國中數學領域

雙語教學資源手冊 英語授課用語

A Reference Handbook for Junior High School Bilingual Teachers in the Domain of Mathematics: Instructional Language in English

〔九年級上學期〕

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單元一 連比
Continued Ratio

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■ 前言 Introduction

本節透過以前所學的比例概念來引入連比，再接著學習三個數兩兩相比的比例求得三數的連比例，並利用連比例的概念處理生活上的實際問題。

■ 詞彙 Vocabulary

單字	中文	單字	中文
continued proportion	連比例	highest common factor	最大公因數
simplest ratio	最簡整數比		

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ **The ratio of the numbers of _______ to _______ is ________.**

例句：**The ratio of the numbers of** apples **to** pears **is** 2:5.

蘋果的數量比梨子的數量為 2 比 5。

■ 問題講解 Explanation of Problems

運算問題的講解

例題一

說明：本題是讓學生了解連比例間的轉換關係。
Students can understand how to convert continued proportion.

（英文）A convenience store would like to renew its signboard. The length and width are x and y cm, respectively, and $x:y=11:4$; the width and length of the side of the board are y and z cm, respectively, and $y:z=4:9$. What would $x:y:z$ (x to y to z) be?

（中文）某便利商店想更換招牌，已知正面招牌的長與寬分別是 x 公分與 y 公分，且 $x:y=11:4$，側面招牌的寬與長分別是 y 公分與 z 公分，且 $y:z=4:9$，求 $x:y:z=$？

Teacher: What can we know from this question?

Student: Ummm...

Teacher: Okay. It tells us that we can substitute the length and width of the front of the signboard and the width and length of the side of the signboard for x, y, and z.

Student: Yes, it does.

Teacher: What else leads can we get from the question?

Student: $x:y=11:4$ and $y:z=4:9$.

Teacher: Well done! To determine $x:y:z$ in continued proportion, we can firstly write down the proportion of $x:y$ and $y:z$.

x	y	z
11	4	
	4	9

Teacher: After writing them down, which unknown number shows up repeatedly?

Student: Only y does.

Teacher: Yes! Because the value of a proportion is fixed, we must make sure the repeated number of y is the same. We can only merge the numbers that have the same value.

Teacher: In $x:y=11:4$ and $y:z=4:9$, y has the same value, we can therefore merge the equations as $x:y:z=11:4:9$ directly.

老師： 來~這題題目給了我們什麼訊息？

學生： 恩…

老師： 好，他是不是告訴了我們正面招牌的長寬和側面招牌的寬長分別用 x、y、z 代替。

學生： 對！

老師： 那它還給了我們什麼線索？

學生： $x:y=11:4$ 和 $y:z=4:9$。

老師： 很棒！回答得很好。那求出 $x:y:z$ 的連比例關係，我們可以先將 $x:y$ 和 $y:z$ 的比例先寫下來。

x	y	z
11	4	
	4	9

老師： 寫下來後發現了哪個未知數在上下列都有重複出現呢？

學生： 只有 y 有重複出現。

老師： 沒錯！因為比例的數值都是固定的，所以我們必須確保重複的數值是否一樣，只有一樣的數值我們才能直接合併。

老師： 也就是在 $x:y=11:4$ 和 $y:z=4:9$ 的比例式中，y 在兩式的數值都是一樣的，所以可以直接合併成 $x:y:z=11:4:9$。

例題二

說明：本題是讓學生了解連比例間的轉換關係。

Students can understand how to convert continued proportion.

（英文）Assume $x:y=3:4$ and $y:z=6:7$,
determine the continued proportion of $x:y:z$.

（中文）設 $x:y=3:4$，$y:z=6:7$，求 $x:y:z$ 三數的連比例式。

Teacher: Firstly, let’s write down $x:y$ and $y:z$ as we did in question one. What shows up repeatedly?

Student: I know, y shows up repeatedly!

Teacher: Good. Now we know that y shows up repeatedly. To make the value that responds to y the same, we should expand the two proportions, and the numbers of y will be the same. We can find the least common multiple of 4 and 6 by using this idea. Does anyone know what the least common multiple of 4 and 6 is?

Student: The least common multiple of 4 and 6 is 12, Sir!

Teacher: Yes, exactly! So, we can expand $x : y = 3 : 4$ as $9 : 12$, and $y : z = 6 : 7$ as 12：14.

x	y	z
3	4	
	6	7
x	y	z
9	12	
	12	14

Teacher: Now, we have the same value of y in the upper and lower lines, and we can merge the two proportions.

Therefore, the continued proportion of $x : y : z$ is 9：12：14.

Teacher: Finally, don't forget to make sure the continued proportion is irreducible. If not, remember to make it irreducible.

老師： 首先，我們將 x 比 y 跟 y 比 z 依序寫下來，透過例題一的方式，請問一下重複出現文字符號是什麼呢？

學生： 我知道，y 重複出現了！

老師： 很好，我們找出相同的文字符號是 y，那為了使對應的 y 值相同，在這邊我們要將這兩個比擴大，使得 y 的值一樣，那我們可以利用最小公倍數的方法，找出 4 和 6 的最小公倍數，那有人知道 4 和 6 的最小公倍數是多少嗎？

學生： 老師，4 和 6 的最小公倍數是 12！

老師： 對！沒錯！所以我們可以將 $x : y = 3 : 4$擴成 9：12，而 $y : z = 6 : 7$ 可以擴成 12：14。

x	y	z
3	4	
	6	7
x	y	z
9	12	
	12	14

老師：　這時上下的 y 值就一樣，那麼就可以將兩個比例合併，因此 $x：y：z$ 的連比為 $9：12：14$。

老師：　最後記得確認連比例是不是最簡整數比，不是的話記得換成最簡整數比。

例題三

說明：本題是要讓學生熟悉連比例的運算方式與性質。

Students can get familiar with the calculation and features of continued proportion.

（英文）If $x：y：z = 2：3：4$, and $x + y + 2z = 39$, what are the values of x, y, and z.

（中文）如果$x：y：z = 2：3：4$，且$x + y + 2z = 39$，求 x、y、z 的值。

Teacher:　We firstly see $x：y：z = 2：3：4$. There are three unknown numbers in continued proportion. Do you remember what to do when there are only two unknown numbers, $x：y = a：b$, in continued proportion?

Student:　I do! We can assume $x = a \cdot r$; $y = b \cdot r$.

Teacher:　Good. But remember, the r here cannot be 0.

Teacher:　We do the same way to the proportion with three unknown numbers. We can see $x：y：z = 2：3：4$ as $x = 2 \cdot r, y = 3 \cdot r,$ and $z = 4 \cdot r$, and substitute them for the equation $x + y + 2z = 39$. Then, we will get:

$2r + 3r + 8r = 39$

$13r = 39$

$r = 3.$

Teacher:　Now substitute $r = 3$ back for $x = 2 \cdot r$; $y = 3 \cdot r$, and $z = 4 \cdot r$.

Finally, we get that $x = 6$; $y = 9$; $z = 12$.

