表格，在表格中，第一列是 X 可能的數值，第二列為 X 它個別所對應的機率。

學生： 有兩種可能：第一種，如果你很幸運猜對此題，你可得 6 分，其所對應的機率為 $P(X=6)=\frac{1}{4}$；另一種，如果你猜錯此題，將被倒扣 x 分，其所對應的機率為 $P(X=-x)=\frac{3}{4}$。表格如下：

X	6	$-x$
P	$\frac{1}{4}$	$\frac{3}{4}$

老師： 根據期望值的定義和題目所述，讓期望值為 0，可得到何種方程式，並解出 x 得值。

學生： 此方程式為 $E(X)=6\times\frac{1}{4}+(-x)\times\frac{3}{4}=0$，並解出 $x=2$。

單元八　二項分布及其應用
Binomial Distributions and Their Applications

臺北市中正高中　鄧宇凱老師

■ 前言 Introduction

本單元將介紹兩種特定的隨機變數：二項分布與幾何分布。描述其隨機變數的分布狀況、期望值與變異數。並舉例這兩種分布的應用。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
binomial random variable	二項隨機變數	**Bernoulli trial**	伯努力試驗
binomial distribution	二項分配	**hypothesis testing**	假設檢定
geometric distribution	幾何分布	**parameter**	參數

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ **Solving for _______ yields _______.**

例句：**Solving for** y **yields** this equation.

解 y 之後可得此方程式。

❷ **_______ be considered _______.**

例句：A coin **is considered** fair if it always comes up heads or tails with equal probabilities when tossed.

若擲一枚硬幣，出現正反兩面的機率總是相同，則此硬幣可視為公正的硬幣。

❸ **_______ must be true for _______.**

例句：The statement P_k **must be true for** all positive integers n.

敘述 P_k 對每個正整數 n 恆成立。

❹ **__________ be subject to ________.**

例句：The random variable X **is subject to** a binomial distribution.

隨機變數 X 符合二項分布。

■ 問題講解 Explanation of Problems

說明

[Binomial variable and its distribution]

For a random variable X, if all of the following conditions are met:

(1) there are a fixed number of trials;

(2) There are only two mutually exclusive outcomes for each trial, i.e., success or failure;

(3) the probability of success is the same on each trial;

(4) each trial is independent of one another;

then X, the number of success(es), is called a binomial random variable.

Here are two examples of binomial random variables:

(1) number of correct guesses at 30 true-false questions when you randomly guess all answers;

(2) number of winning tickets when you buy 10 lottery tickets of the same kind.

Definition: Binomial distribution

When an identical random trial is repeated several times, each time there are only two possibilities: success or failure. The probability of success is p, and each trial is independent of the others. If X is equal to the number of success(es) in n trials, it is called a binomial random variable, its probability distribution is called a binomial distribution and it is denoted as $X \sim B(n, p)$, where n and p $(0 \le p \le 1)$ are its parameters.

If the above trial is done only once, it is called a Bernoulli trial. For a Bernoulli trial, the possible number of success(es) is either 0 or 1 (if the result is "successful", then $X = 1$, otherwise $X = 0$), we call it a Bernoulli random variable denoted as $X \sim B(1, p)$.

In general, if X is a binomial random variable with parameters n and p, i.e., $X \sim B(n, p)$, the probability of obtaining a success x times out of n trials is $P(X = x) = C_x^n p^x (1-p)^{n-x}$.

Probability mass function of a binomial distribution

If $X \sim B(n, p)$, $0 < p < 1$, then the probability mass function of X is as follows:

$b(x) = C_x^n p^x (1-p)^{n-x}, x = 0, 1, \cdots, n$

Now let's verify that $b(x)$ satisfies the conditions of the probability mass function. First, it is obvious that $b(x)$ must be greater than 0. Next, using the binomial theorem, the second condition can be checked as follows:

$$\sum_{x=0}^{n} b(x) = C_0^n p^0(1-p)^n + C_1^n p^1(1-p)^{n-1} + \ldots + C_n^n p^n(1-p)^0 = (p+(1-p))^n = 1$$

If X is a binomial random variable denoted as $X \sim B(n, p)$, the formulas of the expected value $E(X)$ and the variance $Var(X)$ are as follows:

1. $E(X) = np$
2. $\text{Var}(X) = np(1-p)$

[Geometric variables and their distribution]

For a random variable Y, if all of the following conditions are met:

(1) for each trial, there are only two mutually exclusive outcomes, i.e., success or failure;

(2) the probability of success is the same for each trial;

(3) each trial is independent of one another;

(4) Y is the number of trials needed to get the first success.

