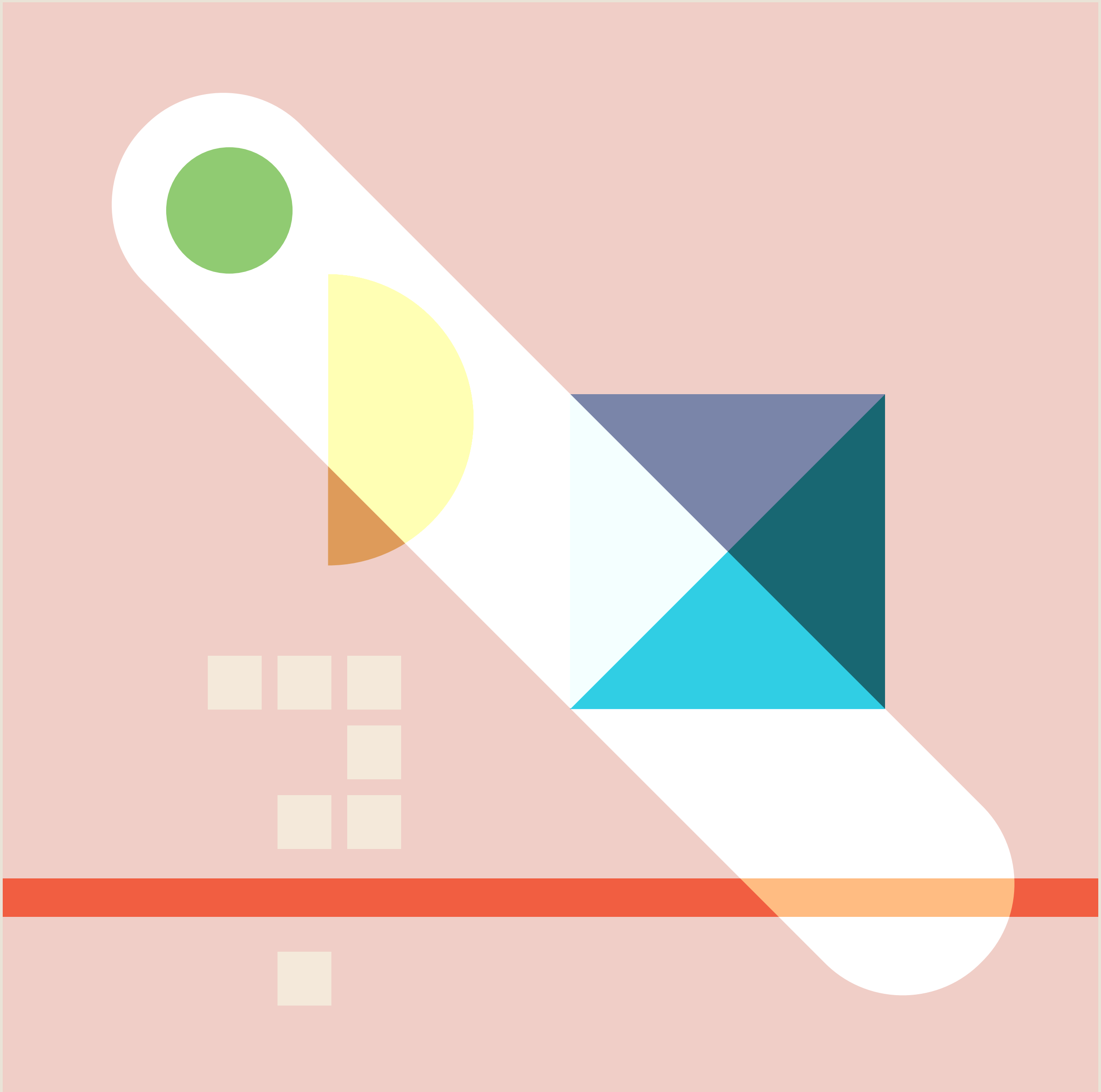
高中數學領域

雙語教學資源手冊 英語授課用語

A Reference Handbook for Senior High School Bilingual Teachers in the Domain of Mathematics: Instructional Language in English

〔 高一上學期 〕



臺師大雙語教學研究中心 發行

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單元一 實數
Real Numbers

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■ 前言 Introduction

在這個數的大家族中，學生已經學過正整數，負整數，小數的英文詞彙，所以在教學前可以跟學生們學複習這些單字。在這節，我們會將更完整的數的家族跟學生介紹，像有理數、無理數來擴充他們這邊的詞彙。我們在備課的時候，也需要多加練習如何用英文將他們的定義說出來，像有理數的定義，或是循環小數的定義。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
positive integer	正整數	density	稠密性
natural number	自然數	irrational number	無理數
integer	整數	proof by contradiction	反證法
origin	原點	law of trichotomy	三一律
rational number	**有理數**	**arithmetic-geometric mean inequality**	**算幾不等式**
repeating decimal	**循環小數**	real number	實數

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ Write________ as a decimal. (Convert ________ to a decimal.)

例句：**Write** $\frac{3}{13}$ **as a decimal.** (**Convert** $\frac{3}{13}$ **to a decimal.**)

試將有理數$\frac{3}{13}$化為小數。

❷ Write the repeating decimal as an irreducible fraction.

例句：**Write** $0.2\overline{35}$ (Read as zero point two three five with the three five repeating.) **as an irreducible fraction.** (Convert $0.2\overline{35}$ to an irreducible fraction.)

試將$0.2\overline{35}$化為最簡分數。

❸ There is always a ____ number between two ____ numbers.

例句(1)：**There is always a** real **number between two** distinct real **numbers.**

在兩相異實數之間必存在一實數。

例句(2)：**There is always a** rational **number between two** rational **numbers.**

在兩相異有理數之間必存在一有理數。

❹ If *a* is _______ and *b* is ______, then *a* + *b* is ________.

例句：If a is rational and b is irrational, then $a+b$ is irrational.

如果a是有理數，b是無理數，則$a+b$是無理數。

❺ Round __________ to the nearest __________.

例句：Use digit-by-digit approximation to **round** $\sqrt{2}$ **to the nearest** hundredth.

用十分逼近法求$\sqrt{2}$的近似值四捨五入到小數點第二位。

❻ Use arithmetic-geometric mean inequality to find the maximum of ________ given that ______________.

例句：**Use arithmetic-geometric mean inequality to find the maximum of** ab **given that** $a+b=6$ and $a \geq 0, b \geq 0$.

利用算幾不等式求出ab的最大值如果$a+b=6$，且$a \geq 0, b \geq 0$。

■ 問題講解 Explanation of Problems

說明

We already learned some properties of integers and decimals in junior high school. In this lesson, we will learn the definition and properties of rational, irrational, and real numbers.
我們在國中有學習了整數，小數的性質及運算的方法，接下來我們要開始認識有理數、無理數及實數是什麼，以及他們的性質。

We define rational numbers as numbers that can be written as the quotients of two integers and irrational numbers as numbers that can't be written as quotients. Then we can use proof by contradiction to show that some numbers, such as $\sqrt{2}$, are irrational. In addition, we use digit-by-digit approximation to approximate the irrational numbers.
我們定義了有理數是那些可以化為兩個整數的商的數，而無理數則不行。接下來我們可以用反證法來證明$\sqrt{2}$是無理數。另外我們能用十分逼近法來找到無理數的近似值。

We also encounter a very useful inequality, the arithmetic-geometric mean inequality, to help us deal with some problems involving getting the maximum value.
最後，我們學到一個有用的算幾不等式來幫助我們解決一些生活中可能會遇到求極值的問題。

運算問題的講解

例題一

說明：如何將循環小數化為分數。

（英文）Write the following repeating decimal $0.\overline{27}$ (read as zero point two seven with the two seven repeating) as an irreducible fraction $\frac{q}{p}$.

（中文）將 $0.\overline{27}$ 化作最簡分數。

Teacher: Before we start to do this question, do you know how to read this repeating decimal, $0.\overline{27}$?

Student: Yes, it is read as zero point two seven with the two seven repeating.

Teacher: Great! Do you think $0.\overline{27}$ is a rational number?

Student: What is a rational number?

Teacher: A rational number is a number that can be written as the quotient of two integers, and of course, the denominator can't be zero.

Student: Oh, I guess, since the question asks us to write that as a fraction, I think it should be a rational number.

Teacher: Nice guess. Now let's try to solve it. At first, assume $x = 0.\overline{27}$, then multiply both sides by 100. What do you find?

Student: The equation becomes $100x = 27.\overline{27}$.

Teacher: Let's place those two equations as follows,

$$x = 0.\overline{27}$$
$$100x = 27.\overline{27}$$

and use the second equation to subtract the first equation. What do we get?

Student: I get $99x = 27$.

Teacher: Then divide both sides by 99.

Student: I got $x = \frac{27}{99}$, which can be simplified to $\frac{3}{11}$.

Teacher: Well done! And do you think $0.\overline{27}$ is a rational number?

Student: Yes, because $0.\overline{27}$ can be written as the quotient of two integers, $0.\overline{27}$ is a rational number. Are all repeating decimals rational?

Teacher: Yes, and you can try to prove it by yourself as a bonus.

Student: Sure, I will definitely figure it out soon.

老師： 在我們開始解這題之前，你知道如何讀 $0.\overline{27}$ 這個循環小數嗎？

學生： 知道，它讀作「零點二七，二七循環」。

老師： 太好了！你認為 $0.\overline{27}$ 是一個有理數嗎？

學生： 什麼是有理數？

老師： 有理數是一個可以寫成兩個整數相除的一個商，當然，分母不能為 0。

學生： 哦，我猜猜看，既然題目說把它寫成一個分數，那它應該是一個有理數。

老師： 很好的猜測。現在讓我們來試著解這個問題。首先，假設$x = 0.\overline{27}$，然後把兩邊都乘以 100。你發現什麼呢？

學生： 算式變成$100x = 27.\overline{27}$。

老師： 我們把這兩個算式排列如下，

$x = 0.\overline{27}$

$100x = 27.\overline{27}$

然後使用第二個算式減去第一個算式。

學生： 得到 $99x = 27$。

老師： 然後兩邊同除以 99。

學生： $x = \frac{27}{99}$，化簡後是 $\frac{3}{11}$。

老師： 做得很好！你覺得 $0.\overline{27}$ 是一個有理數嗎？

學生： 是的，因為 $0.\overline{27}$ 可以寫成兩個整數相除的商，所以$0.\overline{27}$是一個有理數。那是否所有的循環小數都是有理數嗎？

老師： 是的，這些留給你們自己試著證明看看當加分題。

學生： 沒問題，我會盡快將它們解完的。

例題二

說明：是否數線上的數均為有理數呢？我們舉一個在數線上找到的無理數當例子。

（英文）Show that $\sqrt{2}$ is irrational.
（中文）試證明 $\sqrt{2}$ 是無理數。

Teacher: We will use proof by contradiction to show that.

Student: What is "proof by contradiction"?

Teacher: Proof by contradiction is a kind of proof that we assume the conclusion of a statement is false in the beginning. Then we get a result of contradiction after several deductions. Once the contradiction is made, your proof is complete.

Student: It sounds so weird. It looks like our goal is to find a contradiction.

Teacher: Let's see what kind of contradiction we can find here. Assume $\sqrt{2}$ is not irrational.

Student: Do you mean "assume $\sqrt{2}$ is a rational number"?

Teacher: Yes, assume $\sqrt{2}$ is a rational number, $\frac{q}{p}$,where $\frac{q}{p}$ is an irreducible fraction with $p \neq 0$. We can write $\sqrt{2} = \frac{q}{p}$. Cross multiply the equation $\sqrt{2} = \frac{q}{p}$.

Student: I get $p\sqrt{2} = q$. But I didn't see any contradiction.

Teacher: Be patient. Square both sides.

Student: I get $2p^2 = q^2$. So q^2 is a multiple of 2.

Teacher: What kind of number is q, even or odd?

Student: I will say q is even because the square of an odd number is odd.

Teacher: Awesome! Assume $q = 2k$, where k is an integer and substitute q with $2k$ in the equation $2p^2 = q^2$.

Student: Oh, another assumption? I get $2p^2 = (2k)^2 = 4k^2$.
After I divide both sides by 2, and I get $p^2 = 2k^2$.

Teacher: That makes p^2 an even number, too. What kind of number is p, even or odd?

Student: p is even. We have done similar reasoning when we dealt with q.

Teacher: Therefore, both p and q are multiple of 2. Where do you see the contradiction?

Student: It contradicts the fact that $\frac{q}{p}$ is an irreducible fraction. We found the contradiction.

Teacher: Excellent! We also complete this proof. $\sqrt{2}$ is an irrational number. Now show $\sqrt{3}$ is irrational.

Student: It doesn't seem as difficult as before. I think I could prove it on my own now.

Teacher: Wonderful.

老師： 我們要用反證法證明 $\sqrt{2}$ 是無理數。

學生： 反證法是什麼呀？

老師： 反證法是一種證明方式，我們首先假設某個結論是錯誤的，然後經過幾次推導後得到矛盾的結果。當矛盾出現後，就完成反證法了。

學生： 聽起來真奇怪，看起來好像我們的目標是尋找矛盾。

老師： 那我們來看看可以發現什麼矛盾吧。假設 $\sqrt{2}$ 不是一個無理數。

學生： 意思是假設 $\sqrt{2}$ 是一個有理數囉？

老師： 是的，假設 $\sqrt{2}$ 是一個有理數，可被寫成最簡分數的形式 $\frac{q}{p}$，且 $p \neq 0$。我們可以寫成 $\sqrt{2} = \frac{q}{p}$，接著交叉相乘。

學生： $p\sqrt{2} = q$，但我沒發現什麼矛盾的地方。

老師： 耐心點。現在將等號兩邊平方。

學生： $2p^2 = q^2$，所以 q^2 是 2 的倍數。

老師： q 會是偶數還是奇數呢？

學生： q 是偶數，因為奇數的平方也還是奇數。

老師： 很好！現在假設 $q = 2k,$，其中 k 是一個整數，並將其代入算式 $2p^2 = q^2$。

學生： 噢，又是一個假設？
$2p^2 = (2k)^2 = 4k^2$，將等號兩邊同除以 2，得到 $p^2 = 2k^2$。

老師： 這代表 p^2 也是一個偶數。那麼 p 是奇數還是偶數？

學生： p 是偶數，跟 q 是類似的推理。

老師： 因此，p 和 q 都是 2 的倍數，有發現什麼矛盾嗎？

學生： 這違反了 $\frac{q}{p}$ 是最簡分數形式的事實，也就是矛盾的地方。

老師： 太好了！等於我們完成了這個證明：$\sqrt{2}$ 是一個無理數。現在試看看證明 $\sqrt{3}$ 是一個無理數。

學生： 感覺沒有想像中困難，我可以自己完成證明。

老師： 很棒喔！

例題三

說明：此題為試著用十分逼近法求無理數的近似值。

（英文）Use digit-by-digit approximation to round $\sqrt{2}$ to the nearest hundredth.
（中文）用十分逼近法求 $\sqrt{2}$ 的近似值四捨五入到小數點第二位。

Teacher: From the previous example, we already know that $\sqrt{2}$ is an irrational number. Which two integers do you think $\sqrt{2}$ lies between?

Student: Because $1^2 < 2 < 2^2$, we can take the square root of each number in this equality and get $1 < \sqrt{2} < 2$. Therefore, $\sqrt{2}$ lies between 1 and 2.

Teacher: Divide the interval of 1 and 2 into 10 subintervals of equal widths and select the equally spaced numbers, 1.1, 1.2, 1.3, …, and 1.9. Then we square every selected number as follows:

x	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
x^2	1.21	1.44	1.69	1.96	2.25	2.56	2.89	3.24	3.61

Which two decimals do you think $\sqrt{2}$ lies between?

Student: Because $1.4^2 < 2 < 1.5^2$, we can take the square root of each number in this inequality and get $1.4 < \sqrt{2} < 1.5$. Therefore, $\sqrt{2}$ lies between 1.4 and 1.5.

Teacher: The following steps are similar to what we have done earlier. Divide the interval of 1.4 and 1.5 into 10 subintervals of equal widths and select the equally spaced numbers, 1.41, 1.42, 1.43, … and 1.49. Then we square each selected number as follows:

x	1.41	1.42	1.43	1.44	…
x^2	1.9881	2.0164	2.0449	2.0736	…

Actually, we don't need to square each of the 9 numbers because we can draw our conclusion after we check the results of the first two numbers.

Student: Sure, I see that $1.41^2 < 2 < 1.42^2$, so $\sqrt{2}$ lies between 1.41 and 1.42.
The calculation seems to get more and more complicated if we divide the interval of 1.41 and 1.42 into 10 subintervals. Can I just check whether 1.415^2 is larger than 2 or not?

Teacher: Great. That is a good idea. That can save us a lot of time to round $\sqrt{2}$ here.

Student: 1.415^2 is about 2.002, so $\sqrt{2}$ is less than 1.415. The answer to round $\sqrt{2}$ to the nearest hundredth is 1.41.

Teacher: Excellent! You did it.

老師： 我們已經知道$\sqrt{2}$是一個無理數，請問你認為它落在哪兩個整數之間？

學生： 因為$1^2 < 2 < 2^2$，我們可以將不等式兩邊取平方根，得到 $1 < \sqrt{2} < 2$。因此，$\sqrt{2}$落在 1 和 2 之間。

老師： 將 1 和 2 的區間分成 10 等分，並選擇等間距的數字 1.1，1.2，1.3 等，然後對每個選擇的數字算出它的平方數，得到以下結果：

x	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
x^2	1.21	1.44	1.69	1.96	2.25	2.56	2.89	3.24	3.61

$\sqrt{2}$ 落在哪兩個小數之間？

學生： 因為$1.4^2 < 2 < 1.5^2$，我們可以對不等式的數取平方根，得到$1.4 < \sqrt{2} < 1.5$。因此，$\sqrt{2}$ 落在 1.4 和 1.5 之間。

老師： 接下來的步驟與之前所做的類似。將 1.4 和 1.5 的區間等分成 10 等分，並選擇等間距的數字 1.41，1.42，1.43 等，然後對每個選擇的數字算出它的平方數，得到以下結果：

x	1.41	1.42	1.43	1.44	...
x^2	1.9881	2.0164	2.0449	2.0736	...

實際上，我們不需要對這 9 個數字都算出平方數，因為我們可以在檢查前兩個數字的結果後得出結論。

學生： 好的，我觀察到 $1.41^2 < 2 < 1.42^2$，因此$\sqrt{2}$ 落在 1.41 和 1.42 之間。如果將 1.41 和 1.42 的區間再分成 10 等分，計算似乎變得越來越複雜。我們可以只檢查 1.415^2 是否大於 2 來確認嗎？

老師： 很好，這是一個好主意。這樣可以省去四捨五入$\sqrt{2}$的時間。

學生： 1.415^2 約為 2.002，因此$\sqrt{2}$小於 1.415。將$\sqrt{2}$四捨五入到小數點後第二位的答案是 1.41。

老師： 答對了。

應用問題／學測指考題

例題一

說明：這題是取自時下青年會接觸到的電玩遊戲情境，根據預設的條件來為自己的角色獲得最佳的攻擊能力。

（英文）In a video game, you play the role of a wizard/witch. Your ultimate goal is to train your wizard/witch to save the world from evil monsters. The "special ability" to hurt the monsters depends on two skills: intelligence and mentality. At the beginning of the story, you are given 10 magic points to allocate for intelligence and mentality. Each extra point of intelligence for your character will improve 3 units of magical power. Each extra point of mentality will increase 2 units of magic duration.

If the special ability to damage the monsters is the product of magic attack and duration, find a way to allocate your magic points to gain the maximum special ability.

（中文）在某魔法世界電玩中，你要訓練一位巫師（女巫）去拯救世界免於邪惡怪獸的迫害。巫師（女巫）攻擊怪獸的魔法傷害取決於兩個技能，智力及精神力。在遊戲開始時，天神給了你 10 點魔法點數來分配至角色的智力或精神力。每增加一點智力可以提升 3 單位的魔法攻擊力，而每增加一點精神力可以提升 2 單位的魔法持續力。

假設此角色給予怪獸的「魔法傷害」等於「魔法攻擊力×魔法持續力」。試問該如何分配這些魔法點數來獲得最初期角色的最大魔法傷害能力，並求出最高的魔法傷害值。

Teacher: If you have 10 points, how would you allocate those points to your intelligence and mentality?

Student 1: I would place 6 points on intelligence and 4 points on mentality.

Student 2: Maybe mentality is more important to the witch, so I will place 7 points on mentality and 3 points on intelligence.

Teacher: Now calculate your own special ability and compare yours with the results of others.

Student 1: My wizard will have the special ability of 144.

Student 2: Oh, my witch only has 126. So your method is better than mine.

Teacher: Does Student 1 have the best way of allocating points? To solve a question like this, we may use the arithmetic-geometric mean inequality.

Assume I want to increase a units of intelligence and b units of mentality, then the wizard will have $3a$ units increase of magic attack and $2b$ units increase of mentality. So, this allocation will contribute to $3a \times 2b$ which is $6ab$ of the special ability to damage the monsters. Given that $a + b = 10$, we would like to know the maximum of $6ab$.

Student 1: From the arithmetic-geometric mean inequality, we get $\frac{a+b}{2} \geq \sqrt{ab}$

Student 2: Replace $a + b$ by 10; the inequality can be written as $5 \geq \sqrt{ab}$.

Teacher: Now square both sides.

Student: $25 \geq ab$, and multiply both sides by 6, then we get $150 \geq 6ab$.

Student 1: Wow, the maximum of the special ability is 150, which is bigger than my result.

Teacher: We also know from the inequality; the maximum occurs when $a = b$.

Student 1: So when I place 5 points in intelligence and 5 points in mentality for my witch, I will get the most well-prepared witch in this game.

Student 2: Yes, you have found your super witch now.

Teacher: I hope this question also shows you the power of the arithmetic-geometric mean inequality.

老師： 如果你有 10 點，你會如何分配這些點數到智力和精神力上？

學生 1： 我會分配 6 點到智力，4 點到精神力。

學生 2： 也許精神力對於巫師來說比較重要，所以我會分配 7 點給精神力，3 點給智力。

老師： 現在計算你們各自的魔法傷害值，也跟旁邊的同學比較一下。

學生 1： 我的巫師將擁有的魔法傷害值為 144。

學生 2： 噢，我只有 126，所以你的方法比我更好。

老師： 學生 1 的點數分配方式是最佳的嗎？要解這種題目，我們可以使用算幾不等式。

假設我想增加 a 點智力和 b 點精神力，那麼巫師將增加 $3a$ 單位的魔法攻擊力，以及 $2b$ 單位的魔法持續力，而這樣的分配共會得到 $6ab$ 的魔法傷害值。假設 $a + b = 10$, 我們想知道 $6ab$ 的最大值。

學生 1： 先列出算幾不等式 $\frac{a+b}{2} \geq \sqrt{ab}$。

學生 2： 將 $a + b$ 替換為 10，不等式可以寫成 $5 \geq \sqrt{ab}$。

老師： 接著兩邊平方。

學生： $25 \geq ab$，然後將兩邊乘以 6，得到 $150 \geq 6ab$。

學生 1： 哇，魔法傷害值的最大值為 150，比我算出來的還大。

老師： 我們也知道，不等式當 $a = b$ 時，會達到最大值。

學生 1： 所以我的女巫分配 5 點智力和 5 點精神力時，就會是遊戲中最優秀的女巫。

學生 2： 是的，你已經找到了你的超級女巫。

老師： 希望這一題也向你展現了算幾不等式的厲害之處。

單元二 絕對值
Absolute Value

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■ 前言 Introduction

在此節中，雖然絕對值(absolute value)這個詞是第一次學到，但在解絕對值的一次方程式時，還是用到了很多之前在解一元一次方程式會用到的等量加法公理 (add a number to both sides)，及等量減法公理(multiply a number to both sides)的技巧。而在求絕對值不等式的解時，也需提醒學生要考慮「變號」(The inequality changes direction.) 何時會發生。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
absolute value	**絕對值**	**absolute value equation**	**絕對值方程式**
number line	數線	linear equation with one unknown	一元一次方程式
closed interval	閉區間	**absolute value inequalities**	**絕對值不等式**
open interval	開區間		

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ If P is between A and B, and $\overline{PA}:\overline{PB}=$___, find the coordinate of P.

例句：**If P is between A and B, and $\overline{PA}:\overline{PB}=2:5$, find the coordinate of P.**

若 P 點介於 A、B 之間，且 $\overline{PA}:\overline{PB}=2:5$，試求 P 點坐標。

❷ Explain the geometric meaning of _________.

例句(1)：**Explain the geometric meaning of** $|x|=3$. (read as "absolute value of x equals 3")

試說明 $|x|=3$ 的幾何意義。

例句(2)：**Explain the geometric meaning of** $|x-2|$.

試說明 $|x-2|$ 的幾何意義。

❸ Solve the absolute value equation _________.

例句：**Solve the absolute value equation** $|x-2|=5$.

試解下列絕對值方程式，$|x-2|=5$。

❹ Solve the absolute value inequality with, ______.

例句：**Solve the absolute value inequality with,** $|x|\le 2$.

試解下列絕對值不等式，$|x|\le 2$。

❺ The closed interval $[a,b]$ represents all the real numbers x that are between a and b, with a and b included.

例句：**The closed interval** $[0,1]$ **represents all the real numbers x that are between** 0 **and**1, **with** 0 **and** 1 **included**.

閉區間 $[0,1]$ 代表 0 與 1 之間所有的實數，0 與 1 端點這兩個數也包含在內。

■ 問題講解 Explanation of Problems

說明

Here we introduce the geometric meaning of the absolute value and solve an absolute equation with one unknown.
我們在這節介紹了絕對值的幾何意義，及求解含絕對值的一次方程式。

We use interval notation to represent the real numbers lying between two real numbers, and we also use it to represent the solutions to an inequality.
我們用區間符號表示介於兩實數間的所有實數，也用它來表示不等式的解。

Have you ever noticed the numbers on the package you buy? What does $40 \pm 3\text{g}$ mean for the weight of the product? We would interpret it correctly and find a connection with the absolute value we learn here.
在生活中，我們在食品包裝上看到 $a \pm b\%$ 時，我們可以知道此數字代表的意義及它跟絕對值的關係。

運算問題的講解

例題一

說明：如何解含絕對值的一次方程式。

（英文）Solve

(1) $|x-2|=5$　(2) $|x-3|=4$

（中文）試求下列各式的解

(1) $|x-2|=5$　(2) $|x-3|=4$

Teacher: Let's solve $|x-2|=5$ (read as "absolute value of x minus 2 equals 5").
What is the geometric meaning of $|x-2|$?

Student: It is the distance between x and 2 on the number line.

Teacher: Great! From the previous question, what is the geometric meaning of $|x-2|=5$?

Student: It means that the distance between x and 2 on the number line is 5.

Teacher: Can you find a number on a number line that is 5 units away from 2?

Student: Obviously 7 is 5 units away from 2.

Teacher: Yes, let me write $7 = 5 + 2$ on the board. Is there any other number that is also 5 units away from 2?

Student: I can find a number that is 5 units to the left of 2, which is -3.

Teacher: Good job! Let me write $-3 = -5 + 2$ on the board. To solve the equation, we can take out the absolute value sign and rewrite it as $x - 2 = \pm 5$. (positive negative 5) Then, add 2 to both sides. The equation becomes $x = \pm 5 + 2$. Can you see how the equations seem like the ones I wrote on the board?

Student: Yeah, the ones you wrote before, $+5+2$ and $-5+2$ could be combined as $x = \pm 5 + 2$. (positive negative 5 plus 2)

Teacher: To solve the absolute value equation with the form of $|x - a| = b$, you can take out the absolute value and solve $x - a = \pm b$ instead.
Now use this technique to solve $|x - 3| = 4$.

Student: I could take out the absolute value and solve $x - 3 = \pm 4$.
I add 3 to both sides, and I get $x = \pm 4 + 3$. So, $x = 7$ or -1.

Teacher: You did it.