老師： 首先我們先看到$x：y：z=2：3：4$，這裡有三個未知數的連比例式，那大家還記得只有兩個未知數 $x：y=a：b$ 的話我們會怎麼做嗎？

學生： 我知道！可以假設 $x=a\cdot r$；$y=b\cdot r$。

老師： 很好，但要記得這時的 r 不能為 0 喔！

老師： 那三個未知數的比例也是一樣的做法，我們將$x：y：z=2：3：4$ 看成

$x=2\cdot r$、$y=3\cdot r$、$z=4\cdot r$ 代入$x+y+2z=39$，得到

$2r+3r+8r=39$

$13r=39$

$r=3$。

老師： 代回 $x=2\cdot r$、$y=3\cdot r$、$z=4\cdot r$，所以 $x=6$；$y=9$；$z=12$。

單元二 比例線段
Proportional Line Segments

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■ 前言 Introduction

本節首先透過八年級所學的「兩個等高三角形的面積比」來引入，接著將此性質透過探索活動，發展出「平行線所截線段成比例」。當學生有了比例線段的概念後，反過來利用此概念判別所截的線段是否平行，接著再延伸學習三角形兩邊中點連線的性質與應用。

■ 詞彙 Vocabulary

單字	中文	單字	中文
parallel lines	平行線	parallel	平行
proportional line segments	比例線段	midpoint of line	線段中點
area	面積	triangle	三角形

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ If ____and ____are parallel, then __:__ is equal to __:__.

例句(1)：Suppose P and Q are on $\overline{AB}$ 、$\overline{AC}$, respectively. **If $\overline{PQ}$ *and* $\overline{BC}$ are parallel, then $\overline{AP}$ ：$\overline{PB}$ is equal to** $\overline{AQ}$ ：$\overline{QC}$.(graph 1)

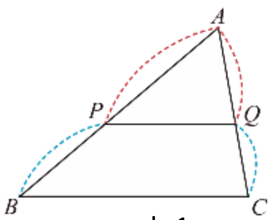
假設P點和Q點分別在$\overline{AB}$線段與$\overline{AC}$線段上。當$\overline{PQ}$線段與$\overline{BC}$線段平行時，$\overline{AP}$線段$\overline{PB}$線段等於$\overline{AQ}$線段比$\overline{QC}$線段。

例句(2)：Suppose P and Q are on $\overline{AB}$ 、$\overline{AC}$, respectively. **If $\overline{PQ}$ *and* $\overline{BC}$ are parallel, then $\overline{AP}$ ：$\overline{AB}$ is equal to** $\overline{AQ}$ ：$\overline{AC}$.(graph 2)

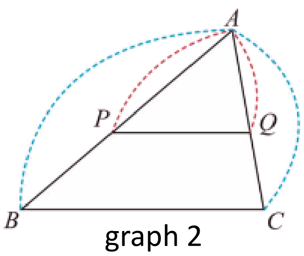
假設P點和Q點分別在$\overline{AB}$線段與$\overline{AC}$線段上。當$\overline{PQ}$線段與$\overline{BC}$線段平行時，$\overline{AP}$線段比$\overline{AB}$線段等於$\overline{AQ}$線段比$\overline{AC}$線段。

例句(3)：Suppose P and Q are on $\overline{AB}$ 、$\overline{AC}$, respectively. **If $\overline{PQ}$ *and* $\overline{BC}$ are parallel**, then $\overline{AB}$ ：$\overline{PB}$ **is equal to** $\overline{AC}$ ：$\overline{QC}$.(graph 3)

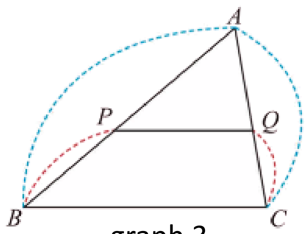
假設P點和Q點分別在$\overline{AB}$線段與$\overline{AC}$線段上。當$\overline{PQ}$線段與$\overline{BC}$線段平行時，$\overline{AB}$線段比$\overline{PB}$線段的比例會等於$\overline{AC}$線段比$\overline{QC}$線段。



graph 1



graph 2



graph 3

❷ If __:__ is equal to __:__, then ____ and ____are parallel.

例句(1)：Suppose P and Q are on $\overline{AB}$ 、$\overline{AC}$. If $\overline{AP}$ ：$\overline{PB}$ **is equal to** $\overline{AQ}$ ：$\overline{QC}$, **then $\overline{PQ}$ and $\overline{BC}$ are parallel.**(graph 1)

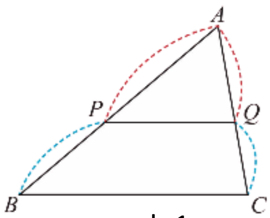
假設P點和Q點分別在$\overline{AB}$線段與$\overline{AC}$線段上。當$\overline{AP}$線段比$\overline{PB}$線段的比例等於$\overline{AQ}$線段比$\overline{QC}$線段的成比例時，$\overline{PQ}$線段與$\overline{BC}$線段平行。

例句(2)：Suppose P and Q are on $\overline{AB}$、$\overline{AC}$. **If** $\overline{AP}$：$\overline{AB}$ **is equal to** $\overline{AQ}$：$\overline{AC}$, **then** $\overline{PQ}$ **and** $\overline{BC}$ **are parallel**.(graph2)

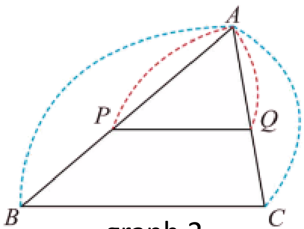
假設 P點和 Q 點分別在 $\overline{AB}$線段與 $\overline{AC}$線段上。當 $\overline{AP}$ 線段比 $\overline{AB}$ 線段的比例等於 $\overline{AQ}$ 線段比 $\overline{AC}$ 線段的成比例時，$\overline{PQ}$線段與$\overline{BC}$線段平行。

例句(3)：Suppose P and Q are on $\overline{AB}$、$\overline{AC}$. **If** $\overline{AB}$：$\overline{PB}$ **is equal to** $\overline{AC}$：$\overline{QC}$, **then** $\overline{PQ}$ **and** $\overline{BC}$ **are parallel**.(graph3)

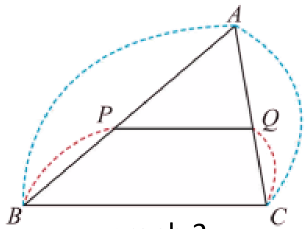
假設 P 點和 Q 點分別在 $\overline{AB}$ 線段與 $\overline{AC}$ 線段上。當 $\overline{AB}$ 線段比 $\overline{PB}$ 線段的比例等於 $\overline{AC}$ 線段比 $\overline{QC}$ 線段的成比例時，$\overline{PQ}$ 線段與 $\overline{BC}$ 線段平行。



graph 1



graph 2

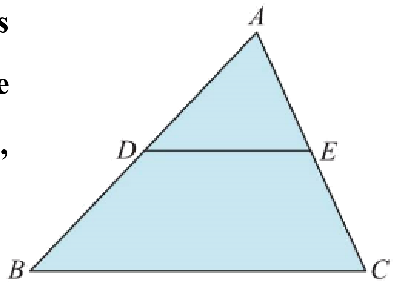


graph 3

❸ **Suppose _____ and _____ are the middle points of two sides _____ and _____ respectively in the triangle, the line segment, _____, is equal to half the length of the third side, _____.**