Definition: Geometric distribution

Repeat identical Bernoulli trials (the probability of success for each trial is p, $0 < p \le 1$). Suppose that each trial is independent, and let Y be the number of trials needed to get the first success. Then Y is called a geometric random variable, and its probability distribution is called its geometric distribution and denoted as $Y \sim G(p)$, where p $(0 < p \le 1)$ is its parameters. The probability mass function of Y is as follows:

$g(y) = P(Y = y) = (1-p)^{y-1} p, y = 1, 2, 3, \ldots$

If Y is a geometric random variable denoted as $Y \sim G(p)$, the formulas of the expected value $E(Y)$ and the variance $Var(Y)$ are as follows:

1. $E(Y) = \dfrac{1}{p}$
2. $\text{Var}(Y) = \dfrac{1-p}{p^2}$

[Hypothesis testing]

In daily life, when encountering a situation that we subjectively think is unreasonable, do we have an objective way to test the rationality of the situation? For instance, we can give a standard probability in advance to discuss whether a coin is fair. When the number of heads or tails exceeds a specific number and the probability of occurrence is less than this standard probability, the coin is considered unfair. There are four considerations as follows:

(1) Hypothesis: The coin is either fair or unfair.

(2) Trials: Toss a coin several times in a row and count the number of heads.

(3) Criteria: Establish the criteria to represent the probability that the conclusion derived from the outcomes is untrue. We usually hope that this value will not be too large.

(4) Critical region: Set up a critical region. If the trial outcomes fall into the critical region, we conclude that the hypothesis is incorrect.

Example 4 will illustrate the steps of hypothesis testing in detail.

運算問題的講解

例題一

說明：寫出隨機變數 X 的機率質量函數並用圖形表示。

（英文）There are 1 white ball and 2 red balls in a box. You draw a random ball from the box, where each ball has an equal chance of being picked. Record the color of each ball drawn and place it back into the box. Do this 3 times. Let X be the number of times where a red ball was picked in these 3 draws. Find the probability mass function.

（中文）盒子裡裝著 1 顆白球和 2 顆紅球，若從盒子中隨意取出一球，每球被取出之機會相等，記錄顏色後放回，重複執行 3 次，令 X 為 3 次當中紅球出現的次數，試求 X 的機率質量函數。

Teacher: As I explained before, list the possible values of X .

Student: The possible values of X are 0, 1, 2, and 3.

Teacher: Complete the following table. What does the probability mass function $p(x)$ stand for?

x	0	1	2	3
$p(x)$				

Student: $p(x)$ stands for the probability of $X = x$ denoted as $P(X = x)$, where $x = 0, 1, 2, 3$.

Teacher: Because the random ball drawn will be placed back into the box after each trial, the probability of picking a red ball is $\frac{2}{3}$ for any of these three draws. On the other hand, the probability of drawing a white ball is $\frac{1}{3}$. Firstly, find the values of $p(0)$ and $p(3)$.

Student: It is clear that $p(3) = (\frac{2}{3})^3 = \frac{8}{27}$. It means that a red ball was picked three times in a row. Also, $p(0) = (\frac{1}{3})^3 = \frac{1}{27}$ means that the white ball was picked in all the three draws.

Teacher: Next, let $p(2) = P(X = 2)$ represent the probability that a red ball is picked exactly 2 times out of the 3 draws. If a red ball was picked the first 2 times, it is denoted as (R, R, W). The probability of the event (R, R, W) can be obtained with $\frac{2}{3} \times \frac{2}{3} \times \frac{1}{3} = (\frac{2}{3})^2 (\frac{1}{3})$. In the event of (R, W, R), the probability of the event (R, W, R) is $\frac{2}{3} \times \frac{1}{3} \times \frac{2}{3} = (\frac{2}{3})^2 (\frac{1}{3})$. There is still one last possible event (W, R, R), whose probability is also $(\frac{2}{3})^2 (\frac{1}{3})$. We can observe that $C_2^3 = \frac{3!}{2!1!} = 3$ represents the number of permutations of (R, R, W).

Hence, we can get $p(2) = 3 \times (\frac{2}{3})^2 (\frac{1}{3}) = \frac{4}{9}$. By the same token, find the values of $p(1)$ and complete the table above.