老師： 讓我們來解 $|x-2| = 5$ 這個方程式。$|x-2|$ 的幾何意義是什麼呢？

學生： 它是數線上 x 與 2 之間的距離。

老師： 很好！那題目 $|x-2| = 5$ 的幾何意義是什麼呢？

學生： 代表數線上 x 與 2 之間的距離為 5。

老師： 你能找到一個距離 2 為 5 單位的數字嗎？

學生： 很明顯 7 和 2 相距 5 單位。

老師： 是的，我把$7 = 5 + 2$寫在黑板上。還有其他數字嗎？

學生： 有的，我可以找到一個從 2 往左邊數 5 單位的數字，也就是-3。

老師： 做得好！我也把$-3 = -5 + 2$寫在黑板上。要解方程式，我們可以去掉絕對值符號，重寫成 $x - 2 = \pm 5$。然後，兩邊同時加上 2，讓方程式變成$x = \pm 5 + 2$。你有注意到這個方程式看起來就像我剛在黑板上寫的那些嗎？

學生： 是的，剛才老師寫的 5+2 及$-5+2$ 可以合寫成 $x = \pm 5 + 2$。

老師： 所以，要解形如 $|x-a| = b$ 的絕對值方程式，你可以去掉絕對值變成解 $x - a = \pm b$。現在試著用這種技巧解 $|x-3| = 4$。

學生： 去掉絕對值，解 $x - 3 = \pm 4$。兩邊同時加上 3，得到$x = \pm 4 + 3$。
所以，$x = 7$ 或 -1。

老師： 答對了，做得很好。

例題二

說明：解絕對值不等式。

（英文）Solve

(1) $|x| \leq 1$　　(2) $|x| > 2$

（中文）試解下列絕對值不等式

(1) $|x| \leq 1$　　(2) $|x| > 2$。

Teacher: What is the geometric meaning of the inequality $|x| \leq 1$?

Student: I can't tell. Should there be a minus sign within the absolute value?

Teacher: There is no minus sign, but we can create one.
The inequality $|x| \leq 1$ can be written as $|x - 0| \leq 1$.

Student: That is so tricky but quite clever. Now I can interpret it as solving for all the numbers that are equal to or less than one unit away from 0.

Teacher: Draw a number line and mark 0 on it. Use a closed dot to label one unit to the right of zero and another closed dot to label one unit to the left of zero. The solution to this inequality includes all the numbers between -1 and 1 with -1 and 1 included. That is why we use closed dots to label 1 and -1 on the number line.
We can write it as $-1 \leq x \leq 1$.

Student: I get it now. I could rewrite $|x| > 2$ as $|x - 0| > 2$ and interpret it as solving for all the numbers that are greater than 2 units away from 0.

Teacher: Draw a number line and mark 0 on it. Use an open dot to label two units to the right of zero and another open dot to label two units to the left of zero. The solution to this inequality includes all the numbers either less than -2 or greater than 2 with -2 and 2 not included. That is why we use open dots to label 2 and -2 on the number line. We can write it as $x < -2$ or $x > 2$.

Teacher: In summary, if $a > 0$, then

(i) The solution to $|x| \leq a$ is $-a \leq x \leq a$.

(ii) The solution to $|x| < a$ is $-a < x < a$.

(iii) The solution to $|x| \geq a$ is $x \geq a$ or $x \leq -a$.

(iv) The solution to $|x|>a$ is $x>a$ or $x<-a$.

On the number line, we use a closed dot to show that the endpoint is included as a solution.

We use an open dot to show that the endpoint is not included as a solution.

老師： $|x| \leq 1$的幾何意義是什麼呢？

學生： 我不知道，絕對值裡不是應該要有個減號嗎？

老師： 這題沒有，但我們可以加入一個。不等式$|x| \leq 1$可以寫成$|x-0| \leq 1$。

學生： 這方法滿難想到的，但是很巧妙。我現在可以理解它的意義就是找出所有距離 0 不超過 1 單位的數字。

老師： 畫一條數線並標記 0。用一個實心點標記 0 的右邊一個單位，再使用一個實心點標記 0 的左邊一個單位。這個不等式的解包括了介於-1和 1 之間的所有數，也包括-1和 1 喔，所以我們用實心點標記。

可以寫成$-1 \leq x \leq 1$。

學生： 我懂了。所以第(2)小題我可以把$|x| > 2$ 寫成 $|x-0| > 2$，並解釋成找出所有距離 0 超過 2 個單位的數字。

老師： 很好，畫一條數線並標記 0。畫一個空心點標記 0 的右邊兩個單位，再用一個空心點標記 0 的左邊兩個單位。這個不等式的解包括了所有小於-2或大於 2 的數，但不包括-2和 2，所以我們用空心點標記。

可以寫成$x < -2$ 或 $x > 2$。

老師： 總結一下，如果$a > 0$，那麼：

(i) $|x| \leq a$ 的解為$-a \leq x \leq a$。

(ii) $|x| < a$解為$-a < x < a$。

(iii) $|x| \geq a$解為$x \geq a$ 或 $x \leq -a$。

(iv) $|x| > a$解為 $x > a$或 $x < -a$。

老師： 在數線上使用實心點表示端點是解的一部分，而使用空心點表示端點不是解的一部分。

應用問題 / 學測指考題

例題一

說明：生活中購買的食物包裝上，會用誤差範圍來說明該商品可能的容量及重量是多少，正是與絕對值有關的應用。

（英文）Allen decided to buy a Christmas present for his mother and picked up a gift box in a store. He noticed that the label said the weight of the box was 200 g ± 1%,

(1) Assume the weight of this box is x grams; what is the possible weight of this box? Use absolute value inequality to show it.

(2) Is the label of this box correct if the weight of this box is measured to be 180 g.

（中文）艾倫想買一件禮物給他的媽媽當聖誕節禮物。他注意到包裝上寫著這個禮盒的重量為 200 克 ±1%，

(1) 若此一禮盒的實際重量為 x 克，試用絕對值不等式寫出 x 的範圍。

(2) 若此一禮盒實際測量的重量為 180 克，試問該標示是否不實？

Teacher: Take a look at 200 g ± 1%. We call the 1% the margin of error. Multiply 200 grams by 1% and we get 2 grams. So, 200 g ± 1% also means the weight of the box is between $200\text{ g} \pm 2\text{ g}$. It means x is between 198 grams and 202 grams. That is $198 \le x \le 202$.

Student: Can I write $200-2 \le x \le 200+2$ instead?

Teacher: That is even better. We can interpret it as all the numbers on the number line less than or equal to 2 units away from 200. So we can use absolute value inequality to rewrite it.

Student: Is that $|x-200| \le 2$?

Teacher: Great! The next question asks if the box has the weight of 180 grams. Is it correct?

Student: 180 is not between 198 and 202. It is far less than the lower bound of the possible value. I would say this box is not labeled correctly.

Teacher: You are correct. Maybe we can consider filing a lawsuit against this company.

Student: Sure, by winning the lawsuit, I can buy another gift for my mother.

Teacher: How sweet you are!

老師： 注意200 g ± 1%的部分，我們稱此盒重的誤差範圍為 1%。將 200 克乘以 1%得到 2 克，所以 200 g ± 1% 也表示盒子的重量在 $200\text{ g} \pm 2\text{ g}$ 之間。因此可以看

作是 x 的範圍在 198 克和 202 克之間，即 $198 \leq x \leq 202$。

學生： 我可以寫成$200 - 2 \leq x \leq 200 + 2$嗎？

老師： 這樣寫更好了。我們可以將它解釋為數線上所有距離 200 不超過 2 單位的數。因此，我們就有辦法用絕對值不等式寫出 x 的範圍。

學生： 是不是 $|x - 200| \leq 2$？

老師： 很棒！下一個問題問如果盒子的重量是 180 克，該標示是否不實？

學生： 180 不在 198 和 202 之間，也遠小於可能的最小值，因此這個盒子標示不實。

老師： 沒錯，或許我們可以考慮對這家公司提起訴訟。

學生： 對，勝訴後我可以再買另一份禮物給媽媽。

老師： 太貼心了吧！

例題二

說明：此題為一關於絕對值不等式的題型，我們可以從代數或函數圖形的角度出發來解題。

（英文）Solve $|4x - 12| \leq 2x$ and write your answer in interval notation. Find the length of this interval.

（中文）請問滿足絕對值不等式 $|4x - 12| \leq 2x$ 的實數 x 所形成的區間，其長度為下列哪一個選項？

(1) 1　(2) 2　(3) 3　(4) 4　(5) 6

（103 年學測單選題 4）

Method 1: Solve this inequality algebraically.

Teacher: To solve the absolute inequality, we can take out the absolute value sign by either adding a negative sign, if we know the value within the absolute value sign is negative, or remaining the same if we know the value within it is positive. That is, we can write $|x|$ as follows:

$$|x| = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$$

Therefore, when solving $|4x - 12| \leq 2x$, we need to consider which x could make $4x - 12 \geq 0$ and which x could make $4x - 12 \leq 0$.

Student: Is that because I would know whether I need to add a negative sign if I take out the absolute value sign?

Teacher: Correct. Consider when $4x - 12 \geq 0$ which means $x \geq 3$, after we take out the absolute value sign the inequality could be written as: $4x-12 \le 2x$. Continue to solve this inequality.

Student: By moving -12 to the right side, I will get $4x \le 2x+12$. Move $2x$ to the left side, and I get $2x \le 12$. Divide both sides by 2, and the answer is $x \le 6$.

Teacher: Don't forget this result assumes $x \ge 3$. So, we can combine $x \le 6$ and $x \ge 3$ into one inequality $3 \le x \le 6$. Now consider when $4x-12 \le 0$ which means $x \le 3$. What should you do when you take away the absolute value sign of this inequality?

Student: I should add a negative sign to it.

Teacher: Good. So let's solve $-(4x-12) \le 2x$ now.

Student: I multiply both sides by -1, and get $4x-12 \ge -2x$.

Teacher: Wonderful, you remembered to change the inequality sign when multiplying both sides by a negative number.

Student: Then, I move $-2x$ to the left side and -12 to the right side. This inequality becomes $6x \ge 12$, so $x \ge 2$.

Teacher: Don't forget this result assumes $x \le 3$. The intersection of the following inequalities $x \ge 2$ and $x \le 3$ is $2 \le x \le 3$. In summary, we know the answer to this inequality is $2 \le x \le 6$. What is the length of this interval?

Student: That is not hard. The answer is 4.

Teacher: Terrific!

方法 1：以代數方法解不等式。

老師： 在解絕對值不等式時，我們會將絕對值去掉再根據絕對值裡的數是正或是負來決定需不需要加負號，若裡面的數是負數則要加負號反之則不用。也就是說，我們可以將 $|x|$ 寫成以下形式：

$$|x| = \begin{cases} x, x \ge 0 \\ -x, x < 0 \end{cases}$$

老師： 為了解 $|4x-12| \le 2x$，我們需要考慮哪個 x 會使得 $4x-12 \ge 0$以及哪個 x 會使得 $4x-12 \le 0$。

學生： 是不是因為我們還要考慮去掉絕對值後，x 是否需要加負號。

老師： 正確。考慮$4x-12 \ge 0$也就是$x \ge 3$ 時，當我們拿掉絕對值後，不等式可以寫成$4x-12 \le 2x$。請繼續解這個不等式。

學生： 將-12移動到右側，得到 $4x \le 2x+12$。將$2x$移動到左側，得到 $2x \le 12$。兩邊同時除以 2，答案為 $x \le 6$。

老師： 不要忘記這個結果的前提是$x \ge 3$。因此，我們可以將$x \le 6$ 和 $x \ge 3$合併成一個不等式 $3 \le x \le 6$。

現在來看當 $4x-12\leq 0$時，也就是 $x\leq 3$的情況。去掉絕對值時，你應該要怎麼做？

學生： 要加負號。

老師： 很好。現在解$-(4x-12)\leq 2x$。

學生： 兩邊同乘以-1，得到$4x-12\geq -2x$ 。

老師： 有記得在兩邊同乘以負數時，要改變不等式符號，很棒！

學生： 然後，將 $-2x$ 移到左側，將-12 移到右側，不等式變成$6x\geq 12$，因此 $x\geq 2$。

老師： 不要忘記這個結果的前提是$x\leq 3$。
以下不等式 $x\geq 2$ 和 $x\leq 3$ 的交集是 $2\leq x\leq 3$。
總結一下以上兩個討論的結果，我們不等式的答案是 $2\leq x\leq 6$。這個區間的長度是多少？

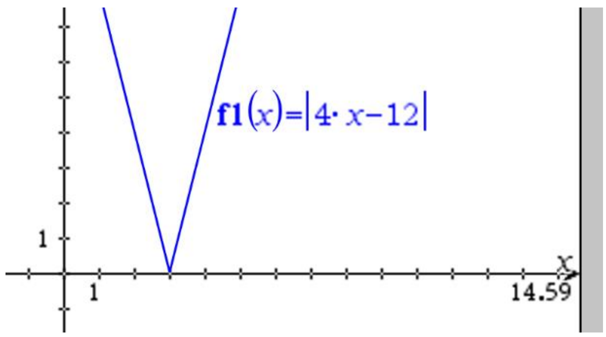
學生： 簡單，答案是 4。

老師： 很好，答對了！

Method 2: Solve this inequality graphically.

Teacher: We can use graphs to solve this inequality, too. First assume $f_1(x)=|4x-12|$ and $f_2(x)=2x$. Graph y_1 on your grids now. Where is the turning point of this function?

Student: The turning point is at $(3,0)$. The graph of $f_1(x)=|4x-12|$ is V-shaped as follows, (See figure 1)



(figure 1)

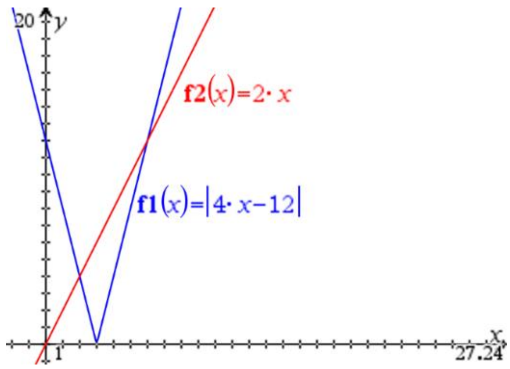
Teacher: Next, graph $f_2(x)=2x$ on the same grid.

Student: $f_2(x) = 2x$ is a line.

I use the red one with the label f_2.

(See figure 2)

(figure 2)



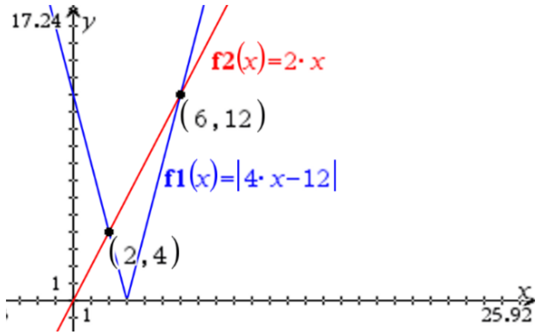
Teacher: We can interpret solving $|4x - 12| \leq 2x$ as answering which part of the graph of f_1 is lower than the graph of f_2. There are two intersections between f_1 and f_2 from the graph. To the right of $x = 3$, we're going to find the intersection between

$y = 2x$ and $y = 4x - 12$ because when $x \geq 3$, $|4x - 12| = 4x - 12$.

Where is the intersection?

Student: The intersection has the coordinates of (6,12) (See figure 3)

(figure 3)

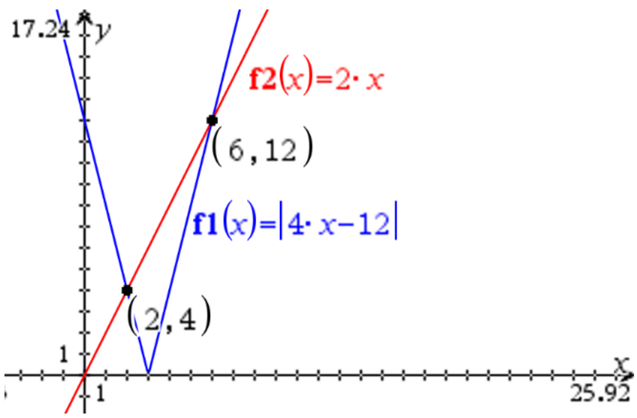


Teacher: Take a look at figure 3. To the left of $x = 3$, we are going to find the intersection between $y = 2x$ and $y = 12 - 4x$ because when $x \leq 3$, $|4x - 12| = 12 - 4x$.

Where is the intersection?

Student: The intersection has the coordinates of (2,4)

(See figure 4).



(figure 4)

From the graph, I can see that the graph of f_2 is above the graph of f_1 when $2 \le x \le 6$.

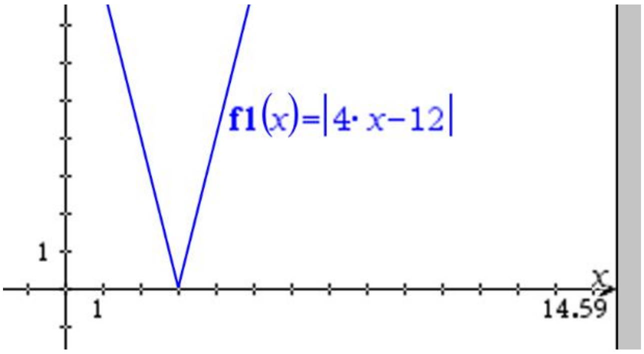
So, the length of the interval is 4.

Teacher: Wonderful! That is correct.

方法 2：以圖形解不等式。

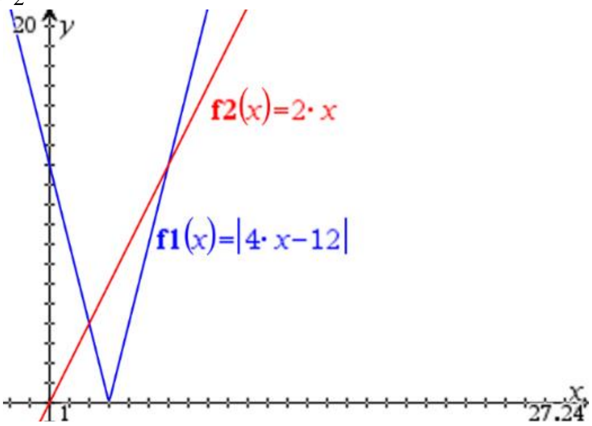
老師： 我們也可以用圖形解這個不等式。先假設 $f_1(x) = |4x - 12|$，以及$f_2(x) = 2x$。現在，在坐標平面上畫出 y_1，該函數的轉折點在哪裡？

學生： 轉折點在 (3,0)。$f_1(x) = |4x - 12|$是一個 v 字型，圖形如下：



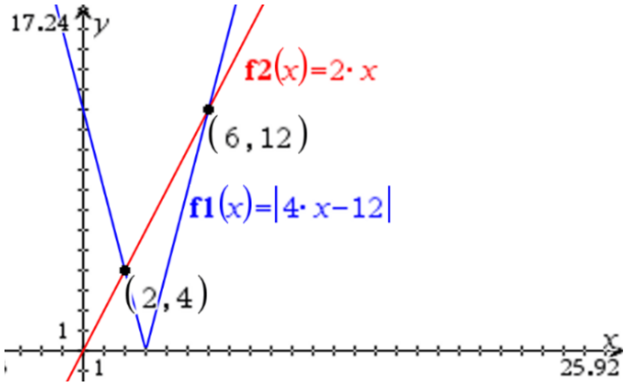
老師： 接下來，在同一個坐標平面上畫出$f_2(x) = 2x$。

學生： $f_2(x) = 2x$是一條直線，我用紅色的線標記。



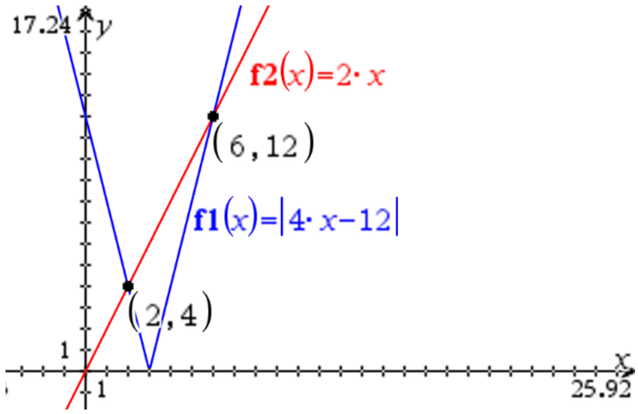
老師： 我們可以把求 $|4x - 12| \le 2x$ 的解解釋為求哪部分的f_1 圖形在f_2 之下。從圖形中可以看出，f_1 和f_2 有兩個交點。因為在$x = 3$的右邊，也就是當 $x \ge 3$時，$|4x - 12| = 4x - 12$，所以我們就可以找到 $y = 2x$ 和 $y = 4x - 12$ 的交點。交點在哪呢？

學生： 交點的坐標為(6, 12)。



老師： 看一下圖三。因為在$x = 3$的左邊，也就是當$x \leq 3$時 ，$|4x - 12| = 12 - 4x$，所以我們就可以找到 $y = 2x$ 和 $y = 12 - 4x$ 的交點。

學生： 交點的坐標為(2, 4)（見圖四）。



從圖形中，我可以看到當 $2 \leq x \leq 6$時，f_2 圖形在 f_1 圖形上方。因此，這個區間的長度是 4。

老師： 很棒！答對了。

單元三 數的運算
Operations of Numbers

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■ 前言 Introduction

國中階段學過和的關於平方乘法公式，在此節會學到關於立方的乘法公式，所以學生的常見到的次方表示法會從 square，擴充到 cube。乘法公式的英文也需要注意它們的順序，像是 “sum of squares” 跟 “square of a sum” 就代表不同的意思。有理化(rationalize)分母也是此節的重點之一，我們在前面學過有理數(rational numbers)後，在這節也會使用到這個字的動詞的用法。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
fraction	分式	**rationalize**	**有理化**
square root	平方根	numerator	分子
radical sign	根號	multiplication formula	乘法公式
expand	**展開**	least common multiple	最小公倍式
factorization	因式分解	**factor**	**因式**
evaluate	**求值**	denominator	分母
simplify	**化簡**		

單字	中譯
difference of squares formula	平方差公式
square of a sum formula	和的平方公式
square of a difference formula	差的平方公式
sum of cubes formula	立方和公式
difference of cubes formula	立方差公式
cube of the sum formula	和的立方公式
cube of the difference formula	差的立方公式
rationalize the denominator	有理化分母

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ Use the multiplication formula to expand ________.

例句：**Use the multiplication formula to expand** $(3x-2y)(3x+2y)$**.**

利用乘法公式展開$(3x-2y)(3x+2y)$。

❷ Factor ________.

例句：**Factor** x^3+125y^3.

因式分解 x^3+125y^3。

❸ Given ________, evaluate ______.

例句：**Given** $a=\sqrt{3}+1, b=\sqrt{3}-1$**, evaluate** a^2+b^2

給定$a=\sqrt{3}+1, b=\sqrt{3}-1$，求出$a^2+b^2$之值。

❹ **Simplify __________.**

例句：**Simplify** $\frac{x+1}{x-3}+\frac{x}{x+1}$

化簡 $\frac{x+1}{x-3}+\frac{x}{x+1}$。

❺ **Which of the following is larger? _____ or _____?**

例句：**Which of the following is larger?** $\sqrt{3}+\sqrt{6}$ **or** $\sqrt{4}+\sqrt{5}$?

試比較 $\sqrt{3}+\sqrt{6}$ 和 $\sqrt{4}+\sqrt{5}$ 的大小。

❻ **Rationalize the denominator of __________ and simplify it.**

例句：**Rationalize the denominator of** $\sqrt{\frac{3}{5}}$ **and simplify it.**

將 $\sqrt{\frac{3}{5}}$ 分母有理化並化簡。

■ 問題講解 Explanation of Problems

說明

In 8th grade, we learned some multiplication formulas related to squares, such as the square of a sum formula and the difference of squares formula. Today, we are going to learn a few more formulas related to cubes, such as the difference of cubes formula and the **cube of the sum formula.**

我們先複習一些國中所學過的乘法公式，接著向學生介紹立方乘法公式。

Given $a>0$, we define $\sqrt{a}$ as the positive square root to the equation of $x^2=a$. There is the property "if $a\geq 0, b\geq 0$, then $\sqrt{a}\cdot\sqrt{b}=\sqrt{ab}$" which can help us simplify some numbers in radical (square root) signs. And when simplifying fractions in square root, we usually rationalize the denominator.

我們可以利用根式運算性質來化簡算式，在化簡過程中習慣會將分母有理化。

We use $\sqrt{4+2\sqrt{3}}$ to represent a solution to $x^2=4+2\sqrt{3}$, but you can see we use the radical signs twice in expressing $\sqrt{4+2\sqrt{3}}$ and it seems complicated. Can we simplify $\sqrt{4+2\sqrt{3}}$ without using the square root twice? We will figure this out later.

當在解 $x^2=4+2\sqrt{3}$ 這個方程式時，會用到雙重根式 $\sqrt{4+2\sqrt{3}}$ 來表達這個解，但雙重根式顯得複雜難懂。在本節會學到化簡雙重根式的方法。

運算問題的講解

例題一

說明：利用乘法公式展開或因式分解各式。

（英文）(a.) Use the multiplication formula to expand

$$(x-3)(x+3)(x^2+3x+9)(x^2-3x+9).$$

(b.) Factor $27x^3+8y^3$.

（中文）(a.) 利用乘法公式展開 $(x-3)(x+3)(x^2+3x+9)(x^2-3x+9)$。

(b.) 因式分解 $27x^3+8y^3$。

Teacher: Instead of using the FOIL method (We expand $(a+b)(c+d)=ac+ad+bc+bd$. ac is the first term, ad is the outer term, bc is the inner term, and bd is the last term) to expand from left to right, we have a better way to expand $(x-3)(x+3)(x^2+3x+9)(x^2-3x+9)$ here.

Student: Really? How?

Teacher: You can separate $(x-3)(x+3)(x^2+3x+9)(x^2-3x+9)$ into two groups. One group with $(x-3)$ and (x^2+3x+9), and the other group with $(x+3)$ and (x^2-3x+9).

Student: What a good idea! By applying the difference of cubes formula, the product of $(x-3)$ and (x^2+3x+9) is x^3-27.

The product of $(x+3)$ and (x^2-3x+9) is x^3+27 by applying the sum of cubes formula.

Teacher: Great! You see the trick here. Which formula can help you expand

$(x^3-27)(x^3+27)$?

Student: The difference of squares formula will work here. So

$(x^3-27)(x^3+27)$

$=(x^3)^2-27^2$

$=x^6-729$

Teacher: Terrific!

Teacher: To solve the question (b), let's review the formula for the sum of cubes formula.

Student: It is $a^3+b^3=(a+b)(a^2-ab+b^2)$.

Teacher: In this question, write $27x^3$ in terms of $(3x)^3$and $8y^3$ in terms of $(2y)^3$.

Student: $27x^3+8y^3=(3x)^3+(2y)^3$.

Teacher: You can see that it is the sum of two cubes, so we can use the sum of cubes formula now. Replace a with $3x$ and b with $2y$ on the right side of this formula.

Student: $(3x)^3+(2y)^3$

$=(3x+2y)((3x)^2-(3x)(2y)+(2y)^2)$

$=(3x+2y)(9x^2-6xy+4y^2)$

Teacher: You did it. Well done!

老師： 不用將此題如$(a+b)(c+d)=ac+ad+bc+bd$這樣從左到右展開，要展開 $(x-3)(x+3)(x^2+3x+9)(x^2-3x+9)$ 這個式子有更好的方法。

學生： 真的嗎？什麼方法？

老師： 可以將 $(x-3)(x+3)(x^2+3x+9)(x^2-3x+9)$ 分成兩組。一組包含$(x-3)$和(x^2+3x+9)，另一組包含$(x+3)$和 (x^2-3x+9)。

學生： 好方法！用立方差公式可以得到$(x-3)$和(x^2+3x+9)的乘積是x^3-27，而立方和的公式可以得到$(x+3)$和(x^2-3x+9)的乘積是x^3+27。

老師： 太棒了！你看懂了。哪個公式可以展開 $(x^3-27)(x^3+27)$？

學生： 可以用平方差公式。所以

$(x^3-27)(x^3+27)$

$=(x^3)^2-27^2$

$=x^6-729$

老師： 太棒了！

老師： 為了解題 (b.)，回顧一下立方和公式。

學生： $a^3+b^3=(a+b)(a^2-ab+b^2)$

老師： 在這個問題中，將$27x^3$ 用 $(3x)^3$ 代替，將$8y^3$ 用$(2y)^3$ 代替。

學生： $27x^3+8y^3=(3x)^3+(2y)^3$。

老師： 可以看到是兩個立方的和，所以現在可以用立方和公式。將公式右邊的a 替換為$3x$，將b替換為$2y$。

學生： $(3x)^3+(2y)^3$
$=(3x+2y)((3x)^2-(3x)(2y)+(2y)^2)$
$=(3x+2y)(9x^2-6xy+4y^2)$

老師： 做得很好！漂亮！

例題二

說明：本題是關於在分數裡，如何將分母的有理化。

（英文）Simplify $\frac{1}{\sqrt{3}-\sqrt{2}}$.