例句：**Suppose** D **and** E **are the middle points of two sides** $\overline{AB}$ **and** $\overline{AC}$ **respectively in the triangle, the line segment,**$\overline{DE}$**, is equal to half the length of the third side,** $\overline{BC}$**.**

因為D點和E點是三角形兩邊的中點，因此兩點的連線段長為第三邊的一半長。



■ 問題講解 Explanation of Problems

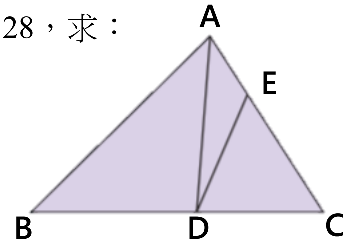
運算問題的講解

例題一

說明：本題是讓學生熟悉當具有相同高時，兩線段的比例與面積的關係。
Students can get familiar through this question with the ratio of two line segments of two triangles having equal heights in relations of their areas.

（英文）In the diagram, above points D and E are on $\overline{BC}$ and $\overline{AC}$, respectively.
If $\overline{BD}$: $\overline{DC}$ = 4 : 3, $\overline{CE}$: $\overline{EA}$ = 5 : 2, and the area of $\triangle ABD$ is 28, find:
(1) the area of $\triangle ADC$.
(2) the area of $\triangle DAE$.

（中文）如圖，$\triangle ABC$ 中，D、E 分別在 $\overline{BC}$ 與 $\overline{AC}$ 上，若 $\overline{BD}$: $\overline{DC}$ = 4 : 3，$\overline{CE}$: $\overline{EA}$ = 5 : 2，且 $\triangle ABD$ 的面積是 28，求：
(1) $\triangle ADC$ 的面積。
(2) $\triangle DAE$ 的面積。



（三上翰林課本 P.72 例題 1）

(1)

Teacher: The first thing that jumps out at us is whether $\triangle ABD$ and $\triangle ADC$ joining each other. Do they have the same height?

Student: They do.

Teacher: Do you still remember what the ratio of the areas of two triangles with equal heights is equal to?

Student: The ratio of their corresponding bases.

Teacher: That is correct. Given $\overline{BD}$: $\overline{DC}$ = 4 : 3, we can say that the ratio of the areas of $\triangle ABD$: $\triangle ADC$ is 4 : 3.

Teacher: Now, we can also assume that the area of $\triangle ABD$ is 4r and the area of $\triangle ADC$ is $3r$, where r is not equal to 0 and $4r$ is equal to 28, that gets us $r = 7$. So, the area of $\triangle ADC$ is $3 \times 7 = 21$.

(2)

Teacher: To answer the second question, we can also take the same approach. Which triangle has the same height as $\triangle DAE$?

Student: I know! It's $\triangle DCE$!

Teacher: Excellent! That can get us the ratio of the areas of $\triangle DCE$: $\triangle DAE$ equal the ratio of the lengths of their bases $\overline{CE}$: $\overline{EA}$ = 5 : 2.

Teacher: Now, we can assume that the area of $\triangle DCE$ is $5r$ and the area of $\triangle DAE$ is $2r$. If we combine $\triangle DCE$ and $\triangle DAE$ together, they become $\triangle ADC$. We've already known that its area is 21.

Teacher: So, we will just need to figure out the following equation

$5r + 2r = 21$

$7r = 21$

$r = 3.$

Teacher: Now, we can solve for the area of $\triangle DCE$, which is $3 \times 5 = 15$.

(1)

老師： 首先觀察一下$\triangle ABC$裡面兩個三角形$\triangle ABD$和$\triangle ADC$是否等高？

學生： 對的。

老師： 那還記得兩個等高三角形面積比會等於甚麼嗎？

學生： 他的對應底邊的比。

老師： 沒錯，因為$\overline{BD}$：$\overline{DC}$ =4：3，所以我們知道$\triangle ABD$*的面積*：$\triangle ADC$的面積為4：3

老師： 接著，我們假設$\triangle ABD$的面積為$4r$、$\triangle$ADC的面積為$3r$，這個r不等於0，那麼$4r$又會等於28，可以得出$r = 7$，因此我們可以得到$\triangle ADC$的面積為$3 \times 7 = 21$。

(2)

老師： 根據前一題的方式，我們先觀察$\triangle DAE$會跟哪一個三角形等高？

學生： 我知道，是$\triangle DCE$！

老師： 很好，所以我們可以得到$\triangle DCE$的面積：$\triangle DAE$的面積 為他們的底邊長之比$\overline{CE}$：$\overline{EA}$ =5：2。

老師： 這時我們假設$\triangle DCE$的面積和$\triangle DAE$的面積分別為$5r$和$2r$，將$\triangle DCE$和$\triangle DAE$合併可以得到$\triangle ADC$，其面積為21。

老師： 所以我們可以做以下計算：

$5r + 2r = 21$

$7r = 21$

$r = 3$。

老師： 將r代回$\triangle DCE$得其面積為 $3 \times 5 = 15$。

例題二

說明：本題是為了讓學生可以活用本章之概念與圖形間的面積轉換關係，進而推知答案。

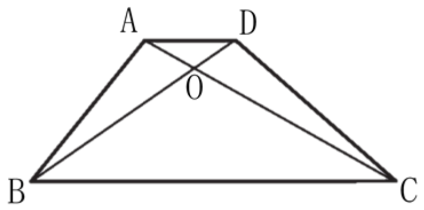
Students can be more flexible through this question about the concept introduced in this chapter in relation to finding areas of different figures by rearranging them. By doing so, they can figure out answers.

（英文）In the diagram of trapezoid $ABCD$, $\overline{AD}$ // $\overline{BC}$. If $\overline{AD}=1.3$ and $\overline{BC}=5.2$

Find the ratio of area of $\triangle AOB$ and $\triangle DOC$.

（中文）如圖，梯形 $ABCD$ 中，$\overline{AD}$ // $\overline{BC}$，若 $\overline{AD}=1.3$，$\overline{BC}=5.2$，

求 $\triangle AOB$ 的面積：$\triangle DOC$ 的面積。



（三上翰林備課課本 P.71 會考觀測站）

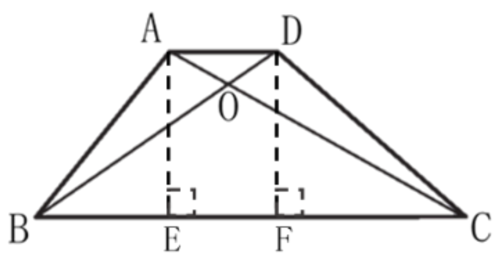
Teacher: Firstly, $\triangle AOB$ is the grand $\triangle ABC$ without the medium $\triangle OBC$.