Student: $p(1)$ means that the probability that a red ball was picked only one time out of the 3 draws. Now, we know that the number of permutations of (R, W, W) is $C_1^3=\frac{3!}{1!2!}=3$. Therefore, $p(1)=C_1^3(\frac{2}{3})\times(\frac{1}{3})^2=\frac{2}{9}$ and the table has been completed as follows:

x	0	1	2	3
$p(x)$	$\frac{1}{27}$	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{8}{27}$

老師： 如上個章節所說的，先列出 X 的可能值。

學生： X 的可能值為 0, 1, 2, 和 3.。

老師： 完成下列表格，其中機率質量函數 $p(x)$ 代表什麼？

學生： $p(x)=P(X=k)$ 代表 $X=k$ 所對應的機率。

老師： 因為是隨機取球且取後放回，所以每次取到紅球的機率都是 $\frac{2}{3}$ ；取到白球的機率都是 $\frac{1}{3}$ 。先計算 $p(0)$ 和 $p(3)$ 。

學生： 顯而易見， $p(3)$ 代表連續三次都取得紅球，因此 $p(3)=(\frac{2}{3})^3=\frac{8}{27}$ 。同理可得， $p(0)$ 代表連續三次都取得白球，因此 $p(0)=(\frac{1}{3})^3=\frac{1}{27}$ 。

老師： 接著我們考慮 $p(2)=P(X=2)$ 的機率，他代表 3 次當中紅球剛好出現 2 次的機率。若紅球出現在前 2 次，即「紅紅白」的狀況，機率是 $\frac{2}{3}\times\frac{2}{3}\times\frac{1}{3}=(\frac{2}{3})^2(\frac{1}{3})$，若是「紅白紅」的狀況，機率是 $\frac{2}{3}\times\frac{1}{3}\times\frac{2}{3}=(\frac{2}{3})^2(\frac{1}{3})$，最後一種狀況是「白紅紅」，機率是 $\frac{1}{3}\times\frac{2}{3}\times\frac{2}{3}=(\frac{2}{3})^2(\frac{1}{3})$ 。

我們觀察一下就會發現，「2 紅 1 白」的排列順序方法數為 $C_2^3=\frac{3!}{2!1!}=3$。因此，我們可以得到 $p(2)=C_2^3\times(\frac{2}{3})^2(\frac{1}{3})=\frac{4}{9}$ 。利用相同的方式，求 $p(1)$ 並完成上述表格。

學生： $p(1)$ 表示 3 次當中紅球剛好出現 1 次的機率，又已知「1 紅 2 白」的排列順序有方法數為 $C_1^3=\frac{3!}{1!2!}=3$。因此，我們可以得到 $p(1)=C_1^3\times(\frac{2}{3})(\frac{1}{3})^2=\frac{2}{9}$。可完成下列表格。

x	0	1	2	3
$p(x)$	$\frac{1}{27}$	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{8}{27}$

例題二

說明：求二項分布 X 的期望值與變異數。

（英文）Two fair dice are rolled simultaneously for a total of 180 times. Let X represent the number of times of getting a sum that is 7 in those 180 times, and find the expected value and variance of X.

（中文）同時擲兩顆公正骰子 180 次，若令 X 代表 180 次當中點數和為 7 的次數，試求 X 的期望值和變異數。

Teacher: First, let's not forget that X is a binomial random variable with the parameters n and p denoted as $X\sim B(n,p)$. Now figure out the parameters n and p.

Student: We know that $n=180$ because there are 180 independent trials in a row (two fair dice are rolled simultaneously). And each trial's outcome is either success (getting a sum that is 7) or failure (getting a sum that isn't 7). Let p be the probability of getting a sum that is 7 for each trial. There are 6 cases where the sum of the two dice is 7. Now we list the 6 possible cases as follows: $(1,6)$, $(2,5)$, $(3,4)$, $(4,3)$, $(5,2)$, and $(6,1)$. We obtain $p=\frac{6}{36}=\frac{1}{6}$ and $X\sim B(180,\frac{1}{6})$.

Teacher: If we use the definition of expected value to find $E(X)$, it can be expressed in the following summation notation:

$$E(X)=\sum_{k=0}^{180}k\cdot P(X=k)=\sum_{k=0}^{180}k\cdot C_k^{180}(\frac{1}{6})^k(\frac{5}{6})^{180-k}.$$

The expression looks complicated, and it will take a long time to find the answer. To save time, we can use the formulas of binomial distribution to find the expected value $E(X)=np$ and the variance $\mathrm{Var}(X)=np(1-p)$.