（中文）化簡 $\frac{1}{\sqrt{3}-\sqrt{2}}$。

Teacher: When we see a fraction as $\frac{1}{\sqrt{3}-\sqrt{2}}$ and find out there is a radical sign in the denominator, we will usually rationalize the denominator. The process of rationalizing the denominator is trying to make the radical sign disappear from the denominator. The difference of squares formula will help us cancel out the radical sign. What does the difference of squares formula say?

Student: It says, $a^2-b^2=(a-b)(a+b)$.

Teacher: Good! To apply the formula, we multiply both the numerator and the denominator by $\sqrt{3}+\sqrt{2}$.

Student: Like this: $\frac{1}{\sqrt{3}-\sqrt{2}}\cdot\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}$?

Teacher: Yes. $\frac{1}{\sqrt{3}-\sqrt{2}}\cdot\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}=\frac{\sqrt{3}+\sqrt{2}}{(\sqrt{3})^2-(\sqrt{2})^2}=\frac{\sqrt{3}+\sqrt{2}}{1}=\sqrt{3}+\sqrt{2}$

Student: I see the trick now. The radical sign in the denominator disappears.

Teacher: I am so glad you saw it.

老師： 看到一個分數形式為 $\frac{1}{\sqrt{3}-\sqrt{2}}$ 且分母有出現根號時，通常會對分母進行有理化。有理化的過程是使分母中的根號消失。用平方差公式可以幫忙消去根號。平方差公式是什麼？

學生： $a^2-b^2=(a-b)(a+b)$。

老師： 很好！用這個公式，將分子和分母都乘以$\sqrt{3}+\sqrt{2}$。

學生： 像這樣：$\frac{1}{\sqrt{3}-\sqrt{2}}\cdot\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}$ ？

老師： 是的，$\frac{1}{\sqrt{3}-\sqrt{2}}\cdot\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}=\frac{\sqrt{3}+\sqrt{2}}{(\sqrt{3})^2-(\sqrt{2})^2}=\frac{\sqrt{3}+\sqrt{2}}{1}=\sqrt{3}+\sqrt{2}$ 。

學生： 現在懂了。分母中的根號消失了。

老師： 太好了。

例題三

說明：本題為乘法公式的運用。

（英文）If the polynomial equation $x^3+ax^2+bx+8=0$ with real coefficients has one root with multiplicity of three, solve for b.

(1) 6 (2) 8 (3) 10 (4) 12 (5) 15

（中文）已知實係數多項式方程式 $x^3+ax^2+bx+8=0$的三根相同，請問 b 的值等於下列哪一個選項？

(1) 6 (2) 8 (3) 10 (4) 12 (5) 15

（改編自 101 年數乙單選第 1 題）

Teacher: If the polynomial equation $x^3+ax^2+bx+8=0$ with real coefficients has one root with multiplicity of three, let's assume the root is k and the equation can be expressed as the cube of $(x-k)$. So $x^3+ax^2+bx+8=(x-k)^3$. Now expand $(x-k)^3$.

Student: I can use the multiplication formula to expand $(x-k)^3$.

$(x-k)^3$

$=x^3-3x^2k+3xk^2-k^3$

Teacher: Great! Now compare the coefficients from both sides. What do you find?

Student: When I compare the constant, I notice that $-k^3=8$. Therefore, $k=-2$.

Teacher: Terrific! That is a really good find. To solve for b, let's compare the x-term.

Student: The coefficient of the x-term is $3\cdot k^2=3\cdot(-2)^2=12$. So $b=12$.

Teacher: Well done!

老師： 如果具實係數的多項式方程式 $x^3+ax^2+bx+8=0$ 三個相同的根，假設此根是 k，則方程式可以表示為 $(x-k)$ 的立方。所以 $x^3+ax^2+bx+8=(x-k)^3$。請展開 $(x-k)^3$。

學生： 可以用乘法公式展開 $(x-k)^3$。

$(x-k)^3$

$=x^3-3x^2k+3xk^2-k^3$

老師： 太棒了！現在比較兩邊的係數，發現什麼了？

學生： 比較常數項時，發現 $-k^3=8$。因此 $k=-2$。

老師： 太好了！觀察仔細。要解 b，比較一下 x 項係數。

學生： x 項的係數是 $3\cdot k^2=3\cdot(-2)^2=12$。所以 $b=12$。

老師： 做得好！

例題四

說明：雙重根式的化簡。

（英文）Simplify $\sqrt{5-\sqrt{24}}$.

（中文）化簡 $\sqrt{5-\sqrt{24}}$。

Student: How should we simplify $\sqrt{5-\sqrt{24}}$? I see a square root of 24 under another square root.

Teacher: Our goal to simplify this is to take out one square root. The way to take out a square root is if the thing under the square root is the square of some number, then the square root can be canceled out.

The question becomes, "Can $5-\sqrt{24}$ be written as $\left(\sqrt{a}-\sqrt{b}\right)^2$?".

Student: How can I start?

Teacher: Let's expand $\left(\sqrt{a}-\sqrt{b}\right)^2$ and compare the result with $5-\sqrt{24}$.

Student: I will use the multiplication formula to expand $\left(\sqrt{a}-\sqrt{b}\right)^2$ as $a-2\sqrt{ab}+b$.

Teacher: Now rewrite $\sqrt{24}=2\sqrt{6}$. We can make the comparison.
Solve $a+b=5$ and $ab=6$
For $ab=6$, substitute $a=5-b$ and we get $(5-b)\cdot b=6$.
Solve $b^2-5b+6=0$
$(b-2)(b-3)=0$
Therefore, $b=2$ or 3.
When $b=2, a=3$. When $b=3, a=2$.

Since $\sqrt{a}-\sqrt{b}\geq 0$ So $a\geq b$.

We can rewrite $5-\sqrt{24}$ as $\left(\sqrt{3}-\sqrt{2}\right)^2$.

Also, $\sqrt{5-\sqrt{24}}=\sqrt{\left(\sqrt{3}-\sqrt{2}\right)^2}=\sqrt{3}-\sqrt{2}$

學生： 該如何簡化 $\sqrt{5-\sqrt{24}}$？看到一個平方根下面還有一個 24 的平方根。

老師： 簡化的目標是消掉一個平方根。消掉平方根的方法是，如果根號下的數字是某個數字的平方，那這個平方根可以被消除。

問題變成這樣，$5-\sqrt{24}$可以寫成 $(\sqrt{a}-\sqrt{b})^2$的形式嗎？

學生： 要怎麼開始？

老師： 展開 $(\sqrt{a}-\sqrt{b})^2$，將結果與 $5-\sqrt{24}$ 進行比較。

學生： 使用乘法公式展開$(\sqrt{a}-\sqrt{b})^2$，得 $a-2\sqrt{ab}+b$。

老師： 現在整理 $\sqrt{24}=2\sqrt{6}$，然後來做比較。
解$a+b=5$和$ab=6$
對於$ab=6$，代換 $a=5-b$，得到 $(5-b)\cdot b=6$。
解$b^2-5b+6=0$
$(b-2)(b-3)=0$
因此，$b=2$ 或 $b=3$。當 $b=2$ 時 $a=3$；當 $b=3$ 時 $a=2$。

由於$\sqrt{a}-\sqrt{b}\geq 0$，所以 $a\geq b$。

我們可以將 $5-\sqrt{24}$ 整理為$\left(\sqrt{3}-\sqrt{2}\right)^2$。

同時$\sqrt{5-\sqrt{24}}=\sqrt{\left(\sqrt{3}-\sqrt{2}\right)^2}=\sqrt{3}-\sqrt{2}$。

單元四 指數與對數
Exponents and Logarithms

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■ 前言 Introduction

國中已經有介紹一些指數的基本概念及正整數指數的指數律(exponential law)，這邊會再次接觸到這些英文名稱，像是 product of powers, quotient of powers 和 power of a power。指數在生活中有許多的應用會是比較多跟其他科目結合，像是細菌(bacteria)的繁殖(生物)，放射性元素半衰期(half-life)(化學)的介紹等等，科學中的英文專有名詞的準備也會是一項挑戰。國中階段有提到將數以科學記號表示，在高中時增加了有效數字的定義，和科學記號的四則運算。本節最後介紹了常用對數定義及常用對數近似值的求法。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
exponent	**指數**	**power**	次方
base	**底數**	***n*th power**	***n*** 次方
scientific notation	**科學記號**	half-life	半衰期
integer	整數	**significant figures**	**有效數字**
rational number	有理數	logarithm	對數
real number	實數	common logarithm	常用對數
exponential law	指數律		

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ (Product of powers rule) To multiply powers with the same base of _______, add the exponents. $a^m \cdot a^n = a^{m+n}$

例句：**To multiply powers with the same base of** 3**, add the exponents.** $3^m \cdot 3^n = 3^{m+n}$

計算 3^m 乘以 3^n 時，將兩指數相加得到 3^{m+n}。

❷ (Quotient of powers rule) To divide powers with the same base of _______, subtract the exponents. $\frac{a^m}{a^n} = a^{m-n}$

例句：**To divide powers with the same base of** 3**, subtract the exponents.** $\frac{3^m}{3^n} = 3^{m-n}$

計算 3^m 除以 3^n 時，將兩指數相減得到 3^{m-n}。

❸ (Power of a power rule) To find a power of a power of _______, multiply the exponents. $(a^m)^n = a^{mn}$**.**

例句：**To find a power of a power of** 3**, multiply the exponents.** $(3^m)^n = 3^{mn}$**.**

計算 3^m 的 n 次方時，將兩指數相乘得到 3^{mn}。

❹ Write ____ in scientific notation with 3 significant figures.

例句：**Write** 123450000 **in scientific notation with 3 significant figures.**

將 123450000 表示成 3 位有效數字科學記號。

❺ Use exponential laws to evaluate _______.

例句：**Use exponential laws to evaluate** $3^5 \cdot 3^9$**.**

利用指數律求出下列式子 $3^5 \cdot 3^9$。

❻ List _____ (the following numbers) from least to greatest.

例句：**List** $0.6^{2/3}$**,** $0.6^{3/4}$**, and** $0.6^{4/5}$ **from least to greatest**.

由小到大列出 $0.6^{2/3}$，$0.6^{3/4}$和 $0.6^{4/5}$的關係。

❼ Simplify ___________.

例句：Simplify $(8^{\sqrt{2}-1})^{\sqrt{2}+1}$.

試化簡 $(8^{\sqrt{2}-1})^{\sqrt{2}+1}$ 。

❽ Write ___ in scientific notation and show how many digits ___ has.

例句：**Write** $10^{3.68}$ **in scientific notation and tell how many digits** $10^{3.68}$ **has.**

將 $10^{3.68}$ 以科學記號表示並判斷 $10^{3.68}$ 的整數部位有幾位數。

❾ We use $log\ a$ (read as "log of a") to represent the solution to $10^x = a$.

例句：**We use** $\log 5$ **to represent the solution to** $10^x = 5$**.**

我們用 $\log 5$ 來表示方程式 $10^x = 5$ 的解。

■ 問題講解 Explanation of Problems

說明

Students have learned how to use powers to express a very large or small number but are restricted to exponents of integers. In this lesson, we will help students understand the exponents of negative integers, rational numbers, and even real numbers. We will evaluate or estimate the value of powers with real numbers. Remember that the exponential laws also hold when the exponents are real numbers.
在國中階段我們介紹了自然數指數，我們在這節先將指數延伸到負整數，定義了有理數指數，接著將再指數範圍推廣到所有實數，而指數律依然成立。

In addition to talking about scientific notation, we will define the “significant figures” and calculate the addition, subtraction, multiplication, and division of numbers in scientific notation.
科學記號的用法在國中也有提到，而在高中階段會再介紹有效數字的定義，跟科學記號的四則運算。

At the end of this lesson, we define what a common logarithm is and use $\log a$ to represent the unique real solution to the equation of $10^x = a, a > 0$.
節末帶入了常用對數的介紹，用 $\log a$ 來表示 $10^x = a, a > 0$ 的唯一實數解。

There are two ways of reading $\log_b a$. We can say, “log base b of a” or “log of a with base b.” But if the logarithm is a common logarithm like $\log 5$, we directly read it as “log of 5”. For a natural logarithm like $\ln 5$, we will read it as “ln of 5.” (read as “l n of 5”)
這邊也附上對數 $\log_b a$ 的英文講法。而在遇到常用對數或自然對數時，我們可以省略底數不說。自然對數就直接念 ln of x。

運算問題的講解

例題一

說明：讓學生練習如何利用指數律將數化簡並求值。

（英文）Simplify the following numbers by using exponential laws.

（中文）利用指數律求出下列的值：

(1) $25^{\frac{1}{2}}$ (2) $\left(\sqrt[3]{3}\right)^2\left(\sqrt[6]{3}\right)^2$ (3) $\dfrac{\left(\sqrt[3]{5^6}\right)}{\left(\sqrt[4]{\sqrt{256}}\right)}$

Teacher: Let's take a look at $25^{\frac{1}{2}}$. We know 25 equals 5^2. Rewrite $25^{\frac{1}{2}}$ as $(5^2)^{\frac{1}{2}}$. How do we apply the power of a power rule?

Student: We multiply the powers together. Since 2 times a half equals 1, the result of the power is 1. So the answer is 5.

Teacher: Let's simplify $\left(\sqrt[3]{3}\right)^2\left(\sqrt[6]{3}\right)^2$. We can rewrite the cubic root of a number as the number to the power of $\frac{1}{3}$. So $\sqrt[3]{3} = 3^{\frac{1}{3}}$. Also, $(\sqrt[3]{3})^2 = (3^{\frac{1}{3}})^2$. Again, you can use the power of a power rule to rewrite $(3^{\frac{1}{3}})^2 = 3^{\frac{2}{3}}$. Now, here is a question for you. When we see a number in a radical sign with the index of 6, how do we rewrite this $\sqrt[6]{3}$ (read as 3 to the sixth root)?

Student: I know the answer. We can take the index to the denominator of the exponent as $3^{\frac{1}{6}}$. That is, we can express $\sqrt[n]{a}$ as $a^{\frac{1}{n}}$.

Teacher: Great. Next, $(\sqrt[6]{3})^2 = (3^{\frac{1}{6}})^2 = 3^{\frac{1}{3}}$. We have simplified the $\left(\sqrt[3]{3}\right)^2$ and $\left(\sqrt[6]{3}\right)^2$ separately. Now we have to multiply them together. So $\left(\sqrt[3]{3}\right)^2\left(\sqrt[6]{3}\right)^2 = 3^{\frac{2}{3}} \cdot 3^{\frac{1}{3}}$. Use the product of powers rule to finish the last step.

Student: Okay. I add the exponents up, and the power is 1. So, the answer is 3^1, which equals 3.

Teacher: You did it!

Teacher: For $\dfrac{\left(\sqrt[3]{5^6}\right)}{\left(\sqrt[4]{\sqrt{256}}\right)}$, you will find that the answer is straightforward later. I simplify the numerator first. By using $\sqrt[n]{a^m} = a^{\frac{m}{n}}$ (read as "the nth root of a to the power of m equals a to the power of $\frac{m}{n}$), $\left(\sqrt[3]{5^6}\right) = 5^{\frac{6}{3}} = 5^2 = 25$. The denominator seems complicated. Notice that 256 is within two radical signs. By using $\sqrt[n]{a} = a^{\frac{1}{n}}$ twice, we can get $\left(\sqrt[4]{\sqrt{256}}\right) = \left(\sqrt{256}\right)^{\frac{1}{4}} = \left(256^{\frac{1}{2}}\right)^{\frac{1}{4}}$. Now, use the law of the power of a power to simplify it.

Student: Multiplying $\frac{1}{4}$ by $\frac{1}{2}$ equals $\frac{1}{8}$. I know $256 = 2^8$. So $\left(256^{\frac{1}{2}}\right)^{\frac{1}{4}} = (2^8)^{\frac{1}{8}} = 2^1 = 2$.

Teacher: Divide 25 by 2 and $\frac{25}{2}$ is the answer.

Student: No one could have imagined the answer as simple as this.

Teacher: You only know it after you solve it by yourself.

Student: That is true.

老師： 看一下 $25^{\frac{1}{2}}$。我們知道 25 等於 5^2。將 $25^{\frac{1}{2}}$ 改寫為 $(5^2)^{\frac{1}{2}}$。一個數乘方的乘方要怎麼計算呢？

學生： 這裡將指數相乘。由於 2 乘以 0.5 等於 1，結果就是 1。所以答案是 5。

老師： 簡化$(\sqrt[3]{3})^2(\sqrt[6]{3})^2$。可以將一個數的立方根改寫為該數的 $\frac{1}{3}$ 次方。所以 $\sqrt[3]{3} = 3^{\frac{1}{3}}$。同樣可以用指數律將 $(\sqrt[3]{3})^2 = (3^{\frac{1}{3}})^2$重寫為 $(3^{\frac{1}{3}})^2 = 3^{\frac{2}{3}}$。現在再問你。看到在根號中的一個數，根指數為 6 時，怎麼重新表達 $\sqrt[6]{3}$？

學生： 我知道。可以把根指數當作指數的分母記為$3^{\frac{1}{6}}$，將$\sqrt[n]{a}$重寫為$a^{\frac{1}{n}}$。

老師： 太棒了。所以是$(3^{\frac{1}{6}})^2 = 3^{\frac{1}{3}}$。剛剛已經分別簡化了$\left(\sqrt[3]{3}\right)^2$和$\left(\sqrt[6]{3}\right)^2$，現在要將它們相乘。所以$\left(\sqrt[3]{3}\right)^2\left(\sqrt[6]{3}\right)^2 = 3^{\frac{2}{3}} \cdot 3^{\frac{1}{3}}$。使用指數律中同底相乘指數相加的規則來完成最後一步。

學生： 好的。將指數相加，結果為 1。所以答案是3^1，等於 3。

老師： 很好！

老師： 在 $\frac{\left(\sqrt[3]{5^6}\right)}{\left(\sqrt[4]{\sqrt{256}}\right)}$ 這題中，化簡後會發現答案非常簡單。首先化簡分子，使用 $\sqrt[n]{a^m}=a^{\frac{m}{n}}$ 的規則，得出 $\left(\sqrt[3]{5^6}\right)=5^{\frac{6}{3}}=5^2=25$ 。分母看起來很複雜，注意 256 在兩個根號裡面。用兩次 $\sqrt[n]{a}=a^{\frac{1}{n}}$ 的規則，可以得到 $\left(\sqrt[4]{\sqrt{256}}\right)=\left(\sqrt{256}\right)^{\frac{1}{4}}=\left(256^{\frac{1}{2}}\right)^{\frac{1}{4}}$ 。現在，用指數律中乘方的乘方將指數相乘這個規則來簡化。

學生： 將 $\frac{1}{4}$ 乘以 $\frac{1}{2}$ 得到 $\frac{1}{8}$。我知道 $256=2^8$。所以 $\left(256^{\frac{1}{2}}\right)^{\frac{1}{4}}=(2^8)^{\frac{1}{8}}=2^1=2$。

老師： 將 25 除以 2 得到 $\frac{25}{2}$ 為答案。

學生： 沒有人猜得到答案會這麼簡單。

老師： 只有自己解出來後才會發現。

學生： 真的。

例題二

說明：本題是跟學生介紹如何利用碳 14 半衰期為 5730 年來推估某古代生物化石已死亡多少年。

（英文）It is known that the half-life of radiocarbon is 5730 years. An archaeologist just found a fossil from a creature that contains 1/8 of the radiocarbon that it originally contained. Work out how long ago this creature died.

（中文）已知碳 14 的半衰期是 5730 年。一名考古學家發現一塊古生物化石的碳 14 含量為原始含量的 1/8，試推論這隻古代生物已死亡多久？

Teacher: Some electrical power comes from nuclear power plants. Do you know what chemical substance is used to generate power in nuclear power plants?

Student: I know. It is uranium-235.

Teacher: We also know that using uranium-235 can produce radiation that pollutes our environment and damages our health. The method to deal with this nuclear waste is to carefully store it for a long time; long enough to make sure the waste decays to nonradioactive and becomes harmless to living things.

Student: How long will that take?

Teacher: We call the time it takes half of a radioactive substance to change to a nonradioactive substance its "half-life." The half-life of uranium-235 is 703,800,000 years.

Student: Wow. When nuclear waste becomes harmless, I will already be a fossil.

Teacher: You sure will be. By the way, when talking about half-life, we can use the half-life of radiocarbon to work out how old a fossil is. For example, assume that the original quantity of radiocarbon of the fossil is P. Then given the half-life of radiocarbon is 5730 years, how much of the radiocarbon is left after t years?

Student: It is $P \cdot \left(\frac{1}{2}\right)^{t/5730}$.

Teacher: Correct. The question says this fossil we find contains 1/8 of its original radiocarbon. Now we solve $P \cdot \left(\frac{1}{2}\right)^{t/5730} = P \cdot \left(\frac{1}{8}\right) = P \cdot \left(\frac{1}{2}\right)^{3}$

Student: So $\frac{t}{5730} = 3$, and $t = 3 \cdot 5730 = 17190$. The fossil is 17190 years old.

Teacher: Yes, you found out how old this fossil is.

老師：有些電力來源是來自核電廠發的電，那麼你們知道核電廠中用什麼原料來發電嗎？

學生：我知道，是鈾-235。

老師：已知用鈾-235 會產生污染環境且有害健康的輻射。目前處理用過核燃料的方法是長時間妥善儲存，用夠長的時間確保用過的燃料衰變成對生物無害的非放射性物質。

學生：需要多久的時間呢？

老師：我們稱放射性物質的一半轉變為非放射性物質所需的時間為「半衰期」。鈾-235 的半衰期為 7,038,000,000 年。

學生：哇！等用過核燃料變得無害，我都要變化石了。

老師：確實。既然提到了半衰期，我們可以利用放射性碳的半衰期來推斷化石的年齡。例如假設化石中的放射性碳的初始量為 P。根據放射性碳的半衰期為 5730 年，經過 t 年後還剩下多少放射性碳？

學生：它是 $P \cdot \left(\frac{1}{2}\right)^{t/5730}$。

老師：正確。題目說發現的化石中的放射性碳含量是原來的 1/8。現在解方程式

$$P \cdot \left(\frac{1}{2}\right)^{t/5730} = P \cdot \left(\frac{1}{8}\right) = P \cdot \left(\frac{1}{2}\right)^{3}$$

學生： 所以 $\frac{t}{5730} = 3$，$t = 3 \cdot 5730 = 17190$。這個化石年齡是 17190 年。

老師： 對，你求出化石年齡了。

例題三

（英文）The initial amount of some bacteria is 1000 in a closed laboratory. If the bacteria grows at the rate of 8% each hour, how many digits of the total number of this bacteria will it have after 100 hours, given that the bacteria maintain the same growth rate? (Hint: $\log 1.08^{100} \approx 3.34$)

（中文）在密閉的實驗室中，開始時有某種細菌 1 千隻，並且以每小時增加 8%的速率繁殖。如果依此速率持續繁殖，則 100 小時後細菌的數量會達幾位數？

（改編自 99 年學測單選第 5 題）

Teacher: How much bacteria is there after 1 hour?

Student: We know this bacteria grows at a rate of 8% each hour, so the amount of bacteria after 1 hour is $1000 \cdot (1+0.08)^1 = 1000 \cdot 1.08$.

Teacher: Good! So when the bacteria keep growing, there will be $1000 \cdot 1.08^{100}$ bacteria after 100 hours.

Student: What is the approximate number of $1000 \cdot 1.08^{100}$?

Teacher: We can use logarithms to approximate 1.08^{100} first. Use this approximation to multiply 1000 later. Take a common logarithm to 1.08^{100}.

Now use the hint $\log(1.08^{100}) \approx 3.34$.

We have $1.08^{100} \approx 10^{3.34}$

So,

$1000 \cdot 1.08^{100}$

$\approx 10^3 \cdot 10^{3.34}$

$= 10^{6.34}$

$= 10^6 \cdot 10^{0.34}$

We also know $1 = 10^0 < 10^{0.34} < 10^1 = 10$, so $10^6 < 10^6 \cdot 10^{0.34} < 10^7$.

We conclude that $10^6 \cdot 10^{0.34}$ has 7 digits

Student: That is quite a big number!

老師： 在 1 小時後，細菌的數量大還有多少？

學生： 已知這種細菌以每小時 8%的速率增長，所以 1 小時後的細菌數量為 $1000\cdot(1+0.08)^1=1000\cdot 1.08$ 。

老師： 很好！所以當細菌不斷增長時，100 小時後會有 $1000\cdot(1+0.08)^1=1000\cdot 1.08$ 的細菌量。

學生： 那麼 $1000\cdot 1.08^{100}$ 大約是多少？

老師： 可以用對數來求近似值。然後把近似值乘以 1000。將 1.08^{100} 取常用對數。

看看提示 $\log(1.08^{100})\approx 3.34$ 。

我們有 $1.08^{100}\approx 10^{3.34}$

因此，

$1000\cdot 1.08^{100}$

$\approx 10^3\cdot 10^{3.34}$

$=10^{6.34}$

$=10^6\cdot 10^{0.34}$

我們也知道 $1=10^0<10^{0.34}<10^1=10$ ，所以 $10^6<10^6\cdot 10^{0.34}<10^7$

我們可以得出 $10^6\cdot 10^{0.34}$ 有 7 位數

學生： 數字真的很大！

例題四

說明：讓學生知道如何利用 pH 值算出溶液中氫離子濃度。

（英文）Some lemonade has a pH of 3.6, and a box of baking soda has a pH of 8.6.
(a) What are their hydrogen-ion concentrations?
(b) How many times greater is the hydrogen-ion concentration of the lemonade than that of the baking soda?

（中文）某檸檬汁的 pH 值為 3.6 而烘焙用小蘇打的 pH 值為 8.6。
(a) 試求它們分別的氫離子濃度。
(b) 檸檬汁的氫離子濃度為小蘇打的幾倍？

Teacher: In chemistry, we measure the concentration of the nitrogen ions in a water-based solution (in moles per liter) to tell how acidic this solution is. Usually, the concentration involves the negative powers of 10, therefore, the pH value used to describe the acidity is defined to be the common negative log of the concentration of the hydrogen ions of the solution. We use the symbol “[H^+]” to indicate the concentration of hydrogen ions. And we use “pH=-log[H^+]” to evaluate the pH value of a given water-based solution. Now we have lemonade with pH of 3.6. What is its hydrogen-ion concentration?

Student: $3.6 = -log[H^+]$. $log[H^+] = -3.6$. So $[H^+] = 10^{-3.6}$.

Teacher: Great! Find the hydrogen-concentration of the baking soda.

Student: $8.6 = -log[H^+]$. $log[H^+] = -8.6$. So $[H^+] = 10^{-8.6}$.

Teacher: Good job. We can compare those two hydrogen-ion concentrations now. How much larger is the hydrogen-ion concentration of the lemonade than that of the baking soda?

Student: Divide $10^{-3.6}$ by $10^{-8.6}$. $\frac{10^{-3.6}}{10^{-8.6}}$ can be simplified by the quotient of powers rule into $10^{-3.6-(-8.6)} = 10^5 = 100,000$. The hydrogen-ion concentration of the lemonade is 100,000 times larger than that of the baking soda.

Teacher: So consider a solution that contains a higher concentration of hydrogen ions. What do you think its pH is? Higher or lower?

Student: Lower. Take this question for example. The lemonade with a higher concentration of hydrogen ions has a lower pH value.

Teacher: Correct! And vice versa, the lower the pH value is, the more acidic the solution is.