Similarly, $\triangle OBC$ is the grand $\triangle DBC$ without the medium $\triangle OBC$.

Teacher: Do you still remember in the previous chapter when two sides are parallel to each other, what happens to their distance?

Student: They have the same distance.

Teacher: That is correct. Now, we can construct two perpendicular lines from Point A and D respectively to Point E and F on $\overline{BC}$ as of the diagram shown below.



Teacher: Because $\overline{AD}$ // $\overline{BC}$, $\overline{AE}=\overline{DF}$.

Teacher: Because we also know that the area of a triangle is one half base times height, the area of $\triangle ABC$ is equal to the area of $\triangle DBC$.

Teacher: Exactly, so $\triangle AOB$: $\triangle DOC$ =

$(\triangle ABC - \triangle OBC):(\triangle DBC - \triangle OBC) = 1:1$.

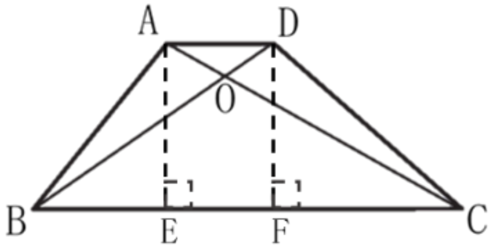
The ratio of the area of $\triangle AOB$: area of $\triangle DOC$ is $1:1$.

老師： 首先，我們先將 $\triangle AOB$ 看成大的 $\triangle ABC$ 減去中的 $\triangle OBC$；另外 $\triangle DOC$ 看成大的 $\triangle DBC$ 減去中的 $\triangle OBC$ 。

老師： 接著，還記得在之前的章節如果兩個直線平行，那兩直線之間的距離有什麼特性？

學生： 距離會相等。

老師： 沒錯！所以我們將點 A 、D 分別作垂線與 $\overline{BC}$ 交於 E 、F 兩點。如下圖：



老師： 因為 $\overline{AD} \,//\, \overline{BC}$ ，所以 $\overline{AE} = \overline{DF}$ 。

老師： 又因為三角形面積公式為 $\frac{底\times高}{2}$，所以 $\triangle ABC$ 的面積等於 $\triangle DBC$ 的面積。

老師： 沒錯，所以 $\triangle AOB：\triangle DOC = (\triangle ABC - \triangle OBC)：(\triangle DBC - \triangle OBC) = 1：1$。那我們就可以知道△ AOB 的面積：△ DOC 的面積為 1：1 。

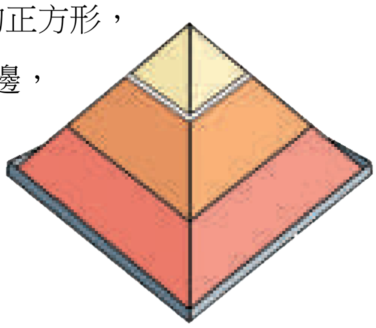
例題三

說明：使學生可以從生活的例子當中計算平行線截比例線段的長度，雖然是立體的，但學生可視為平面來思考。

Students can figure out with real-world examples the length of the proportion of sides when we have parallel lines in a triangle. Although the given diagram is three-dimensional, students can visualize it as if on the paper.

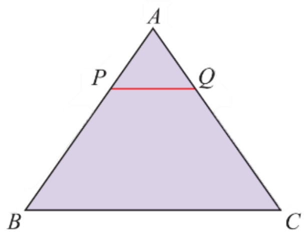
（英文）On the diagram to the right, it is a pyramid-shaped cake. The base layer of the cake is a square, 24cm on each side. A line of whipped cream is circled around the top third of the cake, parallel to the bottom edge. Find the length of the whipped-cream circle.

（中文）右圖是一個金字塔蛋糕，已知底層是每邊 24 公分的正方形，如果在由上而下三分之一處用奶油環繞一圈平行底邊，則這一圈奶油的長度為多少公分？



（三上翰林課本 P.78 例題 3）

Teacher: First, we draw the side view of the cake based on the given description. The drawing would look like this one below.

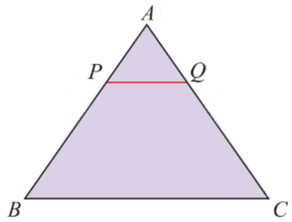


Teacher: Let's say the cream line as $\overline{PQ}$. The base segment on this side is $\overline{BC}$. The question tells us that $\overline{BC} = 24$ *and* $\overline{PQ}$ // $\overline{BC}$. Given the triangle proportionality theorem, $\overline{AP} : \overline{AB} = \overline{PQ} : \overline{BC} = 1 : 3$.

Teacher: We assume that the length of $\overline{PQ}$ is r, so the length of $\overline{BC}$ is going to be $3r$. Because $3r = 24$, that gives us $r = 8$. The length of $\overline{PQ} = 8$.

Teacher: Now, because this is a four-sided cake, the total length of this cream circle is $8 \times 4 = 32$(cm).

老師： 我們先依據題目畫出蛋糕的其中一面，如下圖



老師： 設其中奶油的部分是 $\overline{PQ}$，同一邊的底層是 $\overline{BC}$。

根據題目我們知道 $\overline{BC}=24$ 且 $\overline{PQ}\,//\,\overline{BC}$，根據平行線截比例線段

$\overline{AP}:\overline{AB}=\overline{PQ}:\overline{BC}=1:3$。

老師： 我們將 $\overline{PQ}$ 假設為 r；$\overline{BC}$ *假設為* $3r$ ，又因為 $3r=24$，可解出 $r=8$，因此

$\overline{PQ}=8$。

老師： 然後因為一個蛋糕共四面，所以奶油環繞一圈的總長度為 $8\times4=32(\text{cm})$。

單元三 相似多邊形
Similar Polygon

國立彰化師範大學數學系　盧衵元、邱奕銘

■ 前言 Introduction

本節首先了解某縮放中心對點的縮放，推廣到線段的縮放，再討論平面圖形的縮放。接著 以圖形的縮放引出多邊形相似的意義，最後利用三角形的縮放，導出相似三角形的判別性質。

■ 詞彙 Vocabulary

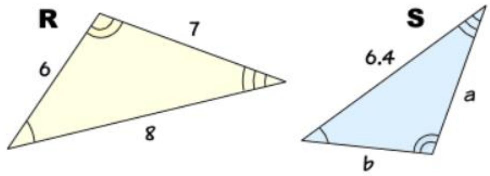
單字	中文	單字	中文
similar polygons	相似多邊形	scaling	比例縮放
corresponding sides	對應邊	corresponding angles	對應角
segment	線段	ratio	比例
polygon	多邊形	pentagon	五邊形
hexagon	六邊形		

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ the lengths A and B are corresponding.

例句：In the following similar triangles, R and S, **the lengths** 7 **and** a **are corresponding**; **the lengths** 8 **and** 6.4 **are corresponding**; **the lengths** 6 **and** b **are corresponding**.