Student: Sure, X is a binomial random variable with the parameters $n=180$ and $p=\dfrac{1}{6}$ denoted as $X\sim B(180,\dfrac{1}{6})$. Therefore, we get:

$$E(X)=np=180\times\frac{1}{6}=30 \text{ and } Var(X)=np(1-p)=180\times\frac{1}{6}\times\frac{5}{6}=25$$

老師：先確認 X 有符合二項分配的條件，符號記為 $X\sim B(n,p)$。並找出參數 n 和 p。

學生：很明顯可知，同時擲兩顆公正骰子180次，可得$n=180$而且各次投擲之間互相獨立。每次的結果不是成功（點數和為 7）就是失敗（點數和不為 7），所以 X 符合二項分布的條件。p 代表每次試驗點數和為 7 的機率，點數和為 7 包括 $(1,6)$、$(2,5)$、$(3,4)$、$(4,3)$、$(5,2)$、$(6,1)$ 共 6 種情況，機率是 $p=\dfrac{6}{36}=\dfrac{1}{6}$，所以 $X\sim B(180,\dfrac{1}{6})$。

老師：如果我們使用期望值的定義，可以將期望值$E(X)$用求和符號表示成

$$E(X)=\sum_{k=0}^{180}k\cdot P(X=k)=\sum_{k=0}^{180}k\cdot C_k^{180}(\frac{1}{6})^k(\frac{5}{6})^{180-k}\text{。}$$

這數學是看起來很複雜，好像需花點時間才能算出答案；為了節省時間，我們可以利用二項分布的期望值 $E(X)=np$ 與變異數 $\mathrm{Var}(X)=np(1-p)$ 的公式算出答案。

學生：是的，因為X的機率分布為參數 (n,p) 的二項分布，以 $X\sim B(180,\dfrac{1}{6})$ 表示。

因此我們可分別求得期望值$E(X)=np=180\times\dfrac{1}{6}=30$與變異數

$$Var(X)=np(1-p)=180\times\frac{1}{6}\times\frac{5}{6}=25\text{。}$$

例題三

說明：求幾何分布 Y 的期望值與變異數。

（英文）Derek is watching cars drive past his window. He knows there's a 25% chance that a passing car will be blue.

(1) Find the probability that the first blue car Derek sees is the fifth car to pass by.

(2) If the first blue car Derek sees is the n-th car to pass by, let Y be the number of n. Find $E(Y)=?$ and $Var(Y)=?$

（中文）德瑞克觀察通過窗戶的汽車，他知道有 25% 的機率汽車的顏色為藍色。

(1) 試求一直到第五輛車，德瑞克才看見藍色汽車通過的機率。

(2) 一直到第 Y 輛車，德瑞克才看見藍色汽車，試求 Y 的期望值和變異數。

Teacher: For question (1), let $p=0.25=\frac{1}{4}$ be the probability that a blue car will pass by, and consider each passing car as a trial. On the other hand, $1-p=0.75=\frac{3}{4}$ means the probability that a non-blue car will pass by. For the first blue car Derek sees to be the fifth car means that none of the first four cars are blue, and the fifth car is blue. Calculate the probability that the first blue car Derek sees is the fifth car.

Student: We know that these five trials are independent. So, we can get the probability of question (1) by calculating the expression: $(1-p)^4\times p=(\frac{3}{4})^4\times\frac{1}{4}=\frac{81}{1024}$.

Teacher: Next, because we know that Y is a geometric random variable with the parameters $p=\frac{1}{4}$ denoted as $Y\sim G(\frac{1}{4})$, we can use the formulas of a geometric distribution: for the expected value $E(Y)=\frac{1}{p}$ and for the variance $\mathrm{Var}(Y)=\frac{1-p}{p^2}$.

Student: Based on the formulas of a geometric distribution, we can plug in $p=\frac{1}{4}$ to get the expected value $E(Y)=\frac{1}{p}=4$ and the $\mathrm{Var}(Y)=\frac{1-p}{p^2}=12$.

老師： 每通過一輛車視為一個試驗，通過是藍色的車的機率為 $p=0.25=\frac{1}{4}$；反之，通過的車不是藍色的機率為 $1-p=0.75=\frac{3}{4}$。一直到第五輛車，德瑞克才看見藍色

汽車通過，代表前四台通過的車輛非藍色，第五台才是藍色的車。計算出一直到第五輛車，德瑞克才看見藍色汽車通過的機率。

學生： 很明顯可知，這有五次試驗且每次試驗之間互相獨立。問題(1) 的答案可得 $(1-p)^4\times p=(\frac{3}{4})^4\times\frac{1}{4}=\frac{81}{1024}$ 。

老師： 我們先確認Y是幾何分布，符號記為$Y\sim G(p)$。其中參數 $p=\frac{1}{4}$。可以利用幾何分布的期望值$E(Y)=\frac{1}{p}$與變異數$\mathrm{Var}(Y)=\frac{1-p}{p^2}$的公式算出答案。

學生： 是的，利用幾何分布期望值與變異數的公式，因此我們可分別求得期望值 $E(Y)=\frac{1}{p}=4$ 與變異數 $\mathrm{Var}(Y)=\frac{1-p}{p^2}=12$ 。

例題四

說明：依據試驗結果判定合理性檢定的假設。

（英文）Flip a coin 20 times in a row. Suppose that the coin rolled is fair. Use a spreadsheet to calculate the respective probabilities that a head is rolled more than k times (k = 0, 1, 2, ..., 20). If the probability that a head is rolled more than k times is less than 0.05, we will conclude that the coin is unfair. Find the minimum value of k.