老師： 化學中，我們用水溶液中氫離子的濃度（以莫耳/公升為單位）來衡量溶液的酸鹼值。濃度通常用 10 的負次方來表達，因此 pH 值被定義為該溶液中氫離子濃度的負常用對數。我們用符號「$[H^+]$」表示氫離子的濃度，並用「$pH=-\log[H^+]$」來計算給定水溶液的 pH 值。現在有一杯 pH 值為 3.6 的檸檬水。它的氫離子濃度是多少？

學生： $3.6 = -log[H^+]$。$log[H^+] = -3.6$。因此$[H^+] = 10^{-3.6}$。

老師： 太好了！現在找出小蘇打的氫離子濃度。

學生： $8.6 = -log[H^+]$。$log[H^+] = -8.6$。因此$[H^+] = 10^{-8.6}$。

老師： 很好。現在可以比較這兩個溶液的氫離子濃度了。檸檬水的氫離子濃度比小蘇打大多少？

學生： 將 $10^{-3.6}$ 除以 $10^{-8.6}$。$\frac{10^{-3.6}}{10^{-8.6}}$ 可以用指數律簡化為 $10^{-3.6-(-8.6)} = 10^5 = 100,000$。檸檬水的氫離子濃度是小蘇打的氫離子濃度的 10 萬倍。

老師： 因此，對於含有更高濃度氫離子的溶液，你認為它的 pH 值是更高還是更低？

學生： 更低。以這題為例，擁有較高濃度氫離子的檸檬水具有較低的 pH 值。

老師： 正確！反之亦然，pH 值越低，溶液越酸。

單元五 多項式函數
Polynomial Functions

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■ 前言 Introduction

本單元描述如何以英文講解多項式概念及命名，多項式的加減法。介紹長除法及綜合除法的過程，帶學生觀察兩種除法的關聯性，介紹因式定理、餘式定理及其應用。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
polynomial function	多項式函數	long division	長除法
dividend	**被除式**	**synthetic division**	**綜合除法**
divisor	**除式**	division algorithm	除法原理
quotient	**商式**	**remainder theorem**	**餘式定理**
remainder	**餘式**	**factor theorem**	**因式定理**
descending order	降冪排列	like terms	同類項
distributive property	分配律		

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ _______ divides evenly into _______.

例句：2 **divides evenly into** 6.

6 被 2 整除。

❷ Evaluate the polynomial at _______.

例句：**Evaluate the polynomial** $f(x)$ **at** $x = 2$.

算出 $f(2)$ 的值。

❸ _____ can be written as ______.

例句：The result **can be written in** a fraction.

這個結果可以寫成分數式來表示。

❹ Substitute ___ for *x*.

例句：**Substitute** 6 **for** x in the polynomial.

把多項式裡的 x 代入 6。

❺ Write _____ in factored form.

例句：**Write** this polynomial function **in factored form.**

把這個多項式函數寫成因式分解的形式。

■ 問題講解 Explanation of Problems

說明

[Polynomials and classification]

A polynomial is an algebraic expression made up of variables and coefficients, such as $3x+2y$, $-4m+2n$…etc. A polynomial with the variable x is defined as a polynomial function of x, for example, $f(x) = 3x + 5$, $f(x) = 2x^4 - 3x^2 - 6$ …etc. The standard form of a polynomial function arranges the terms by degrees in descending order. The standard form is

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where n is a nonnegative integer and $a_n, a_{n-1}, \ldots a_0$ are real numbers.

The degree of the function is the degree of the polynomial. We can classify the polynomial by its highest degree. Let's see some examples.

If $f(x) = 5x$, or $f(x) = 5x + 4$, the degree of $5x$ is 1, and the degree of the constant 4 is 0. Therefore, the higher degree is 1. We say that the degree of $f(x)$ is 1. We can write $deg f(x) = 1$. This $f(x)$ is a linear function. The graph of this kind of function is a line. If $f(x) = x^2 - 2x + 1$, or $f(x) = 3x^2$, the highest degree of terms is 2. Therefore, the degree of $f(x)$ is 2. We can write $deg f(x) = 2$. Such $f(x)$ is a quadratic function. The graph of this kind of function is a parabola.

If $deg f(x) = 3$, then $f(x)$ is a cubic function. If $deg f(x) = 4$, then $f(x)$ is a quartic function. If $f(x) = 5$, a constant, then $f(x)$ is still a polynomial. We can regard 5 as $5x^0$, and it fits the standard form.

[Operations of polynomials]

When adding or subtracting polynomial functions, we combine like terms.

We put all x-terms together, all x^2-terms together, all x^n-terms together, and all constants together. Next, we do simple math on their coefficients. Here's an example:

$g(x) = 4x^2 + 5x$, $h(x) = x^2 - 2x + 1$. Let's work on $g(x) + h(x)$. We combine like terms with parentheses and have $(4x^2 + x^2) + (5x - 2x) + 1$. Next, we do work on the coefficients. Solve 4+1, and 5−2, and the answer is $5x^2 + 3x + 1$. You can try $g(x) - h(x)$ on your own. Note that you need to change the sign when subtracting a negative number.

To multiply two polynomials, we have to use the distributive property. Multiply each term in the first polynomial by each term in the second polynomial. The process is like "distributing" the term. Let's work on $g(x) \cdot h(x) = (4x^2 + 5x) \cdot (x^2 - 2x + 1)$

First, we distribute $4x^2$ to each term of the second polynomial, and we get $4x^4 - 8x^3 + 4x^2$. Second, we distribute $5x$ to each term of the second polynomial, and we get $5x^3 - 10x^2 + 5x$. Last, we simplify the polynomial by adding or subtracting like terms.

The answer is $4x^4 + (-8x^3 + 5x^3) + (4x^2 - 10x^2) + 5x = 4x^4 - 3x^3 - 6x^2 + 5x$

Don't forget the common error in multiplication; remember that a negative number times a negative number equals a positive number, while a negative number times a positive number equals a negative number.

[Division of polynomials] Long division

The long division of polynomials is similar to the long division of numbers. We first identify the dividend and divisor, then find the quotient and remainder. We are dealing with polynomials now, so we might have a quotient and a remainder in terms of polynomials. This is the big difference from the division of numbers.

Here is an example: Divide $x^3 - 2x^2 - 9$ by $x - 3$. The dividend is $x^3 - 2x^2 - 9$, which should be written inside the long division sign, in descending order. There is no x-term in the dividend, and we have to use zero or leave a space for the missing term. That's because we need to line up the subtraction for the following steps. The divisor is $x - 3$, which should be written outside the long division sign.

First, focus on the dividend's first term and the divisor's first term. Pretend that you are dividing x^3 by x, and the quotient is x^2. Write x^2 on the top of x^3.

Then multiply $x - 3$ by x^2. Write the product $x^3 - 3x^2$ under the dividend.

You will get:
$$\begin{array}{r|l} & x^2 \\ \hline x-3 & x^3 - 2x^2 + 0x - 9 \\ & \underline{x^3 - 3x^2} \end{array}$$

Subtract $x^3 - 3x^2$ from $x^3 - 2x^2$, you will have $x^3 - 2x^2 - (x^3 - 3x^2) = x^2$. Copy the x-term of the dividend and write it next to x^2. Make sure that there are 2 terms of dividend for the next step division, since there are 2 terms of the divisor. The numbers should be matched.

$$
\begin{array}{r}
x^2 + \\
x-3 \overline{\smash{\big)} x^3 - 2x^2 + 0x - 9} \\
\underline{x^3 - 3x^2 } \\
x^2 + 0x
\end{array}
$$

Now, you are dividing $x^2 + 0x$ by $x - 3$. Focus on the first term of the dividend and the first term of the divisor. Pretend that you are dividing x^2 by x, and the quotient is x. Repeat what we just did. Write the quotient x on the top. Multiply $x - 3$ by x. Write the product $x^2 - 3x$ under the dividend.

Do subtraction $x^2 + 0x - (x^2 - 3x) = 3x$. Copy the number -9, and write it next to $3x$.

You will obtain:

$$
\begin{array}{r}
x^2 + x \\
x-3 \overline{\smash{\big)} x^3 - 2x^2 + 0x - 9} \\
\underline{x^3 - 3x^2 } \\
x^2 + 0x \\
\underline{x^2 - 3x } \\
3x - 9
\end{array}
$$

You are dividing $3x - 9$ by $x - 3$. Also focus on the first term of the dividend and the divisor. Pretend that you are dividing $3x$ by x. Write the quotient 3 on top.

Multiply $x - 3$ by 3, and you will get $3x - 9$. Write $3x - 9$ under the dividend.

Do subtraction $3x - 9 - (3x - 9) = 0$, and 0 is the remainder. So the quotient is $x^2 + x + 3$, and the remainder is 0.

$$
\begin{array}{r}
x^2 + x + 3 \\
x-3 \overline{\smash{\big)} x^3 - 2x^2 + 0x - 9} \\
\underline{x^3 - 3x^2 } \\
x^2 + 0x \\
\underline{x^2 - 3x } \\
3x - 9 \\
\underline{3x - 9} \\
0
\end{array}
$$

The result can be written in fraction form:

$\frac{x^3-2x^2-9}{x-3} = (x^2 + x + 3) + \frac{0}{x-3}$.

Multiply each side by $x - 3$, and you will get:

$x^3 - 2x^2 - 9 = (x^2 + x + 3)(x - 3) + 0.$

Eliminate the 0 and get the clear relationship between the dividend and the divisor:

$x^3 - 2x^2 - 9 = (x^2 + x + 3)(x - 3)$

Since the remainder is 0, we can say that $x - 3$ divides evenly into $x^3 - 2x^2 - 9$. Like 6 is a multiple of 2, and 6 is divisible by 2. We can say that 2 divides evenly into 6. Also, $x^2 + x + 3$ and $x - 3$ are factors of $x^3 - 2x^2 - 9$. We will learn more about the factor later.

I kept 0 in the form because I would like to introduce the general form of the division algorithm. Let's use $f(x)$ for the dividend, $g(x)$ for the divisor, $q(x)$ for the quotient, and $r(x)$ for the remainder. The outcome $x^3 - 2x^2 - 9 = (x^2 + x + 3)(x - 3) + 0$ can be described as $f(x) = g(x)\ q(x) + r(x)$. This is the division algorithm.

[The Division Algorithm]

If $f(x)$ and $g(x)$ are polynomials such that $g(x) \neq 0$, and the degree of $g(x)$ is less than or equal to $f(x)$, then there is only one polynomial for $q(x)$ and only one polynomial for $r(x)$ so that $f(x) = g(x)\ q(x) + r(x)$, where $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $g(x)$.

[Division of polynomials] Synthetic division

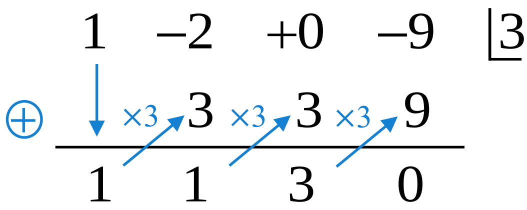
Synthetic division is described as a shortcut of long division. This division only works when the divisor is $x - k$. In order to compare the difference, let me use the last question as an example. Divide $x^3 - 2x^2 - 9$ by $x - 3$.

Write down the coefficients of the dividend in order. If powers of x are missing, you have to put 0 in the dividend. Then identify "*k*". Because the divisor is $x - 3$, the *k* is 3. Set up the array as below. Leave some space between the numbers and the line. We will write some numbers in that space.

$$\begin{array}{cccc|l} 1 & -2 & +0 & -9 & 3 \\ & & & & \\ \hline \end{array}$$

Copy the first number, and write it under the line. Multiply 1 by *k*, which is 3. Then write the product in the diagonal place, see the arrowhead. Add the vertical terms: $-2 + 3 = 1$, and write the sum under the line. Multiply 1 by 3, and put the product in the diagonal place. Do the same steps again.

Add vertical terms: $0 + 3 = 3$, and write the sum under the line. Multiply 3 by 3, and then put the product in the diagonal place. Lastly, add the vertical terms: $-9 + 9 = 0$.



$$\begin{array}{cccc|l} 1 & -2 & +0 & -9 & 3 \\ & 3 & 3 & 9 & \\ \hline 1 & 1 & 3 & 0 & \end{array}$$

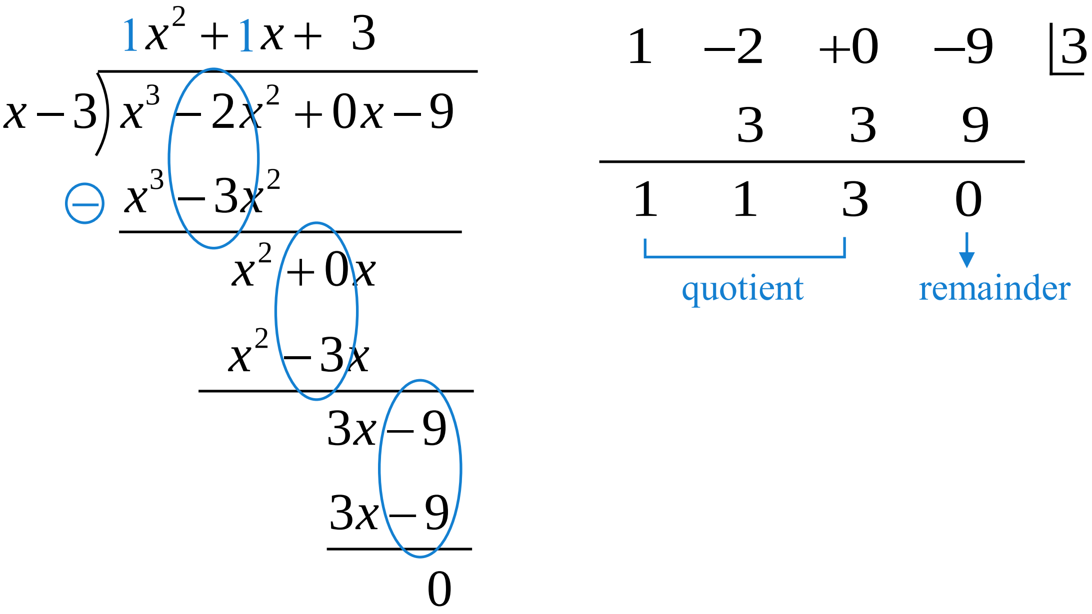
quotient: 1 1 3　　remainder: 0

The last number shows the remainder in the division. The other numbers are coefficients of the quotient. The quotient is $x^2 + x + 3$.

Let's review again. First, you add the vertical terms, which is different from long division. Then, you multiply this sum by "k" and write the product in the diagonal place. Continue until you find the remainder. Which way do you like more?

Most students would like to use synthetic division and not long division because the process looks neat and quick. We only write the coefficients and no x-terms. But synthetic division only works when the divisor is in the form of $x - k$.

Let me show you long division and synthetic division at the same time. What is the relationship between these two ways?



Synthetic division only works on the circled part of the long division.

Numbers are multiplied by "-3" in the long division, but the numbers are multiplied by "$+3$" in the synthetic division. The operations are different: one is subtraction, and the other is addition. However, these differences lead to the same answers: $-2 - (-3) = -2 + 3 = 1,\ 0 - (-3) = 0 + 3 = 3,\ -9 - (-9) = -9 + 9 = 0$. It turns out that 0 is the remainder in both ways.

[Remainder theorem]

Let's work on another example. Please use synthetic division to divide $4x^3 + 10x^2 - 3x + 8$

by $x-1$. You will have the following arrays.

$$\begin{array}{cccc|l} 4 & +10 & -3 & +8 & 1 \\ & 4 & 14 & 11 & \\ \hline 4 & 14 & 11 & 19 & \end{array}$$

19 is the remainder, and the quotient is $4x^2+14x+11$. This polynomial can be written as $4x^3+10x^2-3x+8=(x-1)\left(4x^2+14x+11\right)+19$

Let's name the polynomial $4x^3+10x^2-3x+8$ as $f(x)$ and evaluate $f(1)$. We substitute 1 for x. We will have $4(1)^3+10(1)^2-3(1)+8=19$. Have you noticed that $f(1)=19$ happens to be the remainder?

We have the Remainder Theorem: If a polynomial $f(x)$ is divided by $x-k$, then the remainder is $f(k)$.

From the previous content, we know that $f(x)=g(x)\,q(x)+r(x)$. The divisor is $x-k$, so we can write $f(x)=(x-k)q(x)+r(x)$. Substitute k for x to evaluate $f(k)$, and we have $f(k)=r(x)$. Therefore, the remainder is $f(k)$.

[Factor Theorem]

We know that $x^3-2x^2-9=(x^2+x+3)(x-3)$. $(x-3)$ is a factor of x^3-2x^2-9. Plug in 3 in the polynomial x^3-2x^2-9. We will have $3^3-2\times3^2-9=27-18-9=0$. Let's take another example. $x^2-25=(x+5)(x-5)$, $(x-5)$ is a factor of x^2-25. Plug in 5, and we will have $5^2-25=25-25=0$. This tells us that we can check whether a polynomial has $(x-k)$ as a factor by substituting $x=k$ in the original function. This is the Factor Theorem:

A polynomial $f(x)$ has a factor $(x-k)$, only if $f(k)=0$.

➰ 運算問題的講解 ➰

例題一

說明：練習綜合除法，並由算式中解讀商式與餘式。由此題提醒學生在缺位處應補上 0。

（英文）Use synthetic division to divide $x^4 + 2x$ by $x + 1$. Find the quotient and the remainder.
（中文）求 $x^4 + 2x$ 除以 $x + 1$ 的商式及餘式。

Teacher: First write the array with the coefficients of dividend.
Student: 1, 2.
Teacher: If the powers of x are missing, please put 0 in the dividend to replace the missing terms.
Student: 1, 0, 0, and 2.
Teacher: The constant is 0. You should add 0 at the end.
Student: 1, 0, 0, 2, and 0.
Teacher: Correct. Then identify the k for the divisor. We said that the synthetic division only works when the divisor is in the form of $x-k$.
Student: -1
Teacher: Yes. $x + 1 = x - (-1)$, so k is -1. Please set up the array and try it by yourselves. Remember to add terms in columns and multiply the numbers by -1!
Student:

$$\begin{array}{rrrrr|r} 1 & 0 & 0 & 2 & 0 & -1 \\ & -1 & 1 & -1 & -1 & \\ \hline 1 & -1 & 1 & 1 & -1 & \end{array}$$

Teacher: What is the remainder?
Student: -1
Teacher: What is the quotient?
Student: 1, -1, 1, and 1.
Teacher: They are just coefficients. You have to write the polynomial.
Student: Should I start from x^4 or x^3? I don't know the highest degree of the quotient.

Teacher: Good question. The degree of the dividend is 4, and the degree of the divisor is 1. The degree of the quotient will be the difference of these 2 degrees. 4−1=3, so the degree of the quotient is 3. You should start from $1x^3$.

Student: I get it. The quotient is $x^3 - x^2 + x + 1$.

Teacher: Yes.

老師： 首先，列出被除數的係數。

學生： 1, 2。

老師： 如果缺少 x 的次方項，請在被除數的缺位處補上 0。

學生： 1, 0, 0, 和 2。

老師： 常數項是 0，要在最後加上 0 喔。

學生： 1, 0, 0, 2, 和 0。

老師： 答對了。接著要確定除數的 k 是多少。我們提過，綜合除法只有在除數為 $x-k$ 的形式時才成立。

學生： k 是−1。

老師： 是的，$x+1 = x-(-1)$，因此 k 為−1。請自己試著列式並算算看。記得在直行中進行加法，並將所得的數字都乘以 −1！

學生：

1	0	0	2	0	−1
	−1	1	−1	−1	
1	−1	1	1	−1	

老師： 餘數是多少？

學生： −1。

老師： 商是多少？

學生： 1，−1，1，和 1。

老師： 它們只是係數，你必須寫出多項式。

學生： 我應該從 x^4 開始還是 x^3？我不知道商的最高次項是多少。

老師： 好問題。被除數的次方為 4，除數的次數為 1。商的次數即是這兩個次方的差，4-1=3，因此商的次數為 3。你要從 $1x^3$ 開始寫。

學生： 明白了。商是$x^3 - x^2 + x + 1$。

老師： 沒錯。

例題二

說明：利用因式定理及已知條件找到剩餘的因式。

（英文）Show that $(x+3)$ is a factor of $f(x)=2x^3+3x^2-8x+3$.
Then find the remaining factors of $f(x)$.
（中文）說明 $(x+3)$ 是 $f(x)=2x^3+3x^2-8x+3$ 的因式，並找出剩餘的因式。

Teacher: You can use synthetic division or long division to solve this problem. For those students who choose synthetic division, you have to figure out the correct "k" for the divisor $x-k$. What is the value of k?

Student: $(x+3)=x-(-3)$, so k is -3.

Teacher: Correct. I will give you some time to do division.

Student: I work the synthetic division. This is my answer. The remainder is 0. Therefore $x+3$ is a factor of the polynomial.

$$\begin{array}{rrrr|l} 2 & +3 & -8 & +3 & -3 \\ & -6 & +9 & -3 & \\ \hline 2 & -3 & 1 & 0 & \end{array}$$

Teacher: Correct. Does anyone have a different method?

Student 2: I use long division, but it takes a lot of time.

Student 3: Can I use the factor theorem? I plug in x as -3 in the polynomial: $f(-3)=2(-27)+3(9)-8(-3)+3=0$. According to the factor theorem, $(x+3)$ is the factor.

Teacher: Well, the factor theorem indeed is the fastest way to figure out whether $(x+3)$ is the factor or not. However, the question is not done yet. It is looking for other factors. We use the factor theorem and are happy to get $f(-3)=0$. But, we don't discover new information for the other factors. Both synthetic division and long division leave us some clues. Do you see the quotient?

Student: $2x^2-3x+1$.

Teacher: See if you can factor the quotient.

Student: $x-1$ and $2x-1$.

Teacher: Yes, they are the remaining factors. We can write the polynomial in factored form: $2x^3+3x^2-8x+3=(x+3)(x-1)(2x-1)$

Student: If the quotient cannot be factored, what can I do?

Teacher: The quotient is the remaining factor. If it cannot be factored anymore, leave the answer in the original form.

老師： 這題可以用綜合除法或長除法來算。選擇用綜合除法的學生，你必須找出除數 $x-k$ 中正確的 k。k 的值是多少？

學生： $(x+3)=x-(-3)$，所以 k 為-3。

老師： 正確，現在給大家一些時間來做除法。

學生： 我用了綜合除法，這是我的答案。餘數為 0。因此$(x+3)$是多項式的因數。

$$\begin{array}{rrrr|r} 2 & +3 & -8 & +3 & -3 \\ & -6 & +9 & -3 & \\ \hline 2 & -3 & 1 & 0 & \end{array}$$

老師： 正確。還有其他方法嗎？

學生： 我使用長除法，但需要很長的時間。

學生： 我能用因式定理嗎？如果我把-3 代入多項式中的 x：

$f(-3)=2(-27)+3(9)-8(-3)+3=0$。根據因式定理，$(x+3)$是因式。

老師： 因式定理確實是找出$(x+3)$是否為因式的最快方法。但問題還沒解完，它是要找出其他因式。我們使用因式定理，得到了$f(-3)=0$，但沒辦法找到其他因式。

我們可以看綜合除法和長除法，都留下了一些線索。你看到商了嗎？

學生： $2x^2-3x+1$

老師： 看看你能否因式分解。

學生： $x-1$ 和 $2x-1$。

老師： 答對了，它們就是剩餘的因式。

我們可以將多項式寫成因式形式：$2x^3+3x^2-8x+3=(x+3)(x-1)(2x-1)$

學生： 如果商無法因式分解，該怎麼辦？

老師： 那這個商就是剩下的因式。如果它不能再被分解，答案就保留原始形式。

應用問題 / 學測指考題

例題一

說明： 即使多項式未知，教學生可從其除以$(x-a)$及$(x-b)$的餘式，得知該多項式除以$(x-a)(x-b)$的餘式。

（英文）The remainder is 7 when $f(x)$ is divided by $x-2$, and the remainder is -13 when $f(x)$ is divided by $x+3$.
Find the remainder when $f(x)$ is divided by $(x-2)(x+3)$.

（中文）已知 $f(x)$ 除以 $x-2$ 的餘式為 7，$f(x)$ 除以 $x+3$ 的餘式為 -13，求$f(x)$ 除以 $(x-2)(x+3)$ 的餘式。

Teacher: We can write $f(x)=(x-2)(x+3)\,q(x)+r(x)$ after reading the last sentence. $q(x)$ is the quotient and $r(x)$ is the remainder. The divisor is $(x-2)(x+3)$, and its degree is 2. According to the division algorithm, the degree of the remainder is less than the degree of the divisor. So, the degree of $r(x)$ is at most 1. We can assume that the $r(x)$ is $ax+b$.

Student: What if the degree is actually 0? Can you assume $r(x)$ is just a constant, like a?

Teacher: $ax+b$ is good for these two outcomes. If a is 0, then the remainder is a constant. If a is not 0, then the reminder is a polynomial with degree 1. We can have some flexibility when making this assumption.

Student: Okay.

Teacher: We can assume that $f(x)=(x-2)(x+3)\,q(x)+(ax+b)$. Let’s start with the first information: the remainder is 7 when $f(x)$ is divided by $x-2$. How do you find the remainder without really doing a division?

Student: I forget.

Teacher: The remainder theorem. If a polynomial $f(x)$ is divided by $x-k$, then the remainder is $f(k)$. So what is the remainder?

Student: $f(2)$?

Teacher: Yes. The remainder theorem tells us that $f(2)=7$ is the remainder. What does the other information tell you?

Student: $f(-3)=13$

Teacher: Now plug in the numbers into $f(x)=(x-2)(x+3)\,q(x)+(ax+b)$.

Student: $f(2)=0+2a+b$, and $f(-3)=0-3a+b$.

Teacher: You will have a system of linear equations $\begin{cases} f(2)=2a+b=7 \\ f(-3)=-3a+b=13 \end{cases}$.

Please solve for a and b.

Student: a is 4, and b is -1.

Teacher: What is the remainder?

Student: $4x-1$.

Teacher: Correct.

老師： 根據題目最後一句，我們可以列出等式 $f(x)=(x-2)(x+3)\,q(x)+r(x)$。其中 $q(x)$ 為商、$r(x)$為餘式、被除式為 $(x-2)(x+3)$，其次數為 2。
根據除法原理，餘式的次數必小於除數的次數。因此，$r(x)$ 的次數最多為 1，我們可以假設 $r(x)$ 為 $ax+b$。

學生： 如果次數實際上是 0 呢？你能假設 $r(x)$就是常數，像是 a 嗎？

老師： $ax+b$ 適用於這兩種情況喔。如果a為 0，那麼餘式就是一個常數。
如果 a 不為 0，那麼餘式就是次數為 1 的多項式。這個假設具有彈性。

學生： 了解。

老師： 列出假設 $f(x)=(x-2)(x+3)\,q(x)+(ax+b)$後，從題目第一句開始：$f(x)$ 除以 $x-2$ 的餘式為 7。要怎麼樣在不真正進行除法的情況下找到餘式？

學生： 我忘記了。

老師： 可以使用餘式定理。如果多項式 $f(x)$ 除以 $x-k$,，那麼餘式就會是 $f(k)$。
那麼這題餘式是多少？

學生： $f(2)$？

老師： 是的。餘式定理告訴我們，$f(2)=7$是餘式。其他資訊告訴你什麼？

學生： $f(-3)=13$。

老師： 現在將數字代入 $f(x)=(x-2)(x+3)\,q(x)+(ax+b)$ 。

學生： $f(2)=0+2a+b$、$f(-3)=0-3a+b$。

老師： 會得到一個線性方程組 $\begin{cases} f(2)=2a+b=7 \\ f(-3)=-3a+b=13 \end{cases}$。請解 a 和 b。

學生： $a=4$，$b=-1$。

老師： 所以餘式是多少？

學生： $4x-1$

老師： 正確。

例題二

說明：已知正四角錐的體積及高，教學生利用綜合除法求出剩餘因式，並因式分解得底面積的邊長。

（英文）The expression $4x^3+16x^2+21x+9$ is the volume of a square pyramid. The expression $3x+3$ is the height of the pyramid. What is the side length of the base?

（中文）$4x^3+16x^2+21x+9$ 是一正四角錐的體積，$3x+3$ 為正四角錐的高，請為底面的邊長為何？

Teacher: The formula for the volume of a pyramid is $\frac{1}{3}\times$ base area $\times$ hight. You can write the expression with the given information:

$$4x^3+16x^2+21x+9=\frac{1}{3}\cdot\text{base area}\cdot(3x+3)$$

Student: You can cross out 3.