在相似三角形中，長度為 7 的對應邊為長度 a；長度為 8 的對應邊為長度 6.4；在長度為 6 的對應邊為長度 b。



■ 問題講解 Explanation of Problems

꧁ 運算問題的講解 ꧂

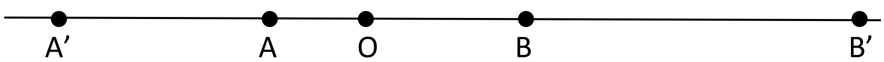
例題一

說明：本題是讓學生了解線段間的縮放關係。

Students can understand scaling of line segments.

（英文）As shown, scale $\overline{AB}$ three times and get $\overline{A'B'}$. The point O is the centre of $\overline{A'B'}$. If the length of $\overline{AA'}$ is 4 cm, and the length of $\overline{BB'}$ is 6 cm, find the length of $\overline{AB}$.

（中文）如圖，$\overline{A'B'}$ 是以 O 點為中心，將 $\overline{AB}$ 縮放 3 倍後的圖形，若 $\overline{AA'}=4$ 公分，$\overline{BB'}=6$ 公分，則 $\overline{AB}$ 是多少公分？



（三上翰林備課課本 P.91 會考觀測站）

Teacher: The question says that $3\overline{AB}$ *equals* $\overline{A'B'}$, $\overline{AA'}$ is 4 cm, and $\overline{BB'}$ is 6 cm.

Teacher: First, $\overline{A'B'}$ can be seen as $\overline{AA'}$+$\overline{AB}$+$\overline{BB'}$.

Teacher: Substitute $3\overline{AB} = \overline{A'B'}$ for the equation, we get $3\overline{AB} = \overline{AA'} + \overline{AB} + \overline{BB'}$.

$3\overline{AB} = \overline{AA'} + \overline{AB} + \overline{BB'}$

$2\overline{AB} = \overline{AA'} + \overline{BB'}$

$2\overline{AB} = 4 + 6 = 10$

$\overline{AB} = 5$(cm).

老師： 根據題目所說 $3\overline{AB} = \overline{A'B'}$，且$\overline{AA'} = 4$公分，$\overline{BB'} = 6$公分

老師： 我們首先將$\overline{A'B'}$看成$\overline{AA'}$ +$\overline{AB}$+$\overline{BB'}$

老師： 代入$3\overline{AB} = \overline{A'B'}$，得到$3\overline{AB} = \overline{AA'} + \overline{AB} + \overline{BB'}$

$3\overline{AB} = \overline{AA'} + \overline{AB} + \overline{BB'}$

$2\overline{AB} = \overline{AA'} + \overline{BB'}$

$2\overline{AB} = 4 + 6 = 10$

$\overline{AB} = 5$(公分)。

例題二

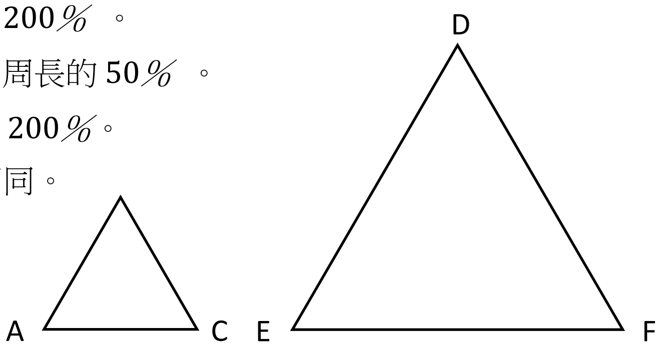
說明：本題是藉生活上的案例，讓學生了解三角形的縮放關係。
Students can understand scaling of triangles through a case that happens in daily life.

（英文）Using a copy machine to scale $\triangle ABC$ as its 200% and get $\triangle DEF$.
The corresponding points of $A, B, and\ C$ are $D, E, and\ F$, respectively. Which statement is NOT true?

(A) The length of $\overline{EF}$ is 200% of the length of $\overline{BC}$.
(B) The perimeter of $\triangle ABC$ is 50% of the perimeter of $\triangle DEF$.
(C) The angle of $\angle D$ is 200% of the angle of $\angle A$.
(D) The angle of $\angle C$ is the same as the angle of $\angle F$.

（中文）用影印機將$\triangle ABC$縮放影印成 200% 得到$\triangle DEF$，其中 A、B、C 的對應點分別為 D、E、F，則下列敘述何者錯誤？

(A) $\overline{EF}$ 的長度是 $\overline{BC}$ 長度的 200% 。
(B) $\triangle ABC$ 的周長是$\triangle DEF$ 周長的 50% 。
(C) $\angle D$ 的度數是$\angle A$ 度數的 200%。
(D) $\angle C$ 的度數與$\angle F$ 的度數相同。



（三上翰林備課課本 P.94 會考觀測站）

Teacher: What are the keywords of this question?
Student: Scale $\triangle ABC$ as its 200% and get $\triangle DEF$.
Teacher: Right. What are the features of scaling a triangle we learned in this chapter?
Student: When scaling a triangle as its r times, the corresponding sides are r times as the original sides, but the corresponding angles remain the same.
Teacher: Good answer. So

$\overline{DE} = 2\overline{AB}$

$\overline{EF} = 2\overline{BC}$

$\overline{DF} = 2\overline{AC}$

And $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$.

The perimeter of $\triangle DEF$ $=2 \times$ the perimeter of $\triangle ABC$.

Teacher: So, the answer is options (C).

老師：　首先我們先想一下這題題目的關鍵字是甚麼？

學生：　將$\triangle ABC$ 縮放影印成 200% 得到$\triangle DEF$

老師：　沒錯，那我們在這章學到了三角形的縮放有甚麼性質？

學生：　當三角形縮放 r 倍時，對應的邊長為原邊長的 r 倍；對應的角度不變。

老師：　回答得很好，所以

$\overline{DE} = 2\overline{AB}$

$\overline{EF} = 2\overline{BC}$

$\overline{DF} = 2\overline{AC}$

且$\angle A = \angle D$，$\angle B = \angle E$，$\angle C = \angle F$。

$\triangle DEF$ 的周長 $= 2 \times \triangle ABC$ 的周長。

老師：　所以錯誤的選項是(C)。

例題三

說明：本題是藉實際題目讓學生了解如何藉由 SAS 相似性質判斷兩三角型是否相似。

Students can understand how to determine whether two triangles are similar using similarity *SAS* through a practical question.

（英文）As shown, in $\triangle ABC$, $\overline{AE}=3, \overline{EB}=5, \overline{AF}=4,$ and $\overline{FC}=2$.

Please answer the following questions:

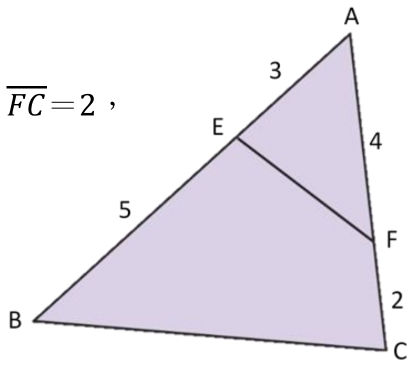
(1) Are $\triangle AEF$ and $\triangle ACB$ similar triangles? Why or why not?