（中文）現有一枚硬幣，連續投擲此硬幣 20 次，假設此硬幣是公正的情況下，利用 Excel 計算出現正面的次數超過 k 次的機率，若此機率小於 0.05，則當出現正面的次數超過 k 次時，判定此硬幣不公正，試求 k 的最小值？

Teacher: Let X be the number of times of rolling a head in these 20 rolls. From what you have already learned, you know that X is subject to a binomial distribution denoted as $X \sim B(20, \frac{1}{2})$. Note that the coin is supposed to be fair. Now express $P(X \le k)$ by sigma notation.

Student: If $X \sim B(20, \frac{1}{2})$, then $P(X = k) = C_k^{20} (\frac{1}{2})^k (\frac{1}{2})^{20-k} = C_k^{20} (\frac{1}{2})^{20}$.

So, we get: $P(X \le k) = \sum_{n=0}^{k} C_n^{20} (\frac{1}{2})^{20}$ for $k = 0, 1, 2, ..., 20$.

Teacher: We can also use the BINOM.DIST $(k, n, p, 1)$ function in the spreadsheet software to obtain the value of $P(X \le k)$. Hence, the probability that a head is rolled more than k times can be obtained by $P(X > k) = 1 - P(X \le k)$.

Student: We plug $n = 20$ and $p = \frac{1}{2}$ into the BINOM.DIST $(k, n, p, 1)$ function in Excel for $k = 0, 1, 2, ..., 20$ respectively. Now we construct a table of $P(X \le k)$ and $P(X > k)$ for $k = 0, 1, 2, ..., 20$ respectively as below:

X=k	P(X≤k)	P(X>k)
0	0.000000953674316	0.999999046325683
1	0.000020027160645	0.999979972839355
2	0.000201225280762	0.999798774719238
3	0.001288414001465	0.998711585998535
4	0.005908966064453	0.994091033935546
5	0.020694732666016	0.979305267333984
6	0.057659149169922	0.942340850830078
7	0.131587982177734	0.868412017822266
8	0.251722335815430	0.748277664184570
9	0.411901473999023	0.588098526000976
10	0.588098526000976	0.411901473999023
11	0.748277664184570	0.251722335815430
12	0.868412017822266	0.131587982177734
13	0.942340850830078	0.057659149169922
14	0.979305267333984	0.020694732666016
15	0.994091033935546	0.005908966064453
16	0.998711585998535	0.001288414001465
17	0.999798774719238	0.000201225280762
18	0.999979972839355	0.000020027160645
19	0.999999046325683	0.000000953674316
20	1.000000000000000	0.000000000000000

Teacher: The minimum value of k obtained from the table is 14. This question is about hypothesis testing. For this problem, let's figure out the four steps mentioned above. First, what is the hypothesis?

Student: Our hypothesis should be: "The coin rolled is fair."

Teacher: Second, what are the trials?

Student: The trials should be: "Flip a coin 20 times and count the number of times where a head is rolled."

Teacher: Third, what is the criterion to conclude that the hypothesis is untrue?

Student: The criterion should be: "If the coin is fair, the probability at which we falsely conclude that the coin is unfair is less than 0.05."

Teacher: Fourth, what is the critical region?

Student: Based on the table, the critical region should be: "If the number of times of rolling a head exceeds 14, we will reject the hypothesis and then conclude that the coin is unfair."

老師： X 代表擲一枚硬幣 20 次，正面出現的次數。根據上一節所學的主題，我們知

道 X 是一個二次分布，記為 $X \sim B(20,\frac{1}{2})$，試用求和符號表示 $P(X \le k)$。

學生：如果 $X \sim B(20,\frac{1}{2})$，可得 $P(X=k)=C_k^{20}(\frac{1}{2})^k(\frac{1}{2})^{20-k}=C_k^{20}(\frac{1}{2})^{20}$，因此可用求和符號表示 $P(X \le k)=\sum_{n=0}^{k} C_n^{20}(\frac{1}{2})^{20}$，$k$ 可以是 0 到 20 的整數。

老師：我們也可以用 EXCEL 軟體中的函數 BINOM.DIST $(k,n,p,1)$ 功能，計算 $P(X \le k)$ 的機率值。因此可得 $P(X>k)=1-P(X \le k)$，表示出現正面的次數超過 k 次的機率。