Teacher: Okay. You will have $4x^3+16x^2+21x+9=\text{base area}\cdot(x+1)$. What's the next step to find out the base area?

Student: Move $(x+1)$ to the other side.

Teacher: Precisely, we divide both sides by $(x+1)$.
So the base area is $(4x^3+16x^2+21x+9)\div(x+1)$. You can do synthetic division again. Please tell me the quotient.

Student: I got it. The quotient is $4x^2+12x+9$.

Teacher: The base of the square pyramid is a square. To find the side length, you can factor the quotient directly or use the formula $(ax+b)^2$ to obtain the coefficients.

Student: $(2x+3)(2x+3)$

Student: The length of the side is $2x+3$.

Teacher: Correct.

老師：四角錐的體積公式為 $\frac{1}{3}\times$ 底面積 $\times$ 高。從題目給的條件，可以寫出以下表達式：

$4x^3+16x^2+21x+9=\frac{1}{3}\cdot$ 底面積 $\cdot(3x+3)$。

學生：3 可以消掉。

老師：好的，這樣變成$4x^3+16x^2+21x+9=$ 底面積 $\cdot(x+1)$。
接下來要算底面積的話，下一步是什麼？

學生：　把 $(x + 1)$ 移到另一邊。

老師：　比較精確的說法是，兩邊同除以$(x + 1)$。

因此，底面積為$(4x^3 + 16x^2 + 21x + 9) \div (x + 1)$。

現在大家再練習一次綜合除法，告訴我商是多少。

學生：　$4x^2 + 12x + 9$。

老師：　四角錐的底面是一個正方形。要找到邊長，直接對商進行因式分解，也可以使用公式 $(ax + b)^2$ 來求係數。

學生：　$(2x + 3)(2x + 3)$，邊長是$(2x + 3)$。

老師：　沒錯，答對了！

單元六 多項式函數及其圖形
Polynomial Functions and Graphs

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■ 前言 Introduction

本單元描述如何以英文講解函數圖形（二次，三次），將二次函數以配方法寫成頂點式，找出圖形的頂點、對稱軸及極值。並以二次函數圖形為例，說明圖形平移與方程式的關係。最後討論三次函數圖形的特徵。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
quadratic function	二次函數	parabola	拋物線
vertex	**頂點**	**axis of symmetry**	**對稱軸**
complete the square	**配方法**	perfect square trinomial	完全平方式
parent function	母函數	**vertical shift**	**垂直平移**
horizontal shift	**水平平移**	ordered pairs	數對
discriminant	判別式		

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ Write ______ by completing the square.

例句：**Write** the function in vertex form **by completing the square.**

用配方法把方程式寫成頂點形式。

❷ Factor _____ out of ______.

例句：**Factor** 3 **out of** the expression $(3x^2-9)$.

把 $(3x^2-9)$ 的 3 因式分解出來。

❸ Shift ______ units _______.

例句：Please vertically **shift** the graph 5 **units** upward.

請把圖形垂直向上移動五個單位。

❹ ____ greater than ____ by _____.

例句：A is **greater than** B **by** 2.

A 比 B 大 2。

■ 問題講解 Explanation of Problems

說明

The general form of a polynomial function is

$$f(x)=a_nx^n+a_{n-1}x^{n-1}+\dots+a_1x+a_0$$

where n is a nonnegative integer and $a_n, a_{n-1},\dots a_0$ are real numbers.

In the last section, we learned to classify the polynomial by its degree.

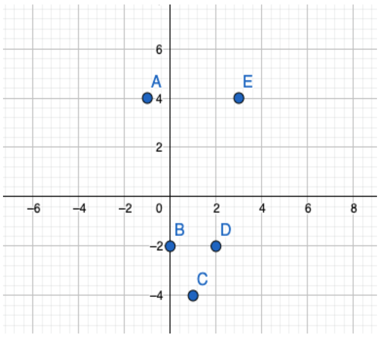
When the degree of $f(x)$ is 1, $f(x)$ is a linear function. The graph of this kind of function is a line. When the degree of $f(x)$ is 2, $f(x)$ is a quadratic function. The graph of this kind of function is a parabola. When the degree of $f(x)$ is 3, $f(x)$ is a cubic function. The graph of this kind of function is a curve with some ups and downs. When the degree of $f(x)$ is 0, for example $f(x) = 3$, the graph of this kind of function is a horizontal line.

[Graphs of Quadratic function] vertex, the axis of symmetry, minimum or maximum of the function

If $f(x) = ax^2 + bx + c$, $a \neq 0$, $f(x)$ is a quadratic function. The graph of a quadratic function is a parabola. If $a>0$, the parabola opens upward, and if $a<0$, the parabola opens downward.

The quadratic function can be written as $f(x) = a(x-h)^2 + k$, the vertex form of a quadratic function. In some textbooks, $f(x) = a(x-h)^2 + k$ is called the "standard form." With this form, we can easily tell the equation of the axis of symmetry and the coordinates of the vertex. Let's take $f(x) = 2(x-1)^2 - 4$ as an example. Plug in x as $-1, 0, 1, 2, 3$, and find the corresponding y values. We will have A$(-1, 4)$, B$(0, -2)$, C$(1, -4)$, D$(2, -2)$, E$(3, 4)$, and these points are on the parabola.

Let's plot these points on GeoGebra, and observe the parabola.



Can you see the shape of this parabola? C$(1, -4)$ is the vertex of the parabola. The A line vertically passing through point C equally divides the parabola into two halves. This line is the axis of symmetry, and its equation is $x = 1$.

In the vertex form $f(x) = a(x-h)^2 + k$, the equation of the axis of symmetry is $x = h$. The point where the axis of symmetry intersects the parabola is the vertex of the parabola, so the x-coordinate of the vertex must be h. When plugging in x as h, the corresponding y value is k. The coordinates of the vertex are (h, k).

In the vertex form $f(x) = a(x-h)^2 + k$, $(x-h)^2$ must be positive or 0. When the parabola opens upward, the leading coefficient $a>0$, so $a(x-h)^2$ must be ≥ 0.

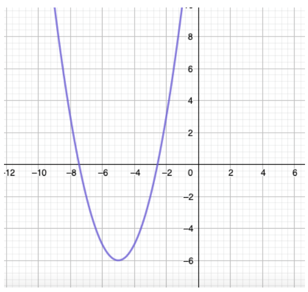
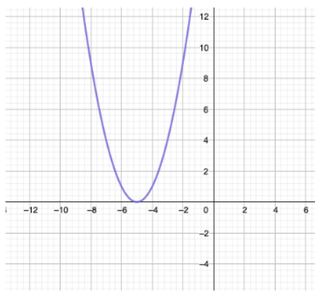
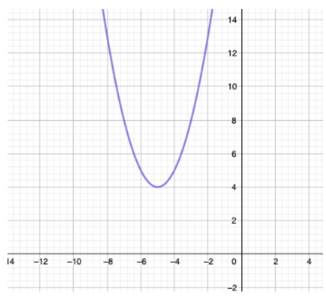
Then $f(x) = a(x-h)^2 + k$ must be $\geq k$. Therefore, k is the minimum value of the function. The parabola keeps going up and up on both sides, so we can't identify the maximum of the function. When the parabola opens downward, the leading coefficient $a<0$, so $a(x-h)^2$ must be ≤ 0. Then $f(x) = a(x-h)^2 + k$ must be $\leq k$. So k is the maximum value of the function. But we cannot identify the minimum value of the function, since the parabola goes down on both sides.

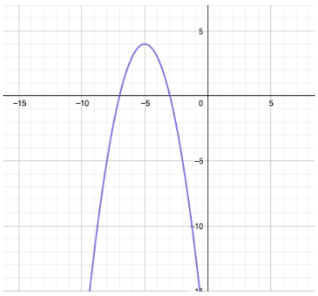
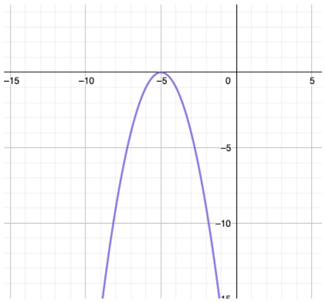
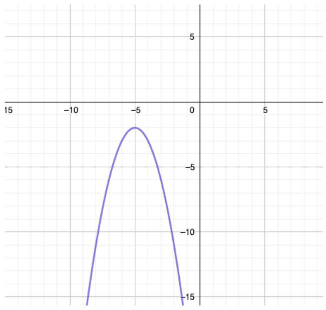
Instead of writing the vertex form, we can also figure out the maximum or minimum from the general form $f(x) = ax^2 + bx + c$.

We can rewrite the general form to $f(x) = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$ by completing the square. The formula for the vertex is $\left(-\frac{b}{2a}, \left(c - \frac{b^2}{4a}\right)\right)$. The y-coordinate $\left(c - \frac{b^2}{4a}\right)$ indicates the maximum or minimum value of the function. You can compare the coefficients and directly plug a, b, and c into the formula. The value of a determines the direction of the parabola. Then you can know whether your answer is the maximum or the minimum.

[Discriminant and graphs]

The coordinates of the vertex are $\left(-\frac{b}{2a}, \left(-\frac{b^2-4ac}{4a}\right)\right)$. We can tell the number of intersections between the quadratic function and the x-axis by the value of $b^2 - 4ac$. "$b^2 - 4ac$" is a discriminant, and noted as D. The value of D and the leading coefficient a can determine the y-coordinate of the vertex. We can tell whether the vertex is above the x-axis, below the x-axis, or just on the x-axis. Additionally, the leading coefficient a tells us the direction of the parabola. Then we can sketch the parabola with this information.

	D>0	D = 0	D<0
a>0 Parabola opens upward	$\left(-\frac{D}{4a}\right)<0$ → y-coordinate is below the x-axis. → vertex is below the x-axis. The graph can be: 	$\left(-\frac{D}{4a}\right) = 0$ → y-coordinate is 0. →vertex is on the x-axis. The graph can be: 	$\left(-\frac{D}{4a}\right)>0$ → y-coordinate is above the x-axis. → vertex is above the x-axis. The graph can be: 

$a<0$ Parabola opens downward.	$\left(-\frac{D}{4a}\right)>0$ → y-coordinate is above the x-axis. → vertex is above the x-axis. The graph can be: 	$\left(-\frac{D}{4a}\right)=0$ → y-coordinate is 0. → vertex is on the x-axis. The graph can be: 	$\left(-\frac{D}{4a}\right)<0$ → y-coordinate is below the x-axis. → vertex is below the x-axis. The graph can be: 
Conclusion	When $D>0$, there are 2 intersections between the parabola and the x-axis.	When $D=0$, there is 1 intersection between the parabola and the x-axis.	When $D<0$, there is no intersection between the parabola and the x-axis.

[Transformations] Translation

Shifts, stretches, shrinks, and reflections are called transformations. In this section, we will introduce translation.

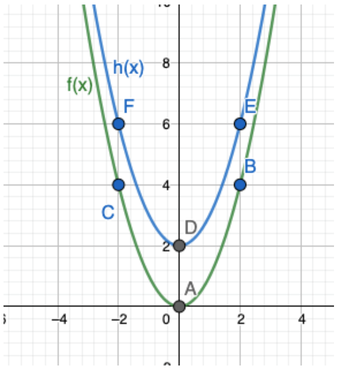
The movement of shifting the whole graph is called translation. It seems that you copy the graph of a function and paste it somewhere else. Everything in the graph stays the same, except the location. Usually, we start with the parent function when introducing a transformation. The parent function is the simplest function of a given family of functions. The parent function of a linear function is $f(x)=x$; the parent function of a quadratic function is $f(x)=x^2$; the parent function of a cubic function is $f(x)=x^3$.

The graph of a function is related to the family of graphs. Sometimes, the graphs are simple transformations of the graph of the parent function.
Take $f(x)=x^2$ and $h(x)=x^2+2$ for example. Graph these two functions on GeoGebra.

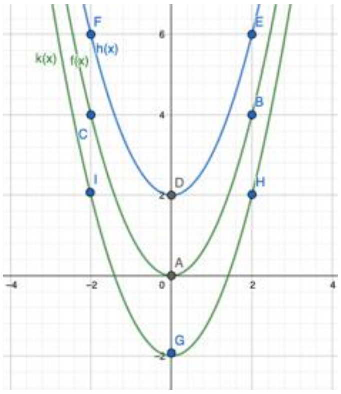
$f(x) = x^2$ is the parent function.

We can get the graph of $h(x) = x^2 + 2$ by shifting $f(x)$ 2 units upward. We can use a table to analyze what happens to individual ordered pairs. Choose 3 points $x = -2, 0, 2$, then plug in $f(x)$ and $h(x)$.

Their corresponding $f(x)$ and $h(x)$ are listed below. $h(x)$ is greater than $f(x)$ by 2. The movements of the points are recorded in the last column.

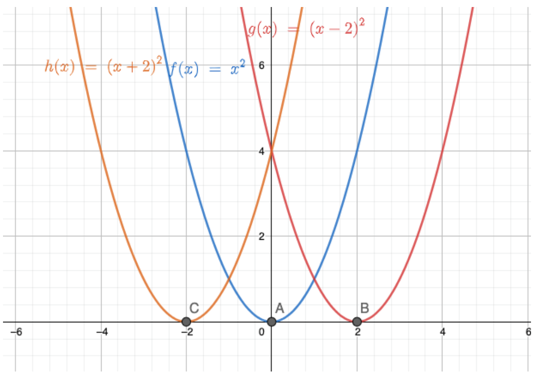


x	$f(x)$	$h(x)$	Shift of points
-2	4	6	C(-2,4) → F(-2,6)
0	0	2	A(0,0) → D(0,2)
2	4	6	B(2,4)→E(2,6)



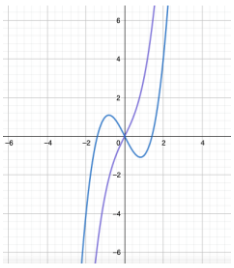
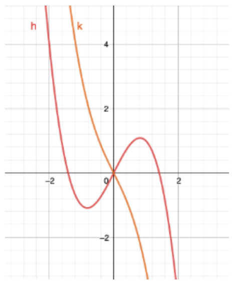
Please graph $k(x) = x^2 - 2$ on the plane, and observe the transformations between $k(x)$ and $f(x)$. We can get the graph of $k(x) = x^2 - 2$ by shifting $f(x)$ 2 units downward. We can find the effect of c on the graph $f(x) = x^2 + c$ graphically and numerically. The value of c makes the vertical shift upward or downward. If c is positive, the graph of the parent function moves c units upward. If c is negative, the graph of parent function moves c units downward.

Next, compare the graphs of $f(x) = x^2$, $h(x) = (x+2)^2$and $g(x) = (x-2)^2$ on GeoGebra. By shifting the graph $f(x) = x^2$ two units to the left, we obtain the graph $h(x) = (x+2)^2$. By shifting the graph $f(x) = x^2$ two units to the right, we obtain the graph $g(x) = (x-2)^2$. In the form $f(x) = (x+c)^2$, the c value in the parentheses makes the graph of the parent function horizontally shift to the left or to the right. If c is positive, the graph of the parent function moves c units to the left. If c is negative, the graph of the parent function moves c units to the right. You can also numerically check the translation by listing numbers in a table.



[Graphs of cubic function]

We are going to observe some graphs of a cubic function. Please graph the functions on GeoGebra. Please focus on the end behavior of the functions. End behavior is the graph of the function when x approaches ∞ or $-\infty$.

Scenario	Cubic function and the leading coefficients are positive.	Cubic function and the leading coefficients are negative.
Examples	Graph $f(x) = x^3 + 2x$ and $g(x) = x^3 - 2x$ on GeoGebra.	Graph $f(x) = -x^3 + 2x$ and $g(x) = -x^3 - 2x$ on GeoGebra.
		
Characteristics of the graph	1. Both graphs fall to the left and eventually rise without bounds as x moves to the right. They have opposite end behaviors. 2. $f(x) \to \infty$ as $x \to \infty$ $f(x) \to -\infty$ as $x \to -\infty$	Both graphs rise to the left and eventually fall without bounds as x moves to the right. They have opposite end behaviors. $f(x) \to -\infty$ as $x \to \infty$ $f(x) \to \infty$ as $x \to -\infty$

Can you explain why the negative leading coefficient makes $f(x)$ fall to the right and the positive leading coefficient makes $f(x)$ rise to the right? Discuss this with your classmates.

Later, you can try to extend this table to polynomials with a higher degree. Graph the functions and observe the characteristics of the graphs. You can find that the degree of the polynomial function (even or odd) and the leading coefficient can determine the end behavior of the function.

運算問題的講解

例題一

說明：教學生使用配方法，將二次函數的一般式改寫成標準式。從方程式中找到頂點坐標、對稱軸及極值。

（英文）For $f(x) = 2x^2 - 8x + 11$, find the coordinates of the vertex, and the equation of axis of symmetry, minimum or maximum of the function.

（中文）針對 $f(x) = 2x^2 - 8x + 11$，找出頂點坐標，對稱軸方程式，及函數的最大或最小值。

Teacher: Write the quadratic function in vertex form by completing the square.

Student: I forgot the first step.

Teacher: Factor out the coefficient of x^2.

Student: The coefficient is 2.

Teacher: Now factor 2 out of x-terms.

Student: $2(x^2 - 4x)$.

Teacher: We have to insert a constant into the parentheses to complete the square. Use the coefficient of x-term. $-\frac{4}{2} = -2$, $(-2)^2 = 4$. Add and subtract 4 within the parentheses to complete the square.

Student: $2(x^2 - 4x + 4 - 4)$. Why do we write $4 - 4$ in the parentheses? They can be crossed out.

Teacher: Please don't cross them out. We need the $+4$ to complete the "perfect square trinomial." The -4 is used to keep the original function. Let's regroup the first three terms, and release the last term out of the parenthesis.

We have $2(x^2 - 4x + 4 - 4) + 11 = 2(x^2 - 4x + 4) - 2 \times 4 + 11 = 2(x^2 - 4x + 4) + 3$. Can you write $(x^2 - 4x + 4)$ into a perfect square?

Student: $x^2 - 4x + 4 = (x - 2)^2$

Teacher: Correct. $2(x^2 - 4x + 4) + 3 = 2(x - 2)^2 + 3$. It is the vertex form. Can you tell the vertex and axis from the function?

Student: Yes. $(x - 2) = 0$ is the axis of symmetry. Plug in x as 2, and the y-value is 3. The coordinates of the vertex are (2, 3).

Teacher: Correct. The extrema of the function are related to the direction of the parabola. If the parabola opens upward, you can find the minimum. Otherwise, you can find the maximum.

Student: Do I have to graph the function? I am tired of plotting points.

Teacher: The graph helps you visualize the answer. Well, actually, you can get the information without graphing. Just check the leading coefficient of the function. Is it positive or negative?

Student: Positive.

Teacher: The parabola opens upward since the leading coefficient is positive. You can find the minimum of the function by the vertex.

Student: The minimum is (2, 3).

Teacher: The minimum of the function is a y-value, not a coordinate point. All numbers on the function are ≥ 3. So we say that the minimum value of the function is 3.

老師： 將二次函數改寫成標準式，請透過配方法來完成。

學生： 我忘了第一步怎麼做了。

老師： 先把 x^2的係數提出來。

學生： 係數是 2。

老師： 接下來把 x 項的 2 因數提出來。

學生： $2(x^2 - 4x)$。

老師： 我們需要在括號中插入一個常數，以完成配方。

使用 x 項的係數，$-\frac{4}{2} = -2$，$(-2)^2 = 4$。在括號中加 4 減 4 以完成配方。

學生： $2(x^2 - 4x + 4 - 4)$。為什麼要在括號中寫「$4 - 4$」？它們可以相減然後等於 0。

老師： 請不要消去它們。我們需要+4以完成「完全平方式」，−4則是用來保留原始函數。讓我們把前三項重新分組，並將最後一項從括號中提出來。

完整寫出來：

$2(x^2 - 4x + 4 - 4) + 11 = 2(x^2 - 4x + 4) - 2 \times 4 + 11 = 2(x^2 - 4x + 4) + 3$。

你能把$(x^2 - 4x + 4)$寫成完全平方式嗎？

學生： $x^2 - 4x + 4 = (x - 2)^2$。

老師： 沒錯。$2(x^2 - 4x + 4) + 3 = 2(x - 2)^2 + 3$，這就是標準式。你能從函數中判斷出頂點和對稱軸嗎？

學生： $(x - 2) = 0$是對稱軸。當 x 等於 2 時，y 的值則會是 3，頂點的坐標是(2, 3)。

老師： 正確。函數的極值與拋物線的開口方向有關。如果拋物線開口朝上，可以找到最小值；相反地，可以找到最大值。

學生： 我必須畫圖嗎？描點好累喔。

老師： 畫圖有助於讓答案視覺化。不過實際上，你也可以不用畫圖得出答案。

只需檢查函數的首項係數，它是正數還是負數？

學生：是正數。

老師：由於首項係數為正，所以拋物線開口會向上。你可以通過頂點找到函數的最小值。

學生：最小值是(2, 3)。

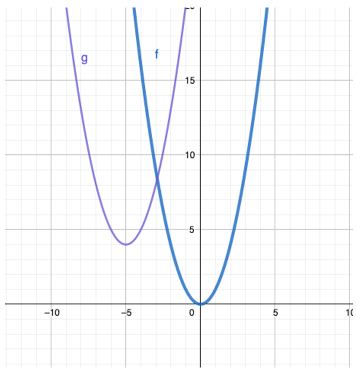
老師：函數的最小值是一個 y 值，而不是一個坐標點。函數中的所有數字都 ≥3。因此，我們說函數的最小值是 3。

例題二

說明：從圖形中辨別平移的方向與單位，找出平移後的函數。

（英文）The graph of $f(x) = x^2$ and $g(x)$ is shown below. $g(x)$ is a translation of $f(x)$. Find the equation of the $g(x)$.

（中文）以下是 $f(x) = x^2$ 和 $g(x)$ 的函數圖形，$g(x)$ 是 $f(x)$ 平移後所得到的結果，請找出 $g(x)$ 的方程式。



Teacher: $g(x)$ is a translation of $f(x)$. Which is the parent function?

Student: $f(x)$

Teacher: Correct. Now we have to observe the movement from $f(x)$ to $g(x)$. How about plotting some points and checking their movement?

Student: Can I choose the vertex?

Teacher: It is a good idea. The movement between vertexes is easy to see. Let's name V_1 for the vertex of $f(x)$, and V_2 for the vertex of $g(x)$. Find their coordinates.

Student: V_1 is (0, 0) and V_2 $(-5, 4)$.

Teacher: How does V_1 move to V_2?

Student: Move 5 units to the left, and then move 4 units up.

Student: Does the order matter? I think that it first moves 4 units up, and then moves to the left.

Teacher: The order is not important. The equation shows the final location of g. The graph of g is a horizontal shift of 5 units to the left and a vertical shift of 4 units upward. We are going to the write the equation.

Do you remember the c value in $f(x) = x^2 + c$? The c determines the vertical change. Moving 4 units up means that c is $+4$.

So far, we know that $g(x) = x^2 + 4$. Next, we need to figure out the horizontal translation. The c value in the parentheses of $f(x) = (x + c)^2$ determines the horizontal change. What is the c value?

Student: Moving 5 units to the left, so c is $+5$.

Teacher: Can you combine these translations and write the equation for g?

Student: $g(x) = (x + 5)^2 + 4$

Teacher: Great.

老師： $g(x)$是 $f(x)$的平移。哪一個是母函數？

學生： $f(x)$。

老師： 正確。現在我們要觀察$f(x)$到$g(x)$的移動。要不要先畫幾個點，觀察函數移動的方式？

學生： 我可以選擇頂點嗎？

老師： 好主意，頂點之間的移動很容易看出來。讓我們暫時命名 $f(x)$的頂點為 V_1，$g(x)$ 的頂點為V_2。找出它們的坐標。

學生： V_1 是 (0, 0)，V_2是 (-5, 4)。

老師： V_1 是如何移動到 V_2 的？

學生： 先向左移 5 個單位，然後向上移 4 個單位。

學生： 順序重要嗎？我認為它先向上移 4 個單位，然後再向左移。

老師： 順序不重要。方程式顯示 $g(x)$ 的最終位置。$g(x)$ 的圖形是向左平移 5 個單位、向上平移 4 個單位。現在我們現在要寫出方程式。

你們還記得 $f(x) = x^2 + c$ 中的 c 值嗎？c 代表垂直方向的變化。向上移動 4 個單位，也就表示 c 是 +4。

到目前為止，可以列出 $g(x) = x^2 + 4$。接下來，我們需要找出水平平移。在$f(x) = (x + c)^2$中，括號內的 c 值是水平變化。c 值是多少？

學生： 向左移動 5 個單位，所以 c 是+5。

老師： 你能將這些平移結合，寫出$g(x)$ 的方程式嗎？

學生： $g(x) = (x + 5)^2 + 4$。

老師： 非常好。

例題三

說明：引導學生練習歸納出三次函數圖形的特徵

（英文）Describe the end behaviors of the graph of $f(x) = -x^3 + 4x - 6$.
（中文）描述 $f(x) = -x^3 + 4x - 6$ 函數圖形在左端點與右端點的圖形特徵。

Teacher: The end behavior of a function refers to how its graph behaves as x approaches positive infinity (∞) or negative infinity ($-\infty$). Can someone explain the end behaviors?

Student: Well, for a cubic function, the end behaviors are opposite. But I can't remember which side goes up and which side goes down.

Teacher: That's okay. We can figure it out by plugging in large numbers for x. Let's start with x approaching positive infinity. Choose a positive number and put it into the function. What do you get?

Student: I'll pick 10. So when I calculate $f(x)$, I get -966.

Teacher: Good. Now, choose an even bigger positive number and find $f(x)$ again.

Student: I'll go with 100. That gives me -999606.

Teacher: Now, imagine plugging in a really huge number for x. Would $f(x)$ be a really big or really small number? And is it positive or negative?

Student: Based on the pattern, $f(x)$ seems to be a negative number like $-99999999...$ with lots of digits.

Teacher: Exactly. It approaches negative infinity ($-\infty$). The right side of the graph goes down indefinitely. Now let's consider x approaching negative infinity.

Student: For x approaching $-\infty$, let's use -99999.

Teacher: What would $f(x)$ be if we substitute $x = -99999$ into the function?

Student: $f(x) = -(-99999)^3 + 4(99999) - 6 = ...$

Teacher: Would it be a positive number or a negative number?

Student: It would be a positive number, and a really large one!

Teacher: Right. It approaches positive infinity (∞). The left side of the graph goes up indefinitely.

老師：「極限行為」是指當 x 趨近於正無限大或負無限大時，函數的圖像所呈現的趨勢。有人可以描述一下這個函數的極限行為嗎？

學生：這是一個三次函數，所以它有相反的極限行為。但我忘了哪一側是向上，哪一側是向下。

老師：我們可以代入數字。當 x 趨近於正無限大時，x 必須是一個很大的正數。你可以選一個正數，然後代入 $f(x)$。你得到什麼？

學生：我選 10，$f(x) = -1000 + 40 - 6 = -966$。

老師：很好。現在選一個更大的正數，算出 $f(x)$。

學生：我代入 100，$f(x) = -1000000 + 400 - 6 = -999606$。

老師：想像一下，如果你代入一個超級大的數字，$f(x)$ 的值會是多少？是一個超級大的數字還是一個超級小的數字？是一個正數還是一個負數？

學生：照這個模式，我覺得 $f(x)$ 是一個負數。$f(x)$ 可能是 -99999999……，有很多位數。

老師：它會趨近於負無限大。如果你把這個函數圖形畫出來，圖形的右側會無限下降。告訴我如何找出左側的極限行為？

學生：當 x 趨近於負無限大時，x 可以是一個例如 -99999 的數字。

老師：如果你將 -99999 代入 $f(x)$，$f(x)$ 的值是多少？

學生：$f(x) = -(-99999)^3 + 4(99999) - 6 = \ldots\ldots$

老師：它會是正數還是負數？

學生：是一個正數，而且很大！

老師：它會趨近於正無限大。如果把這個函數圖形畫出來，圖形的左側會無限上升。

應用問題 / 學測指考題

例題一

說明：利用二次函數的觀念，解決日常生活的數學問題。透過配方法找出二次函數的極值。

（英文）The path of a diver is approximately $y = -\frac{1}{3}x^2 + \frac{8}{3}x + \frac{20}{3}$, where y is the height above water level, and x is the horizontal distance from the end of the diving board. What is the maximum height of the diver?