(2) If $\overline{EF}=3.3$, find the length of $\overline{BC}$.

（中文）如圖，$\triangle ABC$ 中，$\overline{AE}=3$，$\overline{EB}=5$，$\overline{AF}=4$，$\overline{FC}=2$，回答下列問題：

(1) $\triangle AEF$ 與$\triangle ACB$ 是否相似？為什麼？

(2) 若 $\overline{EF}=3.3$，求 $\overline{BC}$。



（三上翰林備課課本 P.105 例題 7）

Teacher: The question wants us to determine whether $\triangle AEF$ *and* $\triangle ACB$ are similar. What are the three triangle similarities we learned in this chapter?

Student: *SSS*, *AA*, and *SAS*.

Teacher: Yes. It's not hard to see that $\angle A$ is the common angle of $\triangle AEF$ and $\triangle ACB$. First, we use similarity *SAS* to check if $\triangle AEF$ *and* $\triangle ACB$ are similar. So, we should determine whether the ratios of the adjacent sides are the same or not.

Teacher: By calculating, we get

$\overline{AE} : \overline{AF} = 3 : 4$

$\overline{AB} : \overline{AC} = \overline{AE} + \overline{EB} : \overline{AF} + \overline{FC}$

$(\overline{AE} + \overline{EB}) : (\overline{AF} + \overline{FC}) = (3+5) : (4+2) = 8 : 6$

$\overline{AB} : \overline{AC} = 8 : 6 = 4 : 3.$

Teacher: Look carefully, if $\overline{AE} : \overline{AF}$ equals $\overline{AB} : \overline{AC}$?

Student: No. The ratios are just the opposite.

Teacher: What if we exchange $\overline{AB}$ and $\overline{AC}$? Does $\overline{AE} : \overline{AF}$ equal $\overline{AC} : \overline{AB}$?

Student: Yes, it does.

Teacher: Now we know that they have the same ratio, we can make a conclusion.

(1) $\angle A$ is the common angle of $\triangle AEF$ and $\triangle ACB$.

(2) $\overline{AE}:\overline{AF}=\overline{AC}:\overline{AB}$

According to (1) and (2), we get that $\triangle AEF \sim \triangle ACB$ (triangle similarity *SAS*).

Teacher: Since $\triangle AEF \sim \triangle ACB$, we can conclude that $\overline{AE}:\overline{EF}=\overline{AB}:\overline{BC}$, and we can get

$\overline{AE}:\overline{EF}=\overline{AB}:\overline{BC}$

$3:3.3=8:\overline{BC}$

$3\times\overline{BC}=26.4$

$\overline{BC}=8.8$.

老師： 題目希望我們求出$\triangle AEF$ 與$\triangle ACB$ 是否相似，那這章我們學到哪三種相似性質？

學生： *SSS*、*AA*、*SAS* 三種相似性質。

老師： 沒錯，那觀察例圖不難發現$\triangle AEF$ 與$\triangle ACB$共用$\angle A$，所以我們首先利用 *SAS* 的相似性質來測試$\triangle AEF$ 與 $\triangle ACB$ 是否相似，所以我們需要先求出 $\angle A$ 的鄰邊比例是否相同。

老師： 計算後得到

$\overline{AE}:\overline{AF}=3:4$

$\overline{AB}:\overline{AC}=\overline{AE}+\overline{EB}:\overline{AF}+\overline{FC}$

$(\overline{AE}+\overline{EB}):(\overline{AF}+\overline{FC})=(3+5):(4+2)=8:6$

$\overline{AB}:\overline{AC}=8:6=4:3$。

老師： 觀察 $\overline{AE}:\overline{AF}$ 是否等於$\overline{AB}:\overline{AC}$？

學生： 沒有，比例剛好相反。

老師： 那我們 $\overline{AB}:\overline{AC}$ 前後對調，$\overline{AE}:\overline{AF}$ 是否等於$\overline{AC}:\overline{AB}$ 呢？

學生： 對的。

老師： 確認比例相同後，我們整理一下結果：

(1)$\angle A$為$\triangle AEF$ 與$\triangle ACB$的共用角

(2) $\overline{AE}:\overline{AF}=\overline{AC}:\overline{AB}$

所以藉由(1)、(2)我們可以得到$\triangle AEF \sim \triangle ACB$ (SAS 相似性質)。

老師： 既然$\triangle AEF \sim \triangle ACB$，所以 $\overline{AE}:\overline{EF}=\overline{AB}:\overline{BC}$，得到

$\overline{AE}:\overline{EF}=\overline{AB}:\overline{BC}$

$3:3.3=8:\overline{BC}$

$3\times\overline{BC}=26.4$

$\overline{BC}=8.8$。

單元四 相似三角形與三角比
Similar Triangles & Trigonometric Ratio

國立彰化師範大學數學系　盧衵元、邱奕銘

■ 前言 Introduction

首先討論兩個相似三角形的對應邊與對應高，及對應邊與面積的關係，接著了解直角三角形的相似關係，並將相似三角形應用在實際的測量問題上，讓學生有學以致用的機會。最後利用三角形相似的概念來學習三角比，認識直角三角形邊長比值的不變性的意義與用 sin、cos、tan 的符號來表示，了解三角比與坡度的關係。

■ 詞彙 Vocabulary

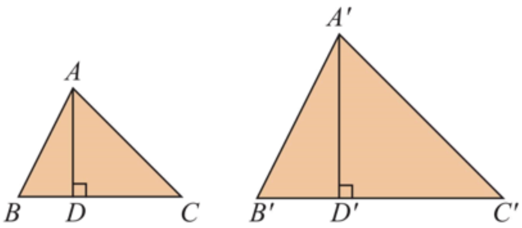
單字	中文	單字	中文
trigonometric ratio	三角比	similar triangles	相似三角形
measurement	測量	right triangle	直角三角形
sine	正弦	cosine	餘弦
tangent	正切		

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ **___:___ is equal to ___:___.**

例句：In similar triangles $\triangle ABC$ and $\triangle A'B'C'$, $\overline{BC}:\overline{B'C'}$ **is equal to** $\overline{AD}:\overline{A'D'}$.

在相似三角形ABC與$A'B'C'$中，線段$\overline{BC}$比線段$\overline{B'C'}$的比例與線段 $\overline{AD}$ 比線段 $\overline{A'D'}$ 的比例相同)



❷ **Suppose the angles of a triangle are respectively to ____, ____ and ____, then the sides of triangle becomes ___:___:___.**

例句：**Suppose the angles of a triangle are respectively to** 30°,60°**and** 90°, **then the ratio of sides of the triangle becomes** $1:\sqrt{3}:2$.

因為三角形的三個角分別為 30°, 60°和 90°，因此三角形的邊長比為 $1:\sqrt{3}:2$。

■ 問題講解 Explanation of Problems

運算問題的講解

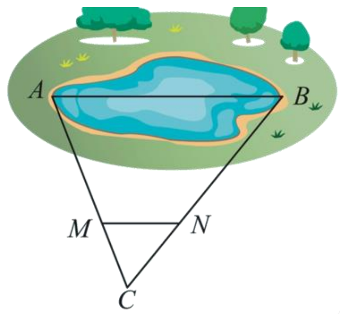
例題一

說明：本題藉由生活化題目讓學生熟悉 SAS 相似性質的實際應用。

This question uses a real-life situation to enable students be familiar with SAS similarity theorem and able to put the theorem into practice.