學生：$k=0,1,..,20$，從 k 等於 0 到 20，我們分別將 $n=20$ 和 $p=\frac{1}{2}$ 輸入 EXCEL 軟體中的函數 BINOM.DIST $(k,n,p,1)$ 可得 $P(X \le k)$ 的機率值。

可得關於 $P(X \le k)$ 與 $P(X>k)$ 表格如下：

X=k	P(X≤k)	P(X>k)
0	0.000000953674316	0.999999046325683
1	0.000020027160645	0.999979972839355
2	0.000201225280762	0.999798774719238
3	0.001288414001465	0.998711585998535
4	0.005908966064453	0.994091033935546
5	0.020694732666016	0.979305267333984
6	0.057659149169922	0.942340850830078
7	0.131587982177734	0.868412017822266
8	0.251722335815430	0.748277664184570
9	0.411901473999023	0.588098526000976
10	0.588098526000976	0.411901473999023
11	0.748277664184570	0.251722335815430
12	0.868412017822266	0.131587982177734
13	0.942340850830078	0.057659149169922
14	0.979305267333984	0.020694732666016
15	0.994091033935546	0.005908966064453
16	0.998711585998535	0.001288414001465
17	0.999798774719238	0.000201225280762
18	0.999979972839355	0.000020027160645
19	0.999999046325683	0.000000953674316
20	1.000000000000000	0.000000000000000

老師：由此表格得知，k 的最小值為 14。這是關於合理性檢定的題目，討論這個主題，找出上述所提所對應的四個步驟。步驟一，欲檢定的假設為何？

學生：假設為：此硬幣是公正的。

老師：步驟二，欲進行的試驗為何？

學生：試驗為：投擲此硬幣 20 次，觀察其出現正面的次數。

老師：步驟三，訂定假設不成立的標準為何？

學生：　訂定假設不成立為：如果這硬幣是公正的，我們誤判他不是公正的機率要小於 0.05。

老師：　步驟四，拒絕區域為何？

學生：　根上述表格，拒絕區域為：如過出現正面的次數超過 14 次，我們就會否定原有的假設，認定此硬幣不公正。

應用問題 / 學測指考題

例題一

說明：這題利用期望值解決日常生活問題。

（英文）Assume that the probability of a baseball team's players making an error in any inning is equal to p $(0<p<1)$, and each event of error in each inning is independent of all others. Let the random variable X represent the number of innings where an error occurs in any of the 9 innings, and let $p_k = P(X=k)$ represent the probability that there are exactly k innings where an error occurred in those 9 innings.

Given $p_4+p_5=\frac{45}{8}p_6$, find the expected value of the number of innings where an error would occur. (Write your answer in the simplest form.)

（中文）假設某棒球隊在任一局發生失誤的機率都等於 p（其中 $0<p<1$），且各局之間發生失誤與否互相獨立。令隨機變數 X 代表一場比賽 9 局中出現失誤的局數，且令 p_k 代表 9 局中恰有 k 局出現失誤的機率 $P(X=k)$。

已知 $p_4+p_5=\frac{45}{8}p_6$，則該球隊在一場 9 局的比賽中出現失誤局數的期望值為？（化成最簡分數）

（107 年指考數甲）

Teacher: We know that the random variable X is subject to a binomial distribution denoted as $X \sim B(n,p)$. Now, find the parameters n.

Student: The 9 innings can be considered 9 trials in a random experiment and p is the probability that an error occurs in any of the 9 innings, so $n=9$.

Teacher: We can use the formula of binomial distribution $p_k = P(X=k) = C_k^n p^k (1-p)^{n-k}$

to express p_4, p_5, and p_6.

We get $p_4 = C_4^9 p^4(1-p)^{9-4}$, $p_5 = C_5^9 p^5(1-p)^{9-5}$, and $p_6 = C_6^9 p^6(1-p)^{9-6}$. Now, solve the equation $p_4 + p_5 = \frac{45}{8} p_6$ for p.

Student: We have simplified the equation and got the solution $p = \frac{2}{5}$.

Teacher: And how can we find the expected value of X, $E(X) = ?$

Student: We can use the expected value formula of a binomial random variable. If $X \sim B(n, p)$, then $E(X) = np = 9 \times \frac{2}{5} = \frac{18}{5}$.