（中文）跳水者路徑的方程式為 $y = -\frac{1}{3}x^2 + \frac{8}{3}x + \frac{20}{3}$，其中 y 是距離水平面的高度，x 是距離跳水板前端的水平距離。請問跳水者離水平面的最高距離為何？

Teacher: Let's analyze the function first. It is a polynomial with a degree of 2. What does the graph look like?

Student: A parabola, the mouth opens downward.

Teacher: Why is it downward?

Student: The leading coefficient is a negative number.

Teacher: Great. The function shows the path of the diver. The maximum height of the diver is the maximum of the function. Where is the maximum?

Student: The vertex of the parabola.

Teacher: Good. We can rewrite the function in the vertex form $f(x) = a(x-h)^2 + k$, by completing the square. You have a few minutes to do it.

Student: I don't like fractions. How should I start?

Teacher: We can factor out $\frac{1}{3}$ or $\frac{-1}{3}$ from the polynomial. Each term has $\frac{1}{3}$ as the factor. If you factor out $\frac{-1}{3}$, don't forget to change the operation sign when necessary. You will have $-\frac{1}{3}(x^2-8x)+\frac{20}{3}$.

Student: Why don't you factor $\frac{20}{3}$? It has $\frac{1}{3}$ as the factor.

Teacher: I will add and subtract a number in the parenthesis in order to make a perfect square trinomial with x^2 and x. I do not want to get confused, so I decide to leave $\frac{20}{3}$ out of the parentheses. This fraction is not helping right now. $\frac{8}{2}=4$, which number should I add in the parentheses to complete the square?

Student: 4 squared. 16!

Teacher: I will have $-\frac{1}{3}(x^2-8x+16-16)+\frac{20}{3}$. Remember to subtract 16. We do this to keep the original function.

I regroup the function as $-\frac{1}{3}(x^2-8x+16)+(-\frac{1}{3})(-16)+\frac{20}{3}$. Please remember to multiply -16 by the factor $-\frac{1}{3}$. Simplify the function.

Student: $-\frac{1}{3}(x^2-8x+16)+\frac{16}{3}+\frac{20}{3}=-\frac{1}{3}(x-4)^2+\frac{36}{3}=-\frac{1}{3}(x-4)^2+12.$

Teacher: What is the vertex?

Student: (4, 12)

Teacher: What is the maximum height of the diver?

Student: 12.

Teacher: Great. How do we interpret the ordered pair (4, 12)? We would say that the diver reaches the maximum height when he is 4 units away from the end of the diving board.

Student: You taught us the formula of vertex, $\left(-\frac{b}{a}, \left(c-\frac{b^2}{4a}\right)\right)$, can we use that?

Teacher: Sure. Let's compare the functions and find a, b, and c. $y=-\frac{1}{3}x^2+\frac{8}{3}x+\frac{20}{3}$, so $a=-\frac{1}{3}$, $b=\frac{8}{3}$, and $c=\frac{20}{3}$. Can you calculate $\left(c-\frac{b^2}{4a}\right)$?

Student: $\frac{20}{3}-\frac{\left(\frac{8}{3}\right)^2}{4\left(-\frac{1}{3}\right)}=\frac{20}{3}+\frac{16}{3}=12$. The same answer.

老師：　讓我們先分析這個函數。它是一個二次多項式函數，圖形長怎樣呢？

學生：　是一個開口向下的拋物線。

老師：　為什麼開口向下？

學生：　因為首項係數是負數。

老師：　很好，函數顯示了跳水員的路徑，跳水員的最高高度是函數的最大值。最大值在哪裡？

學生：　在拋物線的頂點。

老師：　很棒。我們可以通過完成配方來把函數寫成標準式 $f(x)=a(x-h)^2+k$。給你們幾分鐘的時間。

學生：　我不喜歡分數，我應該怎麼開始？

老師：　我們可以從多項式中提取出 $\frac{1}{3}$ 或 $\frac{-1}{3}$，每個項都有 $\frac{1}{3}$ 這個因數。如果你是提取出 $\frac{-1}{3}$，不要忘記改變正負號。因此會得到 $-\frac{1}{3}(x^2-8x)+\frac{20}{3}$。

學生：　為什麼 $\frac{20}{3}$ 不用動？它也有 $\frac{1}{3}$ 這個因數。

老師：　我會加上一個數字並從括號中減去一個數字，以使 x^2 和 x 組成一個完全平方三項式。為了不混淆，我們暫時不考慮 $\frac{20}{3}$ 這個分數，現在還用不到它。$\frac{8}{2}=4$，所以應該在括號裡加入哪個數字來完成配方？

學生：　4 的平方是 16。

老師：　很好，所以可以寫成 $-\frac{1}{3}(x^2-8x+16-16)+\frac{20}{3}$，記得減去 16，這樣才能保留原來的函數。

接下來，把函數重新排序變成 $-\frac{1}{3}(x^2-8x+16)+(-\frac{1}{3})(-16)+\frac{20}{3}$，記得要用 $-\frac{1}{3}$ 乘上−16。簡化函數。

學生： $-\frac{1}{3}(x^2-8x+16)+\frac{16}{3}+\frac{20}{3}$

$=-\frac{1}{3}(x-4)^2+\frac{36}{3}$

$=-\frac{1}{3}(x-4)^2+12.$

老師： 請問頂點是多少？

學生： (4, 12)

老師： 跳水者離水平面的最高距離是多少？

學生： 是 12。

老師： 很好。那我們如何解釋數對 (4, 12)呢？我們可以說，跳水者離跳水板前端的水平距離 4 個單位時，達到最大高度。

學生： 記得老師之前教過頂點公式：$\left(-\frac{b}{a},\left(c-\frac{b^2}{4a}\right)\right)$，也可以用這個嗎？

老師： 當然可以，函數是$y=-\frac{1}{3}x^2+\frac{8}{3}x+\frac{20}{3}$，所以 $a=-\frac{1}{3}$、$b=\frac{8}{3}$、$c=\frac{20}{3}$。接下來計算 $\left(c-\frac{b^2}{4a}\right)$。

學生： $\frac{20}{3}-\frac{\left(\frac{8}{3}\right)^2}{4\left(-\frac{1}{3}\right)}$

$=\frac{20}{3}+\frac{16}{3}$

$=12$。答案相同。

例題二

說明：利用二次函數的觀念，解決日常生活的和經濟相關的問題。帶學生了解成本函數，收入函數，獲利函數的關係，並請學生透過配方法找出二次函數的極值。

（英文）A workshop makes and sells toy bikes. Each toy bike costs \$6 to make, and the workshop's monthly fixed costs are \$50. The price of each toy bike comes from the price function $146 - 10x$, where x is the number of toy bikes sold. Find the monthly cost function, monthly revenue function, and monthly profit function.
Find the maximum monthly profit.

（中文）一間工作坊製造並販賣玩具腳踏車，每輛玩具腳踏車的製作成本為\$6，工作坊每個月固定成本為\$50，玩具腳踏車的售價來自價格函數 $146 - 10x$，x 為售出玩具腳踏車的數量。求每月的成本函數，收入函數，獲利函數，並找到每月獲利函數的最大值。

Teacher: Cost, revenue, and profit are three important words in economics and daily life. Do you remember the sausage roll sale on the school anniversary? We made money by selling sausage rolls and donated the money to a charity. Cost is the money that you spent on raw sausages from the market. The raw sausages, the barbecue grill, wire grid, charcoal…etc. are called fixed costs. Revenue is the total money that we earned from the customers. We agreed to donate the profit to a charity. The profit is the difference between the money we earned and the money we spent. Now, do you understand their relationships?

Student: We didn't donate a lot because we didn't make a big profit.

Teacher: Now, let's come back to this question. Cost is the money that has been spent to produce toy bikes. The business's cost includes the fixed cost and \$6 per toy bike. Can you write the cost function $C(x)$?

Student: $C(x) = 50 + 6x$.

Teacher: Revenue is the income. Revenue is the number of bikes sold times the price. Can you write the revenue function $R(x)$?

Student: The revenue function is $R(x) = x(146 - 10x)$.

Teacher: According to the economic definition, profit is the excess of total revenue over the total cost. You subtract the cost from the revenue. What is the profit function $P(x)$?

Student: $P(x) = R(x) - C(x) = x(146 - 10x) - (50 + 6x) = -10x^2 + 140x - 50$.

Teacher: Great. This question is asking for the maximum profit. We have to find the maximum value of the $P(x)$. Can you write this function into vertex form by completing the square? It should just take you a couple of minutes.

Student: $P(x) = -10x^2 + 140x - 50 = -10(x^2 - 14x) - 50$
$= -10(x^2 - 14x + 7^2 - 7^2) - 50$
$= -10[(x-7)^2 - 49] - 50 = -10(x-7)^2 - 99$

Teacher: I think that you forgot the distributive property on -7^2. You have to multiply -7^2 by (-10), when moving it out of the parentheses.

Student: I see. Let me correct it. $-10(x-7)^2 + 490 - 50 = -10(x-7)^2 + 440$

Teacher: It makes sense now. Where is the vertex of the $P(x)$? What is the maximum profit value?

Student: The vertex is $(7, 440)$. The maximum profit is $440.

Student: I used the vertex formula. I plugged in $a = -10, b = 140, c = -50$ into $\left(-\frac{b^2-4ac}{4a}\right)$.
I got 440 as well.

Teacher: Either way is okay. We can tell that the company will make a maximum profit of $440 after they sell 7 toy bikes. If they sell more than 7 toy bikes, their profits will decrease.

老師： 成本、收益和利潤是經濟學和日常生活中三個重要的詞語。
還記得學校週年慶時的我們賣香腸捲嗎？我們靠香腸捲獲利並將錢捐給了一個慈善機構。成本是你在市場上購買生肉腸花費的錢。生肉腸、烤肉架、鐵絲網、木炭等都被稱為固定成本。收益是我們從客戶那裡賺取的總錢數。我們同意將利潤捐給慈善機構。利潤是我們賺取的錢數與我們花費的錢數之差。現在，你明白它們之間的關係了嗎？

學生： 因為我們沒有賺很多利潤，所以捐得不多。

老師： 現在，讓我們回到這個題目。成本是生產玩具腳踏車所花費的錢。商家的成本包括固定成本和每輛玩具自行車 6 元。你能寫出成本函數$C(x)$嗎？

學生： $C(x) = 50 + 6x$。

老師： 收益是收入。收益是售出的腳踏車數量乘以價格。試著寫出收益函數 $R(x)$。

學生： 收益函數是$R(x) = x(146 - 10x)$。

老師： 根據經濟學的定義，利潤是總收益超過總成本的部分，也就是收益和成本相減。利潤函數 $P(x)$ 是多少？

學生： $P(x) = R(x) - C(x) = x(146 - 10x) - (50 + 6x) = -10x^2 + 140x - 50$。

老師： 太好了。這個問題要求最大利潤。我們必須找到 $P(x)$ 的最大值。你能使用配

方法來將這個函數寫成標準式嗎？這應該只需要你幾分鐘時間。

學生： $P(x) = -10x^2 + 140x - 50 = -10(x^2 - 14x) - 50$

$= -10\left(x^2 - 14x + 7^2 - 7^2\right) - 50$

$= -10[(x-7)^2 - 49] - 50 = -10(x-7)^2 - 99$

老師： 你忘了在-7^2上使用分配律。當你將其從括號中移出時，你必須將-7^2乘上(-10)。

學生： 知道了，我改一下。$-10(x-7)^2 + 490 - 50 = -10(x-7)^2 + 440$。

老師： 現在講得通了。$P(x)$的頂點在哪裡？最大利潤值是多少？

學生： 頂點是$(7, 440)$，最大利潤是 440 元。

學生： 我使用了頂點公式，將$a = -10$、$b = 140$、$c = -50$ 代入$\left(-\frac{b^2-4ac}{4a}\right)$，也得到了 440。

老師： 不管哪種方法都可以。我們可以知道，公司在銷售 7 輛玩具腳踏車後，將獲得最大的利潤，也就是 440 美元；如果他們銷售的玩具腳踏車超過 7 輛，他們的利潤將會減少。

單元七 直線方程式與不等式
Lines and Inequalities

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■ 前言 Introduction

本以英文講解直線方程式的點斜式、斜截式、標準式的不同寫法，並證明點到直線的距離公式。並描述如何以英文介紹二元一次不等式及圖解二元一次聯立不等式，並以不等式解決日常生活問題。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
point-slope form	點斜式	slope-intercept form	斜截式
parallel lines	平行線	perpendicular lines	垂直線
two-variable inequality	二元一次不等式	**half-plane**	**半平面**
system of inequalities	**聯立不等式**	solid line	實線
		dashed line	虛線

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ _____ separate ______ into _____.

例句：The line **separates** the plane **into** two half-planes.

這條線把平面切成兩個半平面。

❷ ____ half-plane which consists of ______.

例句：The **half-plane, which consists of** (1, 4) is our solution.

包含點(1, 4)的那個半平面就是我們的解。

❸ The solution is ______.

例句：**The solution is** the overlap of two-half planes.

解是兩半平面重疊的部分。

❹ Draw ____ intersecting ___ at _____.

例句：**Draw** a horizontal line through point *P*, **intersecting** line *L* **at** point *C*.

通過點 *P* 畫一水平線，相交直線 *L* 於點 *C*。

■ 問題講解 Explanation of Problems

説明

[Review of linear equations]

In junior high school, we learned linear equations. There are different forms of linear equations. Given the slope and y-intercept, you can write the equation in slope-intercept form: $y = mx + b$. Given the slope and a point, or given two points, you can write the equation in point-slope form: $y - y_1 = m(x - x_1)$. You can simplify the equation into the standard form: $ax + by = c$, where a, b, c are real numbers, and a, b are both not zero.

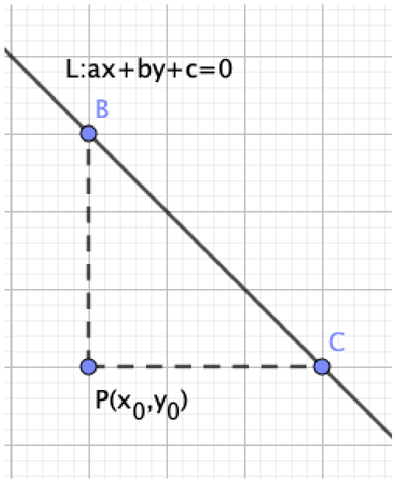
If lines $y = m_1x + a$ and $y = m_2x + b$ are parallel, their slopes must be equivalent, meaning $m_1 = m_2$.

If lines $y = m_1x + a$ and $y = m_2x + b$ are perpendicular, their slopes are negative reciprocals of each other: $m_1 = \frac{-1}{m_2}$.

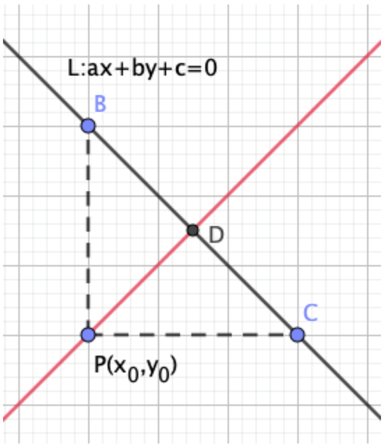
[Distance between point and line]

The distance from point *P* to line *L* is defined as: $d(P, L) = \frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$. It is the length of the perpendicular segment from *P* to *L*. There are many ways to prove this formula, and we will use the area of a right triangle to prove it. In the graph below, line *L*: $ax + by + c = 0$ and point *P* (x_0, y_0) are given. We will construct a right triangle and find the coordinates of the vertex.

First, draw a horizontal line through *P* intersecting *L* at *C*. Its *y*-coordinate is also y_0 because *P* and *C* are on the same horizontal line. Substitute y_0 for *y* in the equation $ax + by + c = 0$, and solve for x. $x = \frac{-c-by_0}{a}$ so the coordinate of C is $(\frac{-c-by_0}{a}, y_0)$. Then draw a vertical line through *P* intersecting *L* at *B*. Its *x*-coordinate is x_0, because *P* and *B* are on the same vertical line. Repeat the step, substitute x_0 for x in the equation $ax + by + c = 0$, and solve for *y*. The coordinate of *B* is $(x_0, \frac{-c-ax_0}{b})$.



Next, construct a line through *P* perpendicular to line *L*, intersecting *L* at point *D*. $\overline{PD} \perp L$



The area of $\Delta PBC = \overline{PB} \times \overline{PC} \times \frac{1}{2} = \overline{PD} \times \overline{BC} \times \frac{1}{2}$, so $\overline{PD} = \frac{\overline{PB} \times \overline{PC}}{\overline{BC}}$.

$$\overline{PB} = \left|\frac{-c - ax_0}{b} - y_0\right| = \left|\frac{-ax_0 - by_0 - c}{b}\right| = \left|\frac{ax_0 + by_0 + c}{b}\right|$$

$$\overline{PC} = \left|\frac{-c - by_0}{a} - x_0\right| = \left|\frac{-ax_0 - by_0 - c}{a}\right| = \left|\frac{ax_0 + by_0 + c}{a}\right|$$

Using the distance formula, $\overline{BC} = \sqrt{\left(x_0 - \frac{-c-by_0}{a}\right)^2 + \left(y_0 - \frac{-c-ax_0}{b}\right)^2} = \sqrt{\left(\frac{1}{a^2} + \frac{1}{b^2}\right)}|ax_0 + by_0 + c|$. Plug in these values and simplify,

$$\overline{PD} = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

Let's see an example. Find the distance between point $P(-1,4)$ and the line $y = 2x - 3$.

First, write the line equation in the form of $ax + by + c = 0$. $y = 2x - 3$ becomes $-2x + y + 3 = 0$. We compare the coefficients and find that $a = -2, b = 1, c = 3$.
The coordinate $P(-1,4)$ tells that $x_0 = -1, y_0 = 4$. We apply the formula and receive:

$$\overline{PD} = \frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}} = \frac{|-2\cdot(-1)+1\cdot4+3|}{\sqrt{(-2)^2+(1)^2}} = \frac{9}{\sqrt{5}} = \frac{9\sqrt{5}}{5}.$$

The distance from the point P to the line $y = 2x - 3$ is $\frac{9\sqrt{5}}{5}$.

[Two- variable inequality]

$y = x + 1$ is a linear equation in two variables. If we change the equal sign, for example, $y > x + 1$ is a linear inequality. An inequality shows the relationships between two quantities, and the relationship can be "greater than, greater than or equal to, less than, or less than or equal to". The mathematics symbols are "$>, \geq, <, \leq$ ", respectively.

The graph of a linear equation is a line, and the graph of a linear inequality is a half-plane of a coordinate plane that is bounded by a line. Let's take $y > x + 1$ as an example.
First, we graph the line equation $y = x + 1$ on the plane (Figure 1). The solutions of $y = x + 1$ lie exactly on the line. However, $y = x + 1$ is the boundary line. This line cuts the coordinate plane into two half-planes: one is above the line, and the other is below the line. One of them is the solution of $y > x + 1$. How do we choose correctly? We have to test with a point. Pick a random point, for example, $(0, 0)$, and check whether it satisfies the inequality. Does $0 > 0 + 1$ satisfy? No, 0 is not greater than 1, which means that $(0, 0)$ is not one of the solutions. The lower half-plane with $(0, 0)$ is not the solution, but what about the other half? The upper half-plane is the solution. Any points in the upper half-plane can be the solution of the inequality. There are infinitely many solutions, so we usually shade the region (Figure 2). Most of the time, $(0, 0)$ is an easy way to test as long as it is not on the boundary. We can try another random point to test the inequality again. Let's try $(1, 4)$ which is on the upper half-plane, and plug $(1, 4)$ in the inequality. Does $4 > 1 + 1$ satisfy? Yes, 4 is greater than 2. $(1, 4)$ is one of the solutions, so the half-plane which consists of $(1, 4)$ is our solution. You can see from the graph, the half-

plane which includes $(1,4)$ is above the boundary line. Lastly, we are going to change the boundary line. Because it is an inequality, the points on the line don't satisfy the inequality. We use a dashed boundary line (Figure 3). If there is an equal sign in the inequality, such as $\geq or \leq$, we use a solid boundary line.

Figure 1

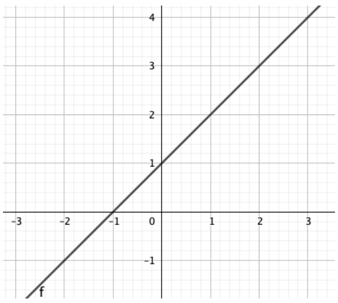


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Figure 2

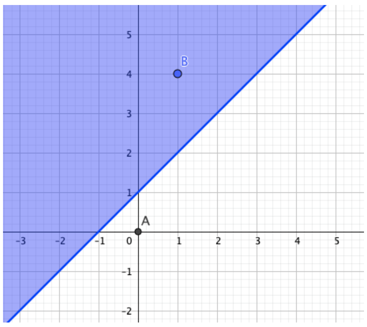

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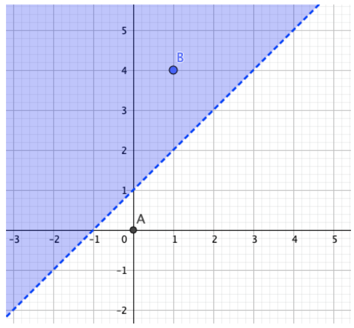
Figure 3



[System of inequalities]

We learned the system of equality before. It is usually formed by two equalities in two variables. The solution of the system of equality is the solution of each equality. The solution will satisfy the equations.

Now, we will learn the system of inequalities, which is formed by two or more inequalities in two variables. A system of inequalities has many solutions, which are a set of points. These points satisfy each inequality. We can solve a system by graphing it. Take $\begin{cases} y + x \leq 8 \\ y - 3 > 2x \end{cases}$ as an example. First, graph the equation $y + x = 8$. Plot intercepts $(0,8)$ and $(8,0)$ on the y-axis and x-axis, then draw a line through these two points. Test the point $(0,0)$ for inequality, $0 + 0 \leq 8$. $(0,0)$ satisfies the inequality, so the half-plane which includes $(0,0)$ is the solution. Shade the lower half-plane. There's an equal sign in the inequality, so the boundary is a solid line (Figure 1). Then graph $y - 3 = 2x$ with points $(0,3)$ and $(1,5)$. Test the point $(0,0)$ for inequality. $0 - 3 < 0$. $(0,0)$ doesn't satisfy the inequality, so the half-plane, which doesn't include $(0,0)$, is the solution. Shade the upper half-plane. The boundary is a dashed line (Figure 2). The solution of the system is the overlapping area. The combined graph should look like Figure 3. Now pick a point in the overlapping region, such as $(0,4)$, and check it in both inequalities. $4 + 0 \leq 8$ and $4 - 3 > 0$, so both inequalities are satisfied.

Figure 1

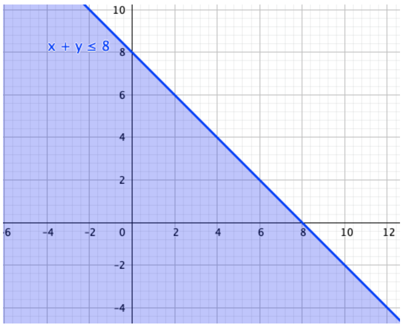


Figure 2

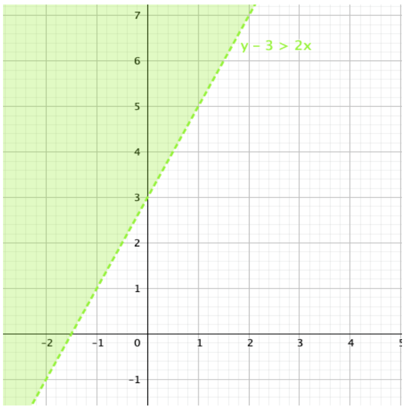
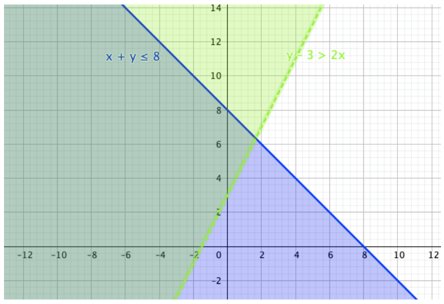


Figure 3



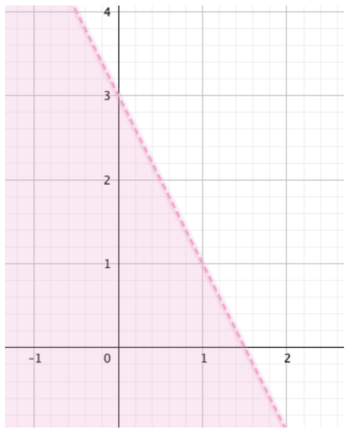
運算問題的講解

例題一

說明：從圖形反推二元一次不等式，並引導學生以不同性質的點去推測不等式，及檢驗所得之答案。

（英文）Write an inequality to graph.

（中文）請寫出圖所代表的二元一次不等式。



Teacher: We need the equation of the boundary line. Two points determine a line. Can you find two points on the boundary line?

Student: $(0, 3)$ and $(1, 1)$.

Teacher: Do you remember how to write the equation of a line? Who has the answer?

Student: The slope is $\frac{1-3}{1-0} = -2$. The equation is $y - 3 = -2(x - 0)$. I use point-slope form.

Teacher: Good. Is there any other way?

Student: I use the form $y = mx + b$. The y-intercept $(0, 3)$ gives the value of b. I can plug in -2 for *m* and 3 for *b*.

Teacher: You both are correct. The boundary line is $y = -2x + 3$. Next, we have to determine the inequality sign, $>, \geq, <$ or $\leq$. What should we do?

Student: We can pick a point and test the inequality.

Teacher: Great. Let's choose the point $(0, 0)$, which is in the shaded region. This point can help determine the inequality sign. Plug $(0, 0)$ into the equation $y = -2x + 3$. We have 0 on the left-hand side and 3 on the right-hand side. Because $0 < 3$, the inequality is $y < -2x + 3$.

Student: The process is reversed. I am confused.

Teacher: The inequality sign is missing, and we use one of the solutions, $(0, 0)$, to find out the missing part. $(0, 0)$ is in the shaded region, so it is one of the solutions. The solution $(0, 0)$ should satisfy the inequality. On the left-hand side of the equation, we have 0. On the right-hand side of the equation, we have 3. Because $0 < 3$, we know the left-hand side is less than the right-hand side. Therefore, $y < -2x + 3$. Let's use the way you already know to check the answer. Does $(0, 0)$ fit the inequality $y < -2x + 3$?

Student: $0 < 3$, yes.

Teacher: Should the half-plane which includes $(0, 0)$ be the solution?

Student: Yes.

Teacher: Now, check whether the half-plane which includes $(0, 0)$ should be colored?

Student: Yes, it should.

Teacher: See, we are correct. The boundary line is dashed, so the inequality is $y < -2x + 3$.

Student: Can we try a point other than $(0, 0)$?

Teacher: Let's try $(2, 2)$. You can tell from the graph that $(2, 2)$ is not the solution because its region is not shaded. $(2, 2)$ will not satisfy the inequality. The left-hand side is 2, and the right-hand side is $-2 \times 2 + 3 = -1$. 2 is greater than -1.
Since $(2, 2)$ will not satisfy the inequality, the inequality sign should be reversed from "$>$" to "$<$". The inequality is still $y < -2x + 3$.

Student: I see, but I am not sure I can do this "reversed" way next time. So I should try the point in the shaded region next time.

Teacher: It's good to plan ahead. You will do better next time.