（英文）In the diagram, Point A and Point B are marked on each side of the lake. Angie would like to know the distance between them. First, she marks Point C on a space by the lake. The distance of $\overline{AC}$ is measured 75m and the distance of $\overline{BC}$ is measured 90m. Point M and Point N are marked close to Point C and intersect $\overline{AC}$ and $\overline{BC}$, respectively. The measure of $\overline{MC}$ = 25m, $\overline{NC}$ = 30m, and $\overline{MN}$=28m, find the width of the lake, $\overline{AB}$.

（中文）如圖，湖邊有 A、B 兩點，安琪想知道它們之間的距離。首先她在湖邊的空地找另一點 C，測得 $\overline{AC}$ 長 75 公尺、$\overline{BC}$長 90 公尺，接著自 C 點出發分別在 $\overline{AC}$、$\overline{BC}$ 上取 M、N 兩點，使得 $\overline{MC}$＝25 公尺，$\overline{NC}$＝30 公尺，此時 $\overline{MN}$＝28 公尺，求湖寬 AB。



（三上翰林備課課本 P.116 例題 4）

Teacher: First, we can easily see that ∠C is common to $\triangle MNC$ and $\triangle ABC$, next we figure out

$\overline{AC}:\overline{BC} = 75:90 = 5:6$

$\overline{MC}:\overline{NC} = 25:30 = 5:6.$

Teacher: Alright, now we know the ratio of $\overline{AC}:\overline{BC}$ and the ratio of $\overline{MC}:\overline{NC}$.

Is $\overline{AC}:\overline{BC}$ equal to $\overline{MC}:\overline{NC}$?

Student: Yes.

Teacher: That is correct. Now, we have the following information:

(1) ∠C is common to $\triangle MNC$ and $\triangle ABC$

(2) $\overline{AC} : \overline{BC} = \overline{MC} : \overline{NC}$

therefore, $\triangle MNC \sim \triangle ABC$ (SAS similarity theorem).

Teacher: Because $\triangle MNC \sim \triangle ABC$, $\overline{AC} : \overline{AB} = \overline{MC} : \overline{MN}$

$\overline{AC} : \overline{AB} = \overline{MC} : \mathrm{MN}$

$75 : \overline{AB} = 25 : 28$

$25 \times \overline{AB} = 75 \times 28 = 2100$

$\overline{AB} = 84$(meters).

老師： 首先，觀察圖形我們不難發現$\triangle MNC$和$\triangle ABC$公用角為$\angle C$，接著計算出

$\overline{AC} : \overline{BC} = 75 : 90 = 5 : 6$

$\overline{MC} : \overline{NC} = 25 : 30 = 5 : 6$。

老師： 好，那算完$\overline{AC} : \overline{BC}$和$\overline{MC} : \overline{NC}$的比例後，$\overline{AC} : \overline{BC}$是否等於$\overline{MC} : \overline{NC}$呢？

學生： 對的。

老師： 沒錯，所以我們接著整理一下資訊：

(1) $\triangle MNC$和$\triangle ABC$公用角為$\angle C$

(2) $\overline{AC} : \overline{BC} = \overline{MC} : \overline{NC}$

所以$\triangle MNC \sim \triangle ABC$(SAS 相似性質)。

老師： 接著利用$\triangle MNC \sim \triangle ABC$，所以

$\overline{AC} : \overline{AB} = \overline{MC} : \overline{MN}$

$75 : \overline{AB} = 25 : 28$

$25 \times \overline{AB} = 75 \times 28 = 2100$

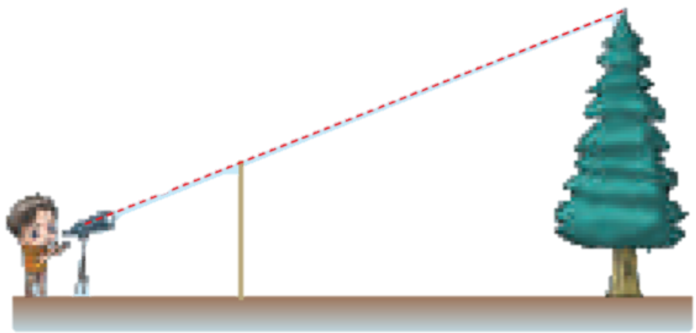
$\overline{AB} = 84$(公尺)。

例題二

說明：本題是藉生活上的案例，讓學生了解三角形的縮放關係。
This question uses a real-world example to lead students to understand the relationship between a triangle and its dilation.

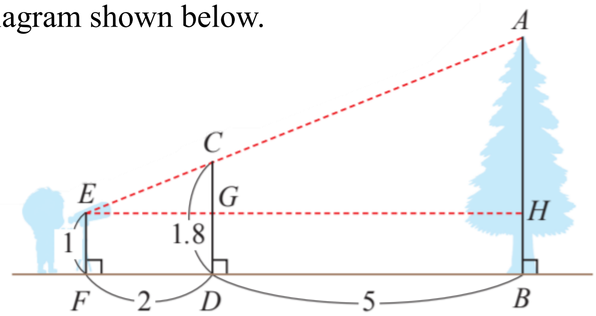
（英文）In the illustration, Zhi-Hao is measuring the height of the tree. He set straight a wooden stick, 1.8m in height, 5 meters away from the tree. Then, he continues to walk further away and finds an observation point right behind the wooden stick. From the telescope, it is observed that the tip of the stick overlaps the treetop. The horizontal distance between the wooden stick and the telescope is measured 2m. The height of the telescope is measured 1m. Find the height of the tree.

（中文）如圖，志豪想要測量樹高，他在樹前 5 公尺垂直豎立了一根長 1.8 公尺的木棍，並繼續往同方向在木棍後方找到觀測點，從望遠鏡看到木棍頂端與樹梢重疊。經測量木棍與望遠鏡的水平距離是 2 公尺，望遠鏡至地面的高度為 1 公尺，求樹高。



（三上翰林備課課本 P.117 例 5）

Teacher: First, let's say the tree height as $\overline{AB}$, the height of the straight wooden stick $\overline{CD}$, and the height of the telescope Point E. From Point E, we draw a perpendicular line to intersect $\overline{CD}$ and $\overline{AB}$ and mark two intersection points as G and H as the diagram shown below.



Teacher: Because both $\overline{CD}$ and $\overline{AB}$ are perpendicular to $\overline{FB}$, $\overline{CD}//\overline{AH}$.

Teacher: Do you still remember what we have learned in the chapter 1-2 about the length of the proportions of sides when we have parallel lines in a triangle?

Student: Yes, we do.