老師： 如我們所知，隨機變數 X 符合二項分布，記為 $X \sim B(n, p)$，先找出 n 和 p 。

學生： 比賽中的 9 局視為 9 次試驗，每局會發生失誤的的機率皆為 p ，因此 $n = 9$ 。

老師： 我們利用二次分布的公式 $p_k = P(X = k) = C_k^n p^k (1-p)^{n-k}$ 表示 p_4, p_5，和 p_6. 我們可得 $p_4 = C_4^9 p^4(1-p)^{9-4}$， $p_5 = C_5^9 p^5(1-p)^{9-5}$，和 $p_6 = C_6^9 p^6(1-p)^{9-6}$. 解方程式 $p_4 + p_5 = \frac{45}{8} p_6$ 的 p 值。

學生： 我們化解方程式後解得 p 值為 $\frac{2}{5}$

老師： 接著找出 X 的期望值 $E(X)$ 。

學生： 我們利用二次分布的期望值的公式，如果 $X \sim B(n, p)$ ，期望值 $E(X) = np = 9 \times \frac{2}{5} = \frac{18}{5}$

例題二

說明：這題利用期望值解決日常生活問題。

（英文）During the Lunar New Year, the Good Luck Department Store prepared many red envelopes for customers to draw and announced that the event would continue until all the red envelopes were given out. There are 5 lots in the box, only 1 of which is marked with "Good Luck", and each lot has an equal chance of being drawn. After each customer draws a lot from the box and records the result, they put the lot back into the box and draw the next one. Each customer is allowed up to 3 draws. When the lot with "Good Luck" is drawn twice in a row, the customer will stop drawing and receive the prize. We can regard each event where a customer gets, or doesn't get, the prize from these three draws as a **Bernoulli trial**.

Assume that the first customer to get a prize in the event is the Xth customer to draw the lots, and let $E(X)$ represent the expected value of the random variable X. Find $E(X)=$? (Round your answer to the nearest integer.)

（中文）大吉百貨春節期準備許多紅包讓顧客抽籤得紅包，並宣稱活動會一直持續到送出所有的紅包。抽籤的籤筒內有 5 支籤、其中只有 1 支籤有標示「大吉」，且每支籤被抽中的機會均等。每位顧客從籤筒中抽取一支籤記錄後，將籤放回籤筒再抽下一回，最多抽取 3 回。當抽取過程中出現連續兩回抽中「大吉」，則該顧客停止抽籤並得到紅包。我們可將每位顧客抽籤是否得到紅包視為一次伯努力試驗。

設整個活動第一個得到紅包的顧客是第 X 位抽籤的顧客，並以 $E(X)$ 表示隨機變數 X 的期望值，則 $E(X)=$？（四捨五入到整數位）

（111 年分科測驗數甲）

Teacher: Now let's illustrate the possible situations in which a customer gets a red envelope.

Student: There are two cases: First, $(+,+)$ means any of the lots with "Good Luck" was drawn for two consecutive times. Second, $(-,+,+)$ means a lot with "Good Luck" wasn't drawn the first time, but lots with "Good Luck" were drawn in the second and third times.

Teacher: Let's calculate the probability p that a customer gets the prize.

Student: Based on the two cases we mentioned previously, $p=\frac{1}{5}\times\frac{1}{5}+\frac{4}{5}\times\frac{1}{5}\times\frac{1}{5}=\frac{9}{125}$.

Teacher: To obtain the result of $E(X)$, we first have to figure out what kind of a specific random variable X is.

Student: Let p stand for the probability that the trial was successful. All the trials are independent. X stands for the number of trials until the first success occurs. The random variable X is a geometric distribution with the parameter $p=\frac{9}{125}$. According to what we have learned about the formula of the expected value of a geometric distribution, we get: $E(X)=\frac{1}{p}=\frac{125}{9}\approx 14$.

老師： 我們先說明顧客如何才能獲得紅包獎項。

學生： 若顧客獲獎有兩種情形：第一種是顧客連續兩次都抽中有「大吉」的籤，第二種是顧客第一次沒抽中「大吉」的籤，接著第二次、第三次皆抽中有「大吉」的籤。

老師： 計算顧客獲獎的機率 p 。

學生： 根據上述之兩種形況，可得 $p=\frac{1}{5}\times\frac{1}{5}+\frac{4}{5}\times\frac{1}{5}\times\frac{1}{5}=\frac{9}{125}$ 。

老師： 若要求 X 的期望值 $E(X)$，我們些確認 X 符合哪種特定的分配。

學生： p 可以代表每次試驗成功的機率，且每次試驗皆獨立。X 表示出現第一次成功，所需試驗的次數，因此 X 的分叫做參數 $p=\frac{9}{125}$ 的幾何分布。根據之前所學幾何分布的期望值公式，我們可得 $E(X)=\frac{1}{p}=\frac{125}{9}\approx 14$ 。