老師： 我們要找到邊界的方程式。兩點決定一直線。能找到兩條邊界上的點嗎？
學生： (0,3) 和 (1,1)

老師： 還記得怎麼寫出一條線的方程式嗎？誰能給出答案？

學生： 斜率是 $\frac{1-3}{1-0} = -2$。方程式是 $y - 3 = -2(x - 0)$。我用點斜式。

老師： 很好。還有其他方法嗎？

學生： 我用$y = mx + b$ 下去代。Y 截距 (0, 3) 得出了 b 的值。我可以將 -2 代入 m，將 3 代入 b。

老師： 你們兩個都對。邊界是 $y = -2x + 3$。接下來要確定不等式的符號，>、≥、< 或≤。該怎麼做？

學生： 可以選一個點來測試不等式。

老師： 太好了。選擇位於陰影區域內的點 (0,0)。這個點可以協助確定不等式的符號。將 (0,0) 代入方程式 $y = -2x + 3$。左側是 0，右側是 3。因為 0<3，所以不等式是 $y < -2x + 3$。

學生： 過程反過來了，我不懂。

老師： 不等號不見了，用其中一個解 (0, 0) 來找不見的部分。(0, 0) 在陰影區域內，所以是其中一個解。解 (0, 0) 應該滿足不等式。我們將 0 放在左側，將 3 放在右側，就像這樣：0__3。因為 0<3，我們知道不見的符號是「<」。用你已知的方法來檢查答案。(0, 0) 是否符合不等式 $y < -2x + 3$？

學生： $0 < 3$，是的。

老師： 包含 (0, 0) 的半平面是解嗎？

學生： 是的。

老師： 現在檢查一下是否要將包含 (0, 0) 的半平面填色？

學生： 是的，要填色。

老師： 你看對了。邊界線是虛線，所以不等式是 $y < -2x + 3$。

學生： 我們可以嘗試除了 (0, 0) 以外的點嗎？

老師： 讓我們嘗試 (2, 2)。從圖表上可以看出，(2, 2) 不是解，因為它所在的區域未被陰影覆蓋。(2, 2) 將不滿足不等式。左側是 2，右側是$-2 \times 2 + 3 = -1$。2 大於 -1。由於 (2, 2) 將不滿足不等式，所以不等式符號應從「>」變為「<」。不等式仍然是 $-2 \times 2 + 3 = -1$。

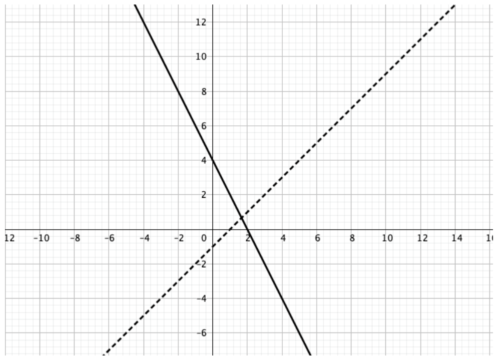
學生： 懂了，但不確定下次還能不能做這種「反過來」的手法。所以下次該拿在陰影區內的點來嘗試。

老師： 預先設想很不錯。下次會更好。

例題二

說明：已知二元一次聯立不等式，鼓勵學生使用不同方式圖示出正確的解。

（英文）The system of inequalities $\begin{cases} y \geq -2x+4 \\ y < x-1 \end{cases}$ is given. Please shade the solution on the coordinate plane.



（中文）已知二元一次聯立不等式 $\begin{cases} y \geq -2x+4 \\ y < x-1 \end{cases}$，請在坐標平面上以陰影表示該聯立不等式的解。

Teacher: Two boundary lines are given, but we don't know which one is which. How can you identify them?

Student: I know an easy way. One is dashed, and the other is solid. The dashed line is $y < x-1$, and the solid line is the other one.

Teacher: Good. Is there any other way?

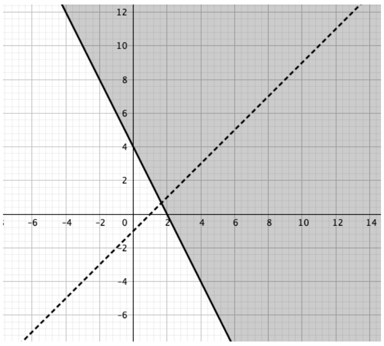
Student: I can find two points on the line and solve the equation of the line.

Teacher: Good. You can use this way when they are both solid lines or both dashed lines. Is there any other way?

Student: I look at the slopes. The slope of the first boundary line $y = -2x+4$ is -2. The solid line is downward from left to right and has a negative slope. The second boundary line $y = x-1$ has a positive slope. The dashed line is upward from left to right and has a positive slope. The dashed line is $y = x-1$.

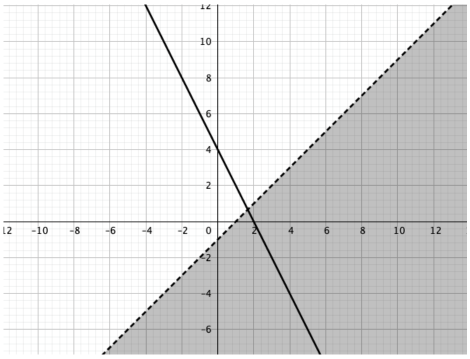
Teacher: Great. These are all correct ways to identify the line. Next, shade the solution for each inequality. Finally, try the test point $(0,0)$.

Student: $0 < 0+4$, so $(0,0)$ doesn't satisfy the first inequality.
The solution should exclude $(0,0)$. The solution is the upper half-plane.



Student: $0 > -1$. $(0, 0)$ doesn't satisfy the second inequality either.

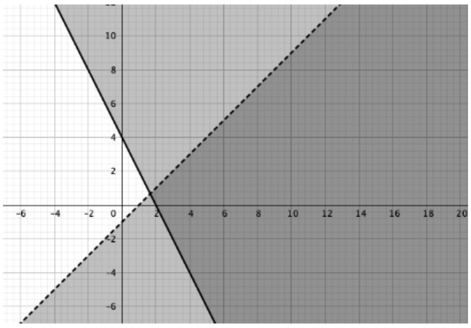
The solution is in the lower half-plane.



Teacher: Yes. The overlapping part is the solution of the system.

I would like to share a shortcut with you.

If the inequality is in slope-intercept form, starting with y ...., shade the upper half-plane of the boundary if $y \geq$ or $y >$ and shade the lower half-plane of the boundary if $y \leq$ or $y <$.



Student: Wow.

Teacher: Let's check the previous example $y < -2x + 3$. The inequality is in slope-intercept form, and $y < \cdots$. The shade region is "below" the boundary line.

Student: Yes. It works. But what if the boundary line is vertical? There's no way to tell the "upper" and "lower" half-plane. What can we do?

Teacher: If so, use the ultimate method: a test point to check the inequality. This method works every time.

老師： 題目給定了兩條邊界，但不知道哪條是哪條。怎麼辨認它們？

學生： 我知道一種簡單的方法。其中一條是虛線，另一條是實線。虛線是 $y < x - 1$，實線就是另一條。

老師： 很好。還有其他方法嗎？

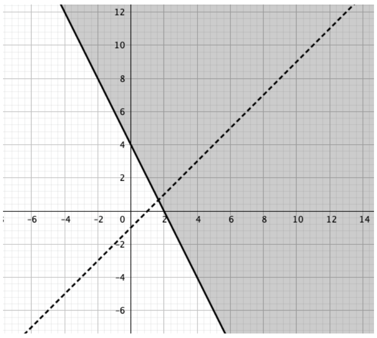
學生： 找到線上的兩個點並解出該線的方程式。

老師： 很好。當它們都是實線或都是虛線時，可以用這種方式。還有其他方法嗎？

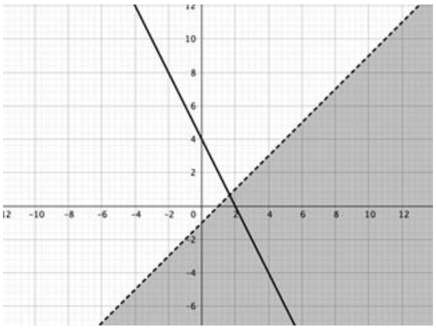
學生： 可以觀察斜率。第一條邊界 $y = -2x + 4$ 的斜率是 -2。實線是從左向右下降，斜率為負。第二條邊界 $y = x - 1$ 斜率為正。虛線是從左向右上升，斜率為正。虛線是 $y = x - 1$。

老師： 太棒了。這些都是辨識線的好方法。接下來對每個不等式填色。最後測試點 (0,0)。

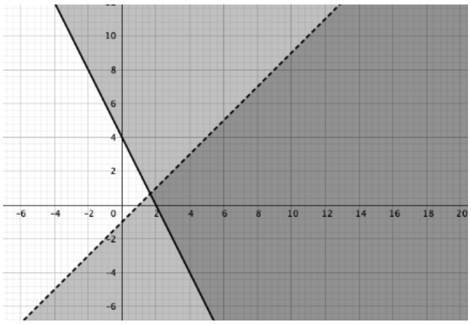
學生： $0 < 0 + 4$，所以 $(0, 0)$ 不滿足第一個不等式。解應該排除 $(0, 0)$。解答是上半平面。



學生： 0>−1。(0, 0) 也不滿足第二個不等式。解在下半平面。



老師： 是的。重疊部分是聯立方程式的解。跟你分享一個快速的方法。如果不等式是以斜率-截距式出現，由 y… 開始，如果出現 $y\geq$ 或 $y>$，則對邊界的上半平面填色；如果是 $y\leq$ 或 $y<$，則邊界的下半平面填色。



學生： 哇。

老師： 檢查前例 $y < -2x + 3$。不等式以斜率-截距式呈現，且 y<⋯。填色的區域在邊界線的「下方」。

學生： 對，很有用。但如果邊界是垂直的呢？沒辦法分「上方」和「下方」的半平面。我們該怎麼做？

老師： 如果變成這樣，那就用必殺技：用測試點來檢查不等式。這種方法絕對有效。

꧁ 應用問題 / 學測指考題 ꧂

例題一

說明：利用地圖上的點與方程式，讓學生練習點到直線的距離公式。

（英文）A town is mapped on a coordinate plane, and the city hall is the original point. The police station is located at the point $(-6, 2)$, and the main street is located at $2x - 3y = 10$. How far is it from the police station to the city hall? What is the shortest distance from the police station to the main street?

（中文）一個小鎮被標示在平面坐標上，原點是市政中心，警察局是$(-6, 2)$，主要街道位於 $2x - 3y = 10$. 請問警察局到市政中心的距離？警察局到主要街道的最短距離？

Teacher: The police station is $(-6, 2)$, and the city hall is $(0, 0)$. So how do you find their distance?

Student: Use the distance formula. But I forgot the formula.

Teacher: The distance formula is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. Try to plug in the numbers.

Student: $\sqrt{(-6)^2 + (2)^2} = \sqrt{40}$, which is 6 point something.

Teacher: Okay. Next, find the shortest distance from the police station to the main street. Actually, it is the length of the perpendicular segment from $(-6, 2)$ to the line. Do you remember the formula?

The formula is $\frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$. It may take you a while to memorize this formula.

Student: $\frac{|-12 - 6 + 10|}{\sqrt{2^2 + (-3)^2}}$

Teacher: You plugged in the wrong number in *c*.

The *c* comes from the equation $ax + by + c = 0$. You have to rewrite the equation to $2x - 3y - 10 = 0$. *c* is -10.

Student: Okay. The distance should be $\frac{|-12-6-10|}{\sqrt{2^2+(-3)^2}} = \frac{28}{\sqrt{13}}$.

Teacher: Please use your calculator to find its value.

Student: It's about 7.77.

Teacher: Correct.

老師： 警察局的坐標是$(-6, 2)$，市政府的坐標是$(0, 0)$。那怎麼找到它們之間的距離？

學生： 用距離公式。但公式忘了。

老師： 距離公式是$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$。試著代入數字。

學生： $\sqrt{(-6)^2 + (2)^2} = \sqrt{40}$，約等於 6 點多。

老師： 好的。接下來是從警察局到主幹道的最短距離。其實是從$(-6, 2)$到該線段的垂直長度。還記得公式嗎？

公式是 $\frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$。你可能需要一些時間來記住這個公式。

學生： $\frac{|-12-6+10|}{\sqrt{2^2+(-3)^2}}$

老師： 你在 c 中插入的數字錯了。c 來自方程式$ax + by + c = 0$。必須將方程式重寫為$2x - 3y - 10 = 0$。c 是-10。

學生： 好的。距離應該是$\frac{|-12-6-10|}{\sqrt{2^2+(-3)^2}} = \frac{28}{\sqrt{13}}$。

老師： 請使用計算機來求其值。

學生： 大約 7.77。

老師： 正確。

例題二

說明： 學習用不等式解決日常生活的問題，引導學生思考變數的性質與可能的解，並在平面上畫出整數解。

（英文）You have NT$150 for grocery shopping. One bottled cola costs NT.20, and one sparkling water costs NT$35. You don’t want to spend more than NT$150. How many bottles of cola or sparkling water can you buy? Write a system of inequality and graph the solutions.

（中文）你帶了 150 元去購物，一瓶可樂 20 元，一瓶氣泡水 35 元。你不想花超過 150 元，你可以買幾瓶的可樂或是氣泡水？請寫出不等式並畫出解。

Teacher: We are going to create an inequality in two variables to represent the relationships between cola, sparkling water, and money. First, we use x for the number of colas, and y for the number of sparkling waters. Can you write an expression to show the total money you spend?

Student: $20x + 35y$.

Teacher: You don't want to spend more than NT$150, so your total money should be less than or equal to 150. We can write an inequality $20x + 35y \leq 150$. Please graph the inequality.

Student: Can I divide the inequality by 5 first? They are all multiples of 5.

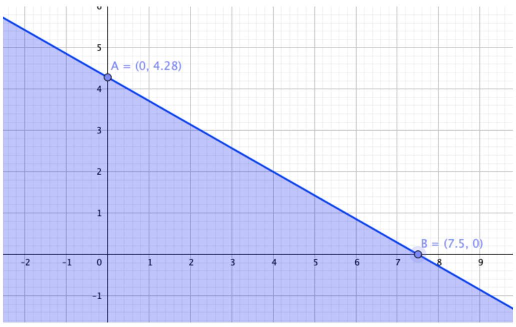
Teacher: Yes, you can simplify the inequality first to make it easier.

Student: I have $4x + 7y \leq 30$.

Teacher: Do you remember how to graph inequality? First, graph the line $4x + 7y = 30$, choose a point to test the inequality, and then shade the correct half-plane. I will give you some time to do that.

Teacher: Who can share their graph and explain?

Student: This is my graph. First, I found the x-intercept and y-intercept. Then I graphed the boundary line. I chose $(0, 0)$ to test the inequality. Because $0 + 0 \leq 30$, the lower half-plane which consists of $(0, 0)$ is the solution. I shaded the region. The inequality has an equal sign, so the boundary should be a solid line.



Teacher: I am impressed! How many answers do you think there are?

Student: Infinitely many!

Teacher: Can x be a negative number? Can y be a negative number?

Student: Um. They shouldn't be negative. They represent the number of drinks.

Teacher: Can they be "0"?

Student: I guess so, as long as you don't buy them.

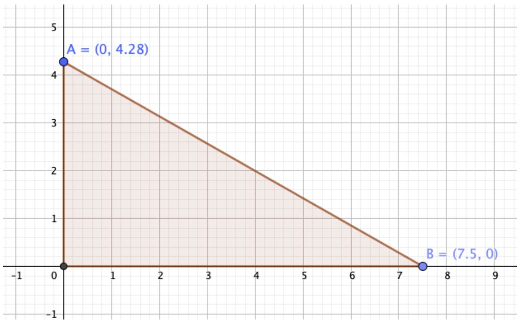
Teacher: Can they be a decimal, such as 1.5 or 2.99?

Student: Well, no, they should be integers. They are the numbers of drinks.

Teacher: How do you revise your graph to reflect what you just said about x and y?

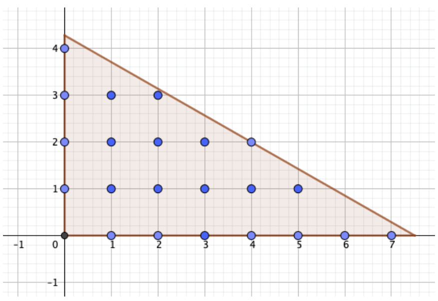
Student: I think the region should be limited to the first quadrant.

Teacher: Like this?



Student: Yes, but I don't know how to graph integer solutions.

Teacher: x and y are whole numbers. They can be 0, 1, 2, 3etc. We can plot the points with whole number coordinates $(0,0),(0,1),(0,2)$ .... until $(0,4)$ on the y-axis. We plot the points $(1,0),(1,1),(1,2),(1,3)$ on $x=1$. Repeat the steps. Remember that we only plot the whole number points in the shaded region. The point $(4,2)$ in on the line, and it satisfies the inequality $20x+35y\leq 150$. The graph is:



Teacher: Can you interpret the coordinate $(1,3)$? What do the numbers mean?

Student: You can buy 1 cola and 3 bottles of sparkling water.

Teacher: Yes. Meanwhile, you spend no more than NT\$150. Here, I would like to revise our inequality a little bit. You clarified that x and y could be positive or 0, so we can add $x\geq 0$ and $y\geq 0$ into inequalities. This is our system of equalities:

$$\begin{cases}20x+35y\leq 150.\\ x\geq 0\\ y\geq 0\end{cases}$$. Done.

老師： 要作出包含可樂、氣泡水和價格關係的雙變數不等式。首先用 x 表示可樂的量，用 y 表示氣泡水的量。能寫出表達總額的表達式嗎？

學生： 是的，$20x+35y$。

老師： 不想花超過 150 元新台幣，所以總額應該小於或等於 150。我們可以寫個不等式 $20x+35y\leq 150$。請繪製這個不等式的圖形。

學生： 可以先將不等式除以 5 嗎？都是 5 的倍數。

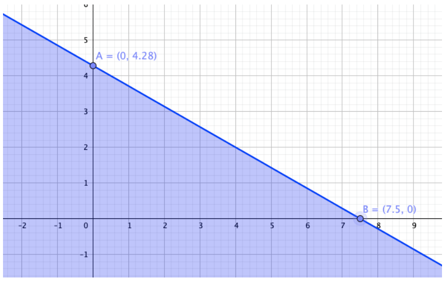
老師：　是的，可以先簡化不等式，這樣更容易處理。

學生：　得出 $4x + 7y \leq 30$。

老師：　還記得怎麼畫不等式的圖形嗎？首先畫出直線 $4x + 7y = 30$，選一個點來測試不等式，然後將正確的半平面塗黑。給你一點時間完成。

老師：　請分享你的圖形然後解釋一下。

學生：　這是我的圖。首先找到 x 軸截距和 y 軸截距，然後畫出邊界。用 (0,0)來測試不等式。因為$0 + 0 \leq 30$，由 (0,0) 組成的下半平面是解的範圍。然後把這個範圍畫上陰影。不等式有一個等號，所以邊界應該是實線。



老師：　太厲害了！你覺得有多少個解？

學生：　有無限多個！

老師：　x 可以是負數嗎？y 可以是負數嗎？

學生：　嗯，它們不應該為負，他們代表飲料的數量。

老師：　可以是「0」嗎？

學生：　應該可以，不買就好了。

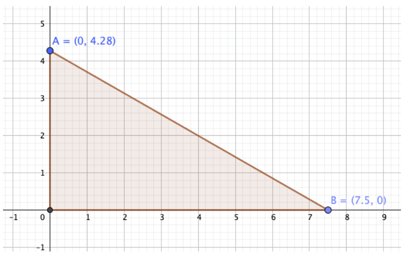
老師：　可以是小數嗎，例如 1.5 或 2.99？

學生：　嗯，不行，應該是整數。代表的是飲料的數量。

老師：　怎麼修改圖形才能反映剛才有關 x 和 y 的論點？

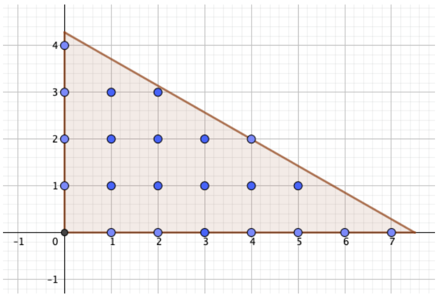
學生：　我認為應該把範圍限制在第一象限。

老師：　像這樣嗎？



學生：　對，但不知道怎麼畫出整數解。

老師：　x 和 y 是整數。可以是 0、1、2、3 等。可以在 y 軸上以整數坐標(0,0)、(0,1)、(0, 2)…直到(0, 4)上標點。在 $x = 1$ 上標出點(1, 0)、(1, 1)、(1, 2)、(1, 3)。然後重複步驟。記得只在陰影範圍內標出整數點。點(4, 2)在直線上，並滿足不等式 $20x + 35y \leq 150$。圖形如下：



老師： 能解釋一下坐標(1, 3)嗎？這些數字代表什麼？

學生： 買 1 罐可樂和 3 瓶氣泡水。

老師： 是的。同時，花費總額不超過 150 元。在這裡想稍微修一下不等式。你說了 x 和 y 可以是正數或 0，所以可以將 $x \geq 0$ 和 $y \geq 0$ 加到不等式中。最後聯立方程式：$\begin{cases} 20x + 35y \leq 150. \\ x \geq 0 \\ y \geq 0 \end{cases}$。完成。

單元八 圓與直線
Circles and Lines

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■ 前言 Introduction

本單元介紹如何以英文講解圓的方程式、圓與直線相交的三種情形，並使用判別式和點到直線距離等兩種方法判斷圓與直線相交的情形，探討過圓上一點的切線和過圓外一點的切線數量及切線方程式。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
origin	坐標原點	center	圓心
radius	半徑	standard form	標準式
tangent line	切線	secant line	割線
point of intersection	交點	**discriminant**	**判別式**
point of tangency or tangent point	**切點**	reciprocal	倒數
point-slope form	點斜式		

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ ______ a circle with center ______.

例句：Draw **a circle with a center** (3, 5) and radius 3 on the plane.

在平面上畫一個圓心在(3, 5)，半徑為 3 的圓。

❷ ______ in standard form.

例句：Please write the equation of the circle **in standard form**.

請寫出圓的標準式。

❸ the distance between _____.

例句：Find the **distance between** point *A* and line *L*.

找出點 *A* 和直線 *L* 的距離。

❹ intersect _____ at ____.

例句：The line **intersects** the circle **at** two points.

這條直線相交圓於兩點。

■ 問題講解 Explanation of Problems

ᘓ 說明 ᘐ

[Equation of a circle]

A circle is the set of all points (x, y) in a plane that are equidistant from a fixed point. The fixed point is the center of the circle. The distance r from the center to any point (x, y) is the radius of the circle. For a circle whose center is $(0, 0)$ and radius is r, the distance between the center $(0, 0)$ and any point (x, y) on the circle can be written as: $r = \sqrt{(x-0)^2 + (y-0)^2}$. This is the distance formula, and you learned it in junior high school. We square both sides of the equation, and we will get: $r^2 = (x-0)^2 + (y-0)^2$. Simplify it, and we get: $r^2 = x^2 + y^2$. This is the standard form of the equation of a circle, with the center at the origin $(0, 0)$ and radius r.

Not every circle has a center $(0, 0)$, so we need a general form of the equation. We can get the equation of the circle whose center is not $(0, 0)$ with the distance formula again. Suppose a circle with radius r with a center (h, k). The distance from the center to any point(x, y) on the circle can be written as: $r = \sqrt{(x-h)^2 + (y-k)^2}$. We square both sides of the equation, and we will get: $r^2 = (x-h)^2 + (y-k)^2$. This is the standard form of the equation of a circle, with the center at (h, k) and radius r.

If we expand the equation and simplify it, we will get
$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0$. You might be confused by the letters in the equation. In short, the standard equation of the circle can be expanded and then simplified to a quadratic equation $x^2 + y^2 + dx + ey + f = 0$. This is the general form of the circle.

[Circle- line Intersection]

You draw a random circle on the paper and then move your ruler around the circle. You can find that sometimes the ruler cuts the circle at two points, sometimes at 1 point, and sometimes there's no intersection between the circle and the ruler. There are three cases of the circle-line intersection in a plane.

Case 1: If the line cuts through the circle, there will be two points of intersection. The line is called "the secant line."

Case 2: If the line is tangent to the circle, there will be only one point of intersection. The line is called "the tangent line."

Case 3: If the line misses the circle, there will be no point of intersection.

Given a circle and a line, how can we identify whether there's any intersection or not? Of course, we can graph the circle and the line on the coordinate plane and see the points of intersection geometrically. There's another algebraic way to solve the problem. Let's look at the following example. Given a circle $x^2 + y^2 + 4x + 10y - 7 = 0$ and a line $y = x + 1$, how many points of intersection are there?

If there is a point of intersection, the coordinates must fit both the circle and the line. You plug the coordinates into the equation of the circle and the line, and both equations will be satisfied.

So, the point of intersection is the solution of the system of equations $\begin{cases} x^2+y^2+4x+10y-7=0 \\ y=x+1 \end{cases}$. We can solve the equations by substitution.

We substitute $x+1$ for y in the equation $x^2+y^2+4x+10y-7=0$.

The equation becomes $x^2+(x+1)^2+4x+10(x+1)-7=0$. Simplify it, and we get: $x^2+8x+2=0$. This is a quadratic equation, and we can find the value of x by the quadratic formula, $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. The value x is the x-coordinates of the point of intersection.

However, the value of x is not our goal, but the number of points of the intersection. We have an easier way to get the answer by checking the discriminant, b^2-4ac. In the quadratic formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, the number b^2-4ac determines the nature of the solution.

Case 1: If $b^2-4ac>0$, $x=\frac{-b+\sqrt{b^2-4ac}}{2a},\frac{-b-\sqrt{b^2-4ac}}{2a}$. There will be two real-number solutions. This finding implies that there are two points of intersection between the line and the circle.

Case 2: If $b^2-4ac=0$, $x=\frac{-b\pm 0}{2a}=\frac{-b}{2a}$. There will be one real-number solution. This finding implies that there is one point of intersection between the line and the circle.

Case 3: If $b^2-4ac<0$, there will be no real-number solution. This finding implies that there is no point of intersection between the line and the circle.

Let's go back to the question. The discriminant $b^2-4ac=64-8>0$. There will be two points of intersection between the line and the circle.

There is another way to determine the position of a line. Remember the formula of the distance between a point and a line? The distance from a point (m,n) to the line $ax+by+c=0$ is $d=\frac{|am+bn+c|}{\sqrt{a^2+b^2}}$.

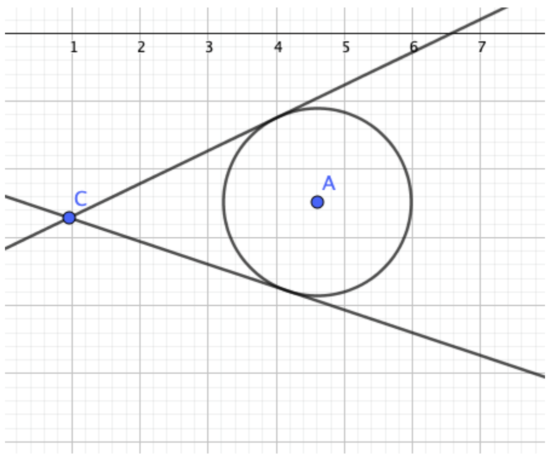
First, we find the distance, written as d, from the center of the circle to the line. Then compare the distance d and the radius of the circle r. If the distance is greater than the radius, $d>r$, the line lies outside the circle. If the distance equals the radius, $d=r$, the line is tangent to the circle. If the distance is less than the radius, $d<r$, the line intersects the circle at two points of intersection.

[Tangents of a circle]

A tangent of a circle is a straight line that intersects the circle at exactly one point P. We name point P as the point of tangency, or the tangent point. The tangent is perpendicular to the radius that joins the center and the point of tangency. How many tangent lines are there? How can we find the equation?

If the point of tangency is given, it means the given point is on the circle, and only one tangent line exists. To find the tangent line's equation, we need the tangent line's slope and the tangent point. For example, given a circle $x^2 + y^2 = 5$ and point $(-1, 2)$ on the circle, find the equation of the tangent line through the point $(-1, 2)$. The center is $(0, 0)$, and the point of tangency is $(-1,2)$. The slope of the radius which joins $(0, 0)$ and $(-1,2)$ is given by: $\frac{\Delta y}{\Delta x} = \frac{2}{-1} = -2$. The tangent line is perpendicular to this radius, so the slope of the tangent line is the negative reciprocal of that of the radius. The slope of the tangent line is $\frac{1}{2}$. With the point $(-1, 2)$ and the slope, the equation can be written in point-slope form: $y - 2 = \frac{1}{2}(x + 1)$, and can be simplified to: $-x + 2y = 5$.

In the previous example, point $(-1, 2)$ lies on the circle $x^2 + y^2 = 5$. What if the given point is an external point outside of the circle? In this case, there are exactly 2 tangents drawn like this:



Two tangent lines are perpendicular to the radius, and the distance between the center and tangent lines is equal to the radius. This will be important information when solving questions.

運算問題的講解

例題一

說明：將圓的一般式改寫為標準式，找出圓心與半徑。

（英文）What are the center and radius of the circle $x^2+y^2-6x+2y+4=0$?
（中文）一圓方程式為 $x^2+y^2-6x+2y+4=0$，找出圓心與半徑。

Teacher: With the standard form, we can easily tell the center and radius. However, we are given a general form. We have to rewrite the equation in standard form. First, we have to set it up to complete the squares.

Student: What do you mean by "set it up"?

Teacher: You put all x^2-terms and x-terms in a parenthesis, and all y^2-terms and y-terms in another parenthesis. Move the constant to the other side of the equation.

Student: Like this: $(x^2-6x)+(y^2+2y)=4$?

Teacher: When you move the constant to the other side, you have to change the sign. It should be -4.

Student: I forgot. Like this: $(x^2-6x)+(y^2+2y)=-4$?

Teacher: Yes. Next, you can complete the square in two parentheses: $(x^2-6x+9)+(y^2+2y+1)$. 9 and 1 are extra numbers on the left of the equation, so you have to add 9 and 1 on the right of the equation, to balance it. $(x^2-6x+9)+(y^2+2y+1)=-4+9+1$. Then please simplify.

Student: $(x-3)^2+(y+1)^2=6$.

Teacher: The standard form of the equation of a circle $(x-h)^2+(y-k)^2=r^2$ tells that the center is (h,k) and the radius r. Can you compare the numbers and find the center and radius now?

Student: The center is $(3,-1)$, $r^2=6$, so $r=\sqrt{6}$.

Teacher: Good.

老師：我們可以藉圓的標準式輕易判斷出圓心和半徑。但我們得到的是一般式。所以要把方程式改為標準式。首先整理算式來產生平方。

學生：「整理」的話要怎麼做？

老師：把所有的 x^2 項和 x 項放在一個括號內，把所有 y^2項和 y 項放在另一個括號內。把常數移到方程式的另一邊。

學生： 像這樣：$x^2-6x)+(y^2+2y)=4$？

老師： 當你把常數移到方程式的另一邊時，就要改變符號。變成是 -4。

學生： 我忘了。像這樣：$(x^2-6x)+(y^2+2y)=-4$？

老師： 對。然後在兩個括號內做成平方：$(x^2-6x+9)+(y^2+2y+1)$。9 和 1 是方程式左側的多出來的數值，所以要同步將 9 和 1 加在方程式的右側來達成等式。

$(x^2-6x+9)+(y^2+2y+1)=-4+9+1$。然後化簡。

學生： $(x-3)^2+(y+1)^2=6$。

老師： 方程式的標準式$(x-h)^2+(y-k)^2=r^2$ 表明圓心是 (h,k)，半徑是 r。現在你能對照數字算出圓心和半徑嗎？

學生： 圓心是 $(3,\ -1)$，$r^2=6$，所以 $r=\sqrt{6}$。

老師： 很好！

例題二

說明：利用點到直線的距離公式，比較圓心到直線的距離與半徑的大小關係，以判別直線和圓的關係。

（英文）Determine whether the given line intersects the given circle at two distinct points, or one point, or does not intersect.

(1) Circle $x^2+y^2-4x+6y-8=0$, and line $x=5$

(2) Circle $x^2+y^2-4x+6y-8=0$, and line $y=x-15$

（中文）判斷直線與圓的相交情形

(1) 圓 $x^2+y^2-4x+6y-8=0$, 和直線 $x=5$

(2) 圓 $x^2+y^2-4x+6y-8=0$, 和直線 $y=x-15$

Teacher: Let’s use the distance formula to find the distance between the center and the line. Compare the distance and the radius. How do you find the radius of the circle?

Student: Change the general form to the standard form.

Teacher: Correct. I will give you some time to do that.

Student: $(x^2-4x+4)+(y^2+6y+9)=8+4+9$,

then I get: $(x-2)^2+(y+3)^2=21$

Teacher: What are the center and radius?

Student: The center is $(2,-3)$, and the radius is $\sqrt{21}$.

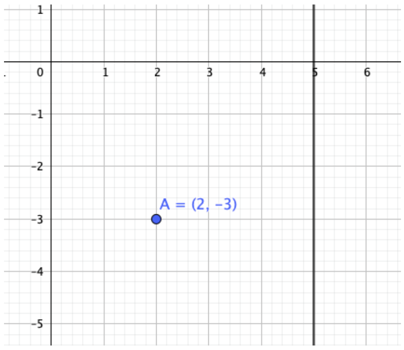
Teacher: Please find the distance between $(2, -3)$ and $x = 5$.

Student: There is no *y*-term in the line equation. There's no *b*!

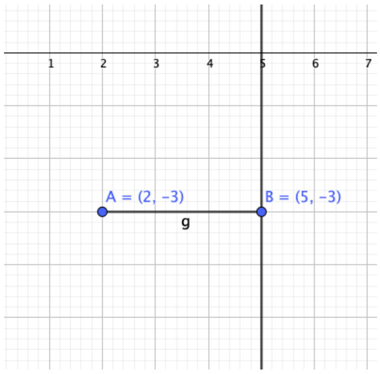
Teacher: You can write the line equation as $x + 0y - 5 = 0$. It will be easy for you to compare the coefficients to get the values of *a*, *b*, and *c*.

Student: Okay. I use the distance formula: $d = \frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$, and I get: $\frac{|1\times 2-0\times(-3)-5|}{\sqrt{1^2+0}} = 3$

Teacher: It's good to use the distance formula, but there's another way I want to show you. I draw the line $x = 5$, and plot the point $(2, -3)$ on a coordinate plane. Like this:



$x = 5$ is a vertical line. You draw a segment perpendicular to the line $x = 5$,



through the point $(2, -3)$. Like this:

The segment will meet at $(5, -3)$. The distance between the point $(2, -3)$ and the line equals the distance between $(2, -3)$ and $(5, -3)$. You can count or use easy math $5 - 2 = 3$. The distance is 3.

Student: Smart. That's faster. Does this work every time?

Teacher: No, this method can be used only for specific lines, like vertical lines and horizontal lines. Now you can compare the distance and the radius.

Student: The distance *d* is 3. The radius is $\sqrt{21}$, which is about 4 point something. $3 < \sqrt{21}$.

Teacher: What is your conclusion?

Student: The distance between the center and the line is less than the radius, so the line cuts through the circle at two points of intersection.

Teacher: Great. Now you can move on to the next question.

Student: The line is $y = x - 15$, and it is not a special line. I have to use the distance formula, right?

Teacher: Correct. You can rewrite the line equation to compare the coefficients.

Student: $x - y - 15 = 0$.

The distance formula is: $d = \frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$, and I get: $\frac{|1\times2+(-1)\times(-3)-15|}{\sqrt{1^2+1^2}} = \frac{10}{\sqrt{2}}$.

Teacher: Please rationalize the answer.

Student: $\frac{10}{\sqrt{2}} = 5\sqrt{2}$. This is d.

Teacher: Compare $5\sqrt{2}$ and the radius $\sqrt{21}$.

Student: Can I use a calculator?

Teacher: Can you use mental math?

Student: Fine. $\sqrt{2}$ is about 1.4, and 5 times 1.4 is 7. $5\sqrt{2}$ is about 7. $\sqrt{21}$ is 4 point something. Thus, $5\sqrt{2}$ is greater than $\sqrt{21}$.

Teacher: What's your conclusion about the intersection?

Student: The distance between the center and the line is greater than the radius, so the line is outside the circle.

Teacher: Correct, so there's no point of intersection between the line and the circle.

老師： 使用距離公式找到圓心到直線之間的距離，並比較這段距離和半徑。怎麼求圓的半徑？

學生： 將一般式轉換為標準式。

老師： 正確。給你一些時間完成。

學生： $(x^2 - 4x + 4) + (y^2 + 6y + 9) = 8 + 4 + 9$，然後得到：

$(x-2)^2 + (y+3)^2 = 21$

老師： 圓心和半徑是什麼？

學生： 圓心是 $(2, -3)$，半徑是 $\sqrt{21}$。

老師： 請找出 $(2, -3)$ 和 $x = 5$ 之間的距離。

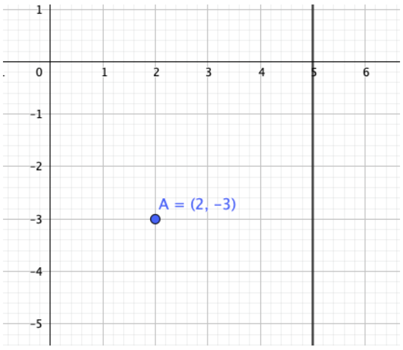
學生： 直線方程中沒有 y 項。沒有 b。

老師： 可以將線方程寫為 $x + 0y - 5 = 0$。這樣可以輕易比較係數，得到 a、b 和 c 的值。

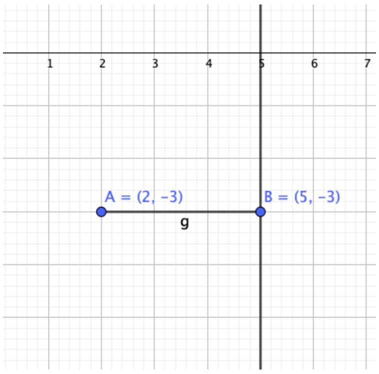
學生： 好的。使用距離公式：$d = \frac{|ax+by+c|}{\sqrt{a^2+b^2}}$，然後我得到：$\frac{|2\times1-3\times0-5|}{\sqrt{1^2+0}} = 3$

老師： 用距離公式是很好，但還有另一種方法。畫一條 $x = 5$ 的直線，在坐標平面上

標記點 $(2,-3)$。像這樣：



$x=5$ 是垂直線。你畫一條垂直於 $x=5$ 的線段，通過點 $(2,-3)$。像這樣：



這個線段會在 $(5,-3)$ 交會。點 $(2,-3)$ 和該直線之間的距離等於點 $(2,-3)$ 和 $(5,-3)$ 之間的距離。可以用數的或簡單計算 $5-2=3$。距離為 3。

學生：妙。這樣更快。這種方法每次都適用嗎？

老師：沒有，這個方法只能用於特定的直線，比如垂直線和水平線。現在可以比較距離和半徑了。

學生：距離 d 是 3。半徑是 $\sqrt{21}$，約為 4 點多。$3<\sqrt{21}.$。

老師：結論是什麼？

學生：圓心和線之間的距離小於半徑，所以線在圓上有兩個交點。

老師：很好。現在可以繼續下一題了。

學生：這條線是 $y=x-15$，且不是一條特殊線。這裡要用距離公式對嗎？

老師：正確。你可以重寫直線方程以便比較係數。

學生：$x-y-15=0$。

距離公式是：$d=\frac{|ax+by+c|}{\sqrt{a^2+b^2}}$，然後我得到：$\frac{|2\times1-3\times(-1)-15|}{\sqrt{1^2+1^2}}=\frac{10}{\sqrt{2}}$。

老師：請將答案有理化。

學生：$\frac{10}{\sqrt{2}}=5\sqrt{2}$ 為 d。

老師：比較 $5\sqrt{2}$ 和半徑 $\sqrt{21}$。

學生：可以用計算機嗎？

老師： 可以心算嗎？
學生： 好。$\sqrt{2}$大約是 1.4，而 5 乘以 1.4 是 7。$5\sqrt{2}$ 大約是 7。$\sqrt{21}$ 是 4 點多，所以，$5\sqrt{2}$ 大於 $\sqrt{21}$。
老師： 關於交點，你的結論是什麼？
學生： 圓心和直線之間的距離大於半徑，所以線在圓外。
老師： 正確，所以直線和圓之間沒有交點。

↜ 應用問題 / 學測指考題 ↝

例題一

說明：已知直線和圓的關係，反推可能的值為何？

（英文）Given the equation of the circle $x^2+y^2+2x-4y=8$, find the value of m for which the line $2x-3y=m$ intersects the circle at 2 points.
（中文）已知圓方程式$x^2+y^2+2x-4y=8$，找出 m 值，使得直線 $2x-3y=m$ 與圓交於兩點。

Teacher: Please write the general form in standard form. I'll give you some time to do that.
Student: $(x^2+2x+1)+(y^2-4y+4)=8+4+1=13$, $(x+1)^2+(y-2)^2=13$.
Teacher: What are the center and radius?
Student: The center is $(-1,2)$, and the radius is $\sqrt{13}$.
Teacher: Next, use the distance formula to find out the distance between the center and the line. You can rewrite the line equation first.
Student: The line is: $2x-3y-m=0$.
The distance between the center (-1,2) to the line is: $\frac{|-2-6-m|}{\sqrt{4+9}}$.
Teacher: The question said that the line intersects the circle at 2 points. So which one is greater, *d* or *r*?
Student: $r>d$.
Teacher: Okay. We can use the above information to write an inequality: $\frac{|-2-6-m|}{\sqrt{4+9}}<\sqrt{13}$, then solve this inequality. We have $\frac{|-2-6-m|}{\sqrt{13}}<\sqrt{13}$, then multiply $\sqrt{13}$ on both sides.
Student: The denominator $\sqrt{13}$ will be crossed out, and I will get: $|-8-m|<13$.

Teacher: Good. Take away the absolute value sign. You will have an inequality $-13 < -8 - m < 13$. Do you remember the next step? You have to keep m in the middle of the inequality.

Student: Plus 8, to cancel out -8.

Teacher: Good. Plus 8, and you will get: $-13 + 8 < -8 - m + 8 < 13 + 8$. Simplify it, and the inequality is: $-5 < -m < 21$. What is the next step?

Student: Change negative m to positive m.

Teacher: Yes, we multiply -1 on the inequality. We have $5 < m < -21$. But is -21 greater than 5? It looks weird, right? When we multiply a negative number, remember to change the sign from "less than" to "greater than." Eventually, we will have $5 > m > -21$

老師： 請將一般式轉換為標準式。我給你一些時間完成。

學生： $(x^2 + 2x + 1) + (y^2 - 4y + 4) = 8 + 4 + 1$=13, $(x + 1)^2 + (y - 2)^2 = 13$.

老師： 圓心和半徑是？

學生： 圓心是$(-1,2)$，半徑是 $\sqrt{13}$。

老師： 接下來，使用距離公式找出圓心和直線之間的距離。你可以先重寫直線方程式。

學生： 直線是：$2x - 3y - m = 0$.。

距離是：$\frac{|-2-6-m|}{\sqrt{4+9}}$。

老師： 題目說直線與圓相交於兩點。那麼哪個大，d 還是 r？

學生： $r > d$。

老師： 好的。可以用上述資訊寫出不等式：$\frac{|-2-6-m|}{\sqrt{4+9}} < \sqrt{13}$，然後解不等式。有 $\frac{|-2-6-m|}{\sqrt{13}} < \sqrt{13}$，然後兩邊乘以 $\sqrt{13}$。

學生： 分母的 $\sqrt{13}$ 會被消掉，我得到：$|-8 - m| < 13$。

老師： 很好。去掉絕對值符號，會得到一個不等式 $-13 < -8 - m < 13$。還記得下一步嗎？必須讓 m 保持在不等式的中間。

學生： 加上 8，抵消掉-8。

老師： 很好。加上 8，你會得到：$-13 + 8 < -8 - m + 8 < 13 + 8$。簡化讓不等式變成：$-5 < -m < 21$。下一步是？

學生： 將負的 m 改成正的 m。

老師： 是的，將不等式乘以 -1。得到 $5 < m < -21$。但是 -21 大於 5 嗎？看起來很奇怪吧？當乘以一個負數時，記得將符號從「小於」改為「大於」。最後會得到 $5 > m > -21$。

例題二

說明：給定圓上一點 (c, d)，推導出圓切線方程式的一般式。

（英文）Given a circle $x^2 + y^2 = a$ and point (c, d) on the circle, find the equation of the tangent line through the point (c, d).

（中文）給定一圓 $x^2 + y^2 = a$ 與圓上一點 (c, d)，找出通過點的切線方程式。

Student: There are no numbers. How do I find the equation?

Teacher: This question needs you to derive the general form of the tangent lines. You can pretend that letters a, c, and d are known numbers. The only variables are x and y.

Student: Okay.

Teacher: I suggest that you scratch the circle and the tangent point, which may help you think. What is the center of the circle?

Student: (0, 0).

Teacher: Now draw a circle, and label the center (0, 0). Plot a random point on the circle, and label it (c, d).

Student: Done.

Teacher: Now connect the segment between the center and the point (c, d). What is the slope of the segment?

Student: $\frac{d}{c}$.

Teacher: Now construct a tangent line through the point (c, d). Remember that the tangent line should be perpendicular to the segment you drew. What is the slope of the tangent line?

Student: I am not sure.

Teacher: The slope of the tangent line is the negative reciprocal of $\frac{d}{c}$, because they are perpendicular.

Student: $\frac{-c}{d}$?

Teacher: Yes. With the given point (c, d), and slope $\frac{-c}{d}$, can you construct the line equation?

Student: $y - d = \frac{-c}{d}(x - c)$. Is this the answer?

Teacher: You are not there yet. Please simplify the equation by multiplying d on both sides.

Student: $dy - d^2 = -cx + c^2$.

Teacher: We move x-terms and y-terms together and move the c and d to the other side of the equation. We will get: $cx + dy = c^2 + d^2$. The point (c, d) is on the circle $x^2 + y^2 = a$. This means that the point (c, d) will satisfy the equation.

Student: $c^2 + d^2 = a$. Aha! You can replace $c^2 + d^2$.

Teacher: Yes. The general form of a tangent line is $cx + dy = a$.

Student: I can directly plug in the coordinates next time with this general form.

學生：沒有數字，怎麼求方程式？

老師：這題要你推導出切線的一般式。你可以假設未知數 a、c 和 d 是已知的數字，唯一的變量是 x 和 y。

學生：好。

老師：我建議你將圓和切點都畫出來，有助於思考。圓的圓心是什麼？

學生：(0, 0)。

老師：現在畫一個圓，並標記圓心為 $(0, 0)$。在圓上標記一個隨機點，並標記為 (c, d)。

學生：完成了。

老師：現在連接圓心和點(c, d) 之間的線段。這個線段的斜率是多少？

學生：$\frac{d}{c}$。

老師：現在穿過點 (c, d) 畫一條切線。記住切線應該跟你畫的線段垂直。切線的斜率是多少？

學生：我不確定。

老師：切線的斜率是 $\frac{d}{c}$ 的負倒數，因為它們是垂直的。

學生：$\frac{-c}{d}$？

老師：是的。根據給定的點 (c, d) 和斜率 $\frac{-c}{d}$，能求出直線方程嗎？

學生：　$y-d=\frac{-c}{d}(x-c)$。這是答案嗎？

老師：　還沒完成。請簡化方程式，兩邊同乘以 d。

學生：　$dy-d^2=-cx+c^2$。

老師：　將 x 項和 y 項移到一起，並將 c 和 d 移到方程式的另一邊。
我們會得到：$cx+dy=c^2+d^2$。點 (c,d) 在圓 $x^2+y^2=a$上。這表示點 (c,d) 將滿足這個方程式。

學生：　$c^2+d^2=a$。啊哈！你可以換掉 c^2+d^2 。

老師：　是的。切線的一般式是 $cx+dy=a$。

學生：　下次可以直接使用這個一般式來代入坐標。

例題三

說明：利用圓切線的觀念，解決日常生活的問題。

（英文）Mindy is designing a farm fence on a scratch app. The equation of a circular farm is $x^2+y^2-4x-4y+7=0$. She would like to draw tangent lines of the circle from a fixed point $(0,0)$, and build a wood farm fence along the tangent line. Find the equation of the tangent lines drawn from the origin to the circle.

（中文）Mindy 在一個繪圖應用程式上設計農場的欄杆。
圓形農場的方程式是$x^2+y^2-4x-4y+7=0$，她想從一點 $(0,0)$ 畫圓的切線，沿著這條切線蓋木製的農場欄杆，請找出切線方程式。

Teacher: Write the equation of the circle in standard form.

Student: $(x^2-4x+4)+(y^2-4y+4)=-7+4+4=1$, $(x-2)^2+(y-2)^2=1$.

Teacher: What is the center? What is the radius?

Student: The center is $(2,2)$, and the radius is 1.

Teacher: We said that we need the tangent line's slope and the tangent point to find the tangent line's equation. But now, we don't have a point of tangency. The point $(0,0)$ is outside of the circle.

Student: So what can we do?

Teacher: We can start with the slope, and let's assume the slope of the tangent line as m. With the fixed point $(0,0)$, we can write the equation of the tangent line in the point-slope form: $y-0=m(x-0)$. Simplify it, and we get the equation of tangent line

$y = mx$. The radius is the distance between the center (2, 2) and the tangent line. Can you apply the distance formula to find the distance between the center (2, 2) and the tangent line?

Student: I have to rewrite the equation first: $mx - y = 0$. The distance is $\frac{|2m-2|}{\sqrt{m^2+1}}$.

Teacher: The distance equals 1. Please write an equation with this information.

Student: $\frac{|2m-2|}{\sqrt{m^2+1}} = 1$.

Teacher: Good. Please solve the equation for m.

Student: There are absolute values and square roots at the same time. What should I do?

Teacher: Try cross multiplication first.

Student: You mean $|2m-2| = \sqrt{m^2+1}$?

Teacher: Yes. Square both sides, and we will get: $(2m-2)^2 = (\sqrt{m^2+1})^2$.
Please simplify this.

Student: $4m^2 + 4 - 8m = m^2 + 1$. Then I get the equation: $3m^2 - 8m + 3 = 0$

Teacher: Solve this quadratic equation.

Student: $3m^2 - 8m + 3 = (m-3)(3m+1) = 0$, so $m = 3$ or $\frac{-1}{3}$.
Are there two answers?

Teacher: Yes, two tangent lines exist through a point outside the circle. You can write the equation.

Student: The point is (0, 0), and I will use point-slope form…

Teacher: You can skip that step. In the beginning, we just derived that the equation of the tangent line is $y = mx$. You can just substitute 3 or $\frac{-1}{3}$ for m.

Student: I see. The tangent lines are $y = 3x$ and $y = \frac{-1}{3}x$.

老師： 寫出圓的標準式。

學生： $(x^2 - 4x + 4) + (y^2 - 4y + 4) = -7 + 4 + 4 = 1,\ (x-2)^2 + (y-2)^2 = 1$。

老師： 圓心是什麼？半徑是多少？

學生： 圓心是 (2, 2)，半徑是 1。

老師： 我們說過，我們需要切線的斜率和切點來找到切線的方程式。但現在我們沒有切點。點 (0, 0) 在圓外面。

學生： 那麼該怎麼做？

老師： 可以從斜率開始，假設切線的斜率為 m。根據定點 (0, 0)，可以用點斜式來寫

出切線的方程式：$y-0=m(x-0)$。簡化後得到切線的方程式 $y=mx$。
半徑是圓心 $(2,2)$ 到切線的距離。能用距離公式來找到圓心 $(2,2)$ 和切線之間的距離嗎？

學生： 要先重新寫出方程式：$mx-y=0$。距離是 $\frac{|2m-2|}{\sqrt{m^2+1}}$。

老師： 距離等於 1。請根據資訊寫出方程式。

學生： $\frac{|2m-2|}{\sqrt{m^2+1}}=1$。

老師： 好的。請解方程式找出 m。

學生： 這個方程式同時有絕對值和平方根，該怎麼辦？

老師： 先試試交叉相乘。

學生： 你是指 $|2m-2|=\sqrt{m^2+1}$？

老師： 是的。兩邊平方，得到： $(2m-2)^2=(\sqrt{m^2+1})^2$。請簡化方程式。

學生： $4m^2+4-8m=m^2+1$。然後得到方程式：$3m^2-8m+3=0$

老師： 解二次方程式。

學生： $3m^2-8m+3=(m-3)(3m+1)=0$，所以 $m=3$或 $\frac{-1}{3}$。有兩個答案嗎？

老師： 是的，通過圓外一點會有兩條切線。可以寫出方程式。

學生： 這個點是 $(0,0)$，使用點斜式...

老師： 可以跳過這步。一開始我們推導出切線的方程式是 $y=mx$。你可以直接將 3 或 $\frac{-1}{3}$ 代入 m。

學生： 懂了。切線的方程式是 $y=3x$ 跟 $y=\frac{-1}{3}x$。

國內外參考資源 More to Explore

國家教育研究院樂詞網	
查詢學科詞彙 https://terms.naer.edu.tw/search/	
教育雲：教育媒體影音	
為教育部委辦計畫雙語教學影片 https://video.cloud.edu.tw/video/co_search.php?s=%E9%9B%99%E8%AA%9E	
Oak Teacher Hub	
國外教學及影音資源，除了數學領域還有其他科目 https://teachers.thenational.academy/	
CK-12	
國外教學及影音資源，除了數學領域還有自然領域 https://www.ck12.org/student/	
Twinkl	
國外教學及影音資源，除了數學領域還有其他科目，多為小學及學齡前內容 https://www.twinkl.com.tw/	

Khan Academy	
可汗學院，有分年級數學教學影片及問題的討論 https://www.khanacademy.org/	
Open Textbook (Math)	
國外數學開放式教學資源 http://content.nroc.org/DevelopmentalMath.HTML5/Common/toc/toc_en.html	
MATH is FUN	
國外教學資源，還有數學相關的小遊戲 https://www.mathsisfun.com/index.htm	
PhET: Interactive Simulations	
國外教學資源，互動式電腦模擬。除了數學領域，還有自然科 https://phet.colorado.edu/	
Eddie Woo YouTube Channel	
國外數學教學影音 https://www.youtube.com/c/misterwootube	

國立臺灣師範大學數學系陳界山教授網站	
國高中數學雙語教學相關教材 https://math.ntnu.edu.tw/~jschen/index.php?menu=Teaching_Worksheets	
2023 年第四屆科學與科普專業英文(ESP)能力大賽	
科學專業英文相關教材，除了數學領域，還有其他領域 https://sites.google.com/view/ntseccompetition/%E5%B0%88%E6%A5%AD%E8%8B%B1%E6%96%87%E5%AD%B8%E7%BF%92%E8%B3%87%E6%BA%90/%E7%9B%B8%E9%97%9C%E6%95%99%E6%9D%90?authuser=0	

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[十年級上學期]

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[10th grade 1st semester]

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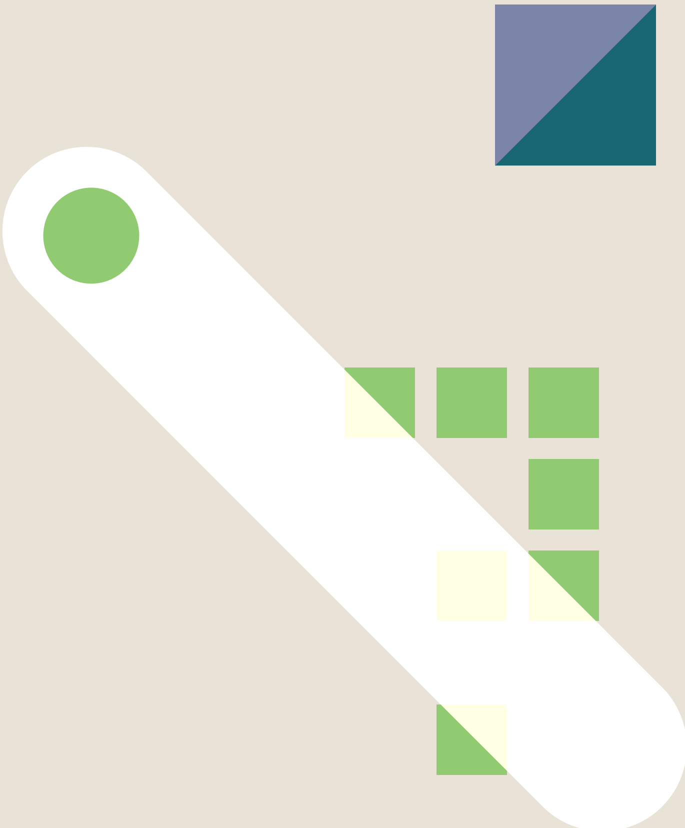

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