國內外參考資源 More to Explore

國家教育研究院樂詞網	
查詢學科詞彙 https://terms.naer.edu.tw/search/	
教育雲：教育媒體影音	
為教育部委辦計畫雙語教學影片 https://video.cloud.edu.tw/video/co_search.php?s=%E9%9B%99%E8%AA%9E	
Oak Teacher Hub	
國外教學及影音資源，除了數學領域還有其他科目 https://teachers.thenational.academy/	
CK-12	
國外教學及影音資源，除了數學領域還有自然領域 https://www.ck12.org/student/	
Twinkl	
國外教學及影音資源，除了數學領域還有其他科目，多為小學及學齡前內容 https://www.twinkl.com.tw/	

Khan Academy	可汗學院，有分年級數學教學影片及問題的討論 https://www.khanacademy.org/
Open Textbook (Math)	國外數學開放式教學資源 http://content.nroc.org/DevelopmentalMath.HTML5/Common/toc/toc_en.html
MATH is FUN	國外教學資源，還有數學相關的小遊戲 https://www.mathsisfun.com/index.htm
PhET: Interactive Simulations	國外教學資源，互動式電腦模擬。除了數學領域，還有自然科 https://phet.colorado.edu/
Eddie Woo YouTube Channel	國外數學教學影音 https://www.youtube.com/c/misterwootube

國立臺灣師範大學數學系陳界山教授網站	
國高中數學雙語教學相關教材 https://math.ntnu.edu.tw/~jschen/index.php?menu=Teaching_Worksheets	
2024 年第五屆科學與科普專業英文(ESP)能力大賽	
科學專業英文相關教材，除了數學領域，還有其他領域 https://sites.google.com/view/ntseccompetition/%E5%B0%88%E6%A5%AD%E8%8B%B1%E6%96%87%E5%AD%B8%E7%BF%92%E8%B3%87%E6%BA%90/%E7%9B%B8%E9%97%9C%E6%95%99%E6%9D%90?authuser=0	
Desmos Classroom	
國外教學資源，也有免費繪圖功能 https://teacher.desmos.com/?lang=en	

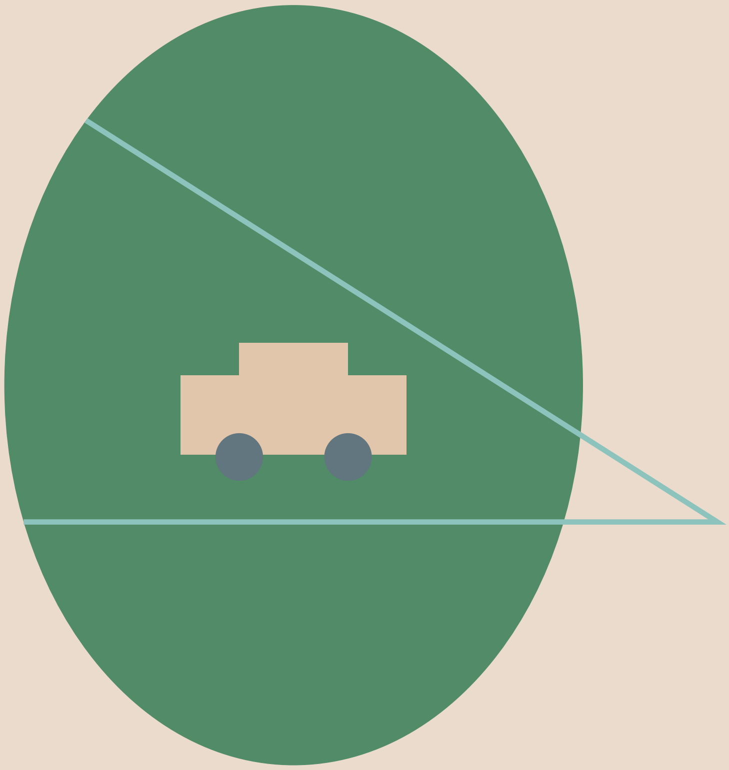
高中數學領域雙語教學資源手冊：英語授課用語

[十二年級下學期]

A Reference Handbook for Senior High School Bilingual Teachers in the Domain of Mathematics: Instructional Language in English

[12th grade 2nd semester]

- 研編單位：國立臺灣師範大學雙語教學研究中心
- 指導單位：教育部師資培育及藝術教育司
- 撰稿：蕭煜修、吳柏萱、林佳葦、鄧宇凱
- 學科諮詢：單維彰、鄭章華
- 語言諮詢：李壹明
- 綜合規劃：王宏均
- 排版：吳依靜
- 封面封底：JUPE Design



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