Teacher: Now, by applying this triangle proportionality theorem to $\triangle AEH$, we can get:

$\overline{EG} : \overline{EH} = \overline{CG} : \overline{AH}$

$\overline{EG} : (\overline{EG} + \overline{GH}) = (\overline{CD} - \overline{GD}) : (\overline{AB} - \overline{HB})$

$2 : (2+5) = (1.8-1) : (\overline{AB} - 1)$

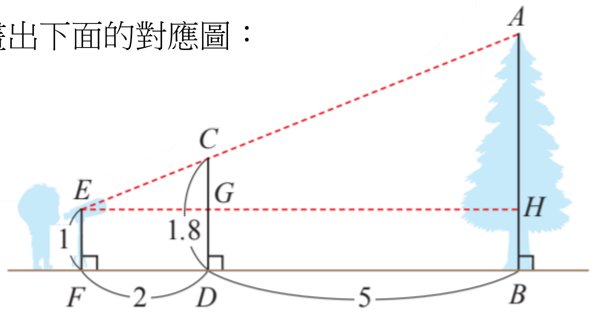
$2 : 7 = 0.8 : (\overline{AB} - 1)$

$2 \times (\overline{AB} - 1) = 2 \times \overline{AB} - 2 = 5.6$

$2 \times \overline{AB} = 7.6$

$\overline{AB} = 3.8.$

老師：首先，樹高是 $\overline{AB}$、垂直木棍是 $\overline{CD}$、望遠鏡是 E 點，自 E 點作垂線分別交 $\overline{CD}$ 與 $\overline{AB}$ 於 G、H 兩點。並畫出下面的對應圖：



老師：因為 $\overline{CD}$ 和 $\overline{AB}$ 皆垂直 $\overline{FB}$，所以 $\overline{CD}//\overline{AH}$。

老師：還記得我們在 1-2 有學過三角形的平行線截比例線段性質嗎？

學生：有！

老師：那在$\triangle AEH$中，我們利用三角形的平行線截比例線段性質可以得到

$\overline{EG} : \overline{EH} = \overline{CG} : \overline{AH}$

$\overline{EG} : (\overline{EG} + \overline{GH}) = (\overline{CD} - \overline{GD}) : (\overline{AB} - \overline{HB})$

$2 : (2+5) = (1.8-1) : (\overline{AB} - 1)$

$2 : 7 = 0.8 : (\overline{AB} - 1)$

$2 \times (\overline{AB} - 1) = 2 \times \overline{AB} - 2 = 5.6$

$2 \times \overline{AB} = 7.6$

$\overline{AB} = 3.8$。

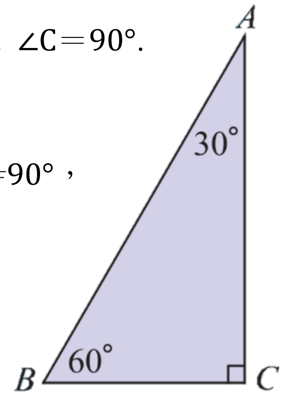
例題三

說明：本題學生熟悉並實用特殊直角三角形的邊長比。

Students can learn and know how to make good use of side ratios in special right triangles.

（英文）In the diagram of the right triangle ABC, ∠A＝30°, ∠B＝60°, ∠C＝90°. If $\overline{AB}=6$, find the length of $\overline{AC}$ and $\overline{BC}$.

（中文）如圖，直角三角形 ABC 中，$\angle A=30°$，$\angle B=60°$，$\angle C=90°$，若 $\overline{AB}=6$，求 $\overline{AC}$、$\overline{BC}$ 的長。



（三上翰林備課課本 P.119 例題 6）

Teacher: Before we dive in, please tell me how many side ratios of a special right triangle can have. We learned this in the previous section.

Student: 30° – 60° – 90° and 45° – 45° – 90°, total two types.

Teacher: That is correct. Do you still remember what is the ratio of the corresponding sides for each?

Student: Yes, the ratio of the corresponding sides for $30° - 60° - 90°$ is $1 : \sqrt{3} : 2$ and for $45° - 45° - 90°$ it is $1 : 1 : \sqrt{2}$.

Teacher: Excellent! Obviously, the triangle in question is the $30° - 60° - 90°$ triangle, so we can get:

$\overline{AB} : \overline{AC} : \overline{BC} = 6 : \overline{AC} : \overline{BC} = 2 : \sqrt{3} : 1$

$6 : \overline{AC} = 2 : \sqrt{3}$

$2 \times \overline{AC} = 6\sqrt{3}$

$\overline{AC} = 3\sqrt{3}$

$6 : \overline{BC} = 2 : 1$

$2 \times \overline{BC} = 6$

$\overline{BC} = 3.$

老師：在解這題之前，先問問大家在這章的前段小節我們學到了哪幾個特殊直角三角形的邊長比？

學生：30° – 60° – 90°、45° – 45° – 90°兩種。

老師： 沒錯，那大家還記得兩者分別對應的邊長比嗎？

學生： 記得，$30°-60°-90°$對應的邊長比為$1:\sqrt{3}:2$、$45°-45°-90°$對應的邊長比為$1:1:\sqrt{2}$

老師： 很好，那在這題的三角形很明顯是$30°-60°-90°$的三角形，所以我們可以得到

$\overline{AB}:\overline{AC}:\overline{BC}=6:\overline{AC}:\overline{BC}=2:\sqrt{3}:1$

$6:\overline{AC}=2:\sqrt{3}$

$2\times\overline{AC}=6\sqrt{3}$

$\overline{AC}=3\sqrt{3}$

$6:\overline{BC}=2:1$

$2\times\overline{BC}=6$

$\overline{BC}=3$。

單元五 點、直線與圓之間的位置關係
The Relationships Between the Positions of Points, Lines and Circles

國立新竹科學園區實驗高級中等學校　印娟娟老師

■ 前言 Introduction

本章節的數學名詞比較多，建議老師以平面圖形標示名詞讓學生易於記憶，同時讓學生即使在某單字或用語不熟悉的情況下，仍能掌握老師教授的內容。本節內容分為三個核心概念，老師可依學生學習狀況，分別討論每一個核心概念，並配合相關例題。

■ 詞彙 Vocabulary

單字	中文	單字	中文
center of circle	圓心	segment (of a circle)	弓形
radius	半徑	sector	扇形
diameter	直徑	tangent line	切線
central angle	圓心角	point of tangency	切點
chord	弦	secant line	割線
arc (length)	弧(長)	major arc	優弧
circumference of (a) circle	圓周	minor arc	劣弧
external	外部的；外面的	associate	相關
corresponding	相對應的	term	詞彙

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ ______ be a portion of ____________.

例句：A circular arc **is a portion of** a circle.

弧是圓的一部分。

❷ divide ______ into__________.

例句：The chord $\overline{AB}$ **divide**s the circle **into** two unequal arcs.

弦 $\overline{AB}$ 將圓分成兩個不相等的弧。

❸ Similarly, _________.

例句：**Similarly**, we can apply the same theorem in the following part.

同樣地，我們可以在以下部份中運用相同的定理。

❹ Refer to ___________

例句：**Refer to** the figure on the right, find the measures of the indicated angles.

如右圖所示，找出標示的各角的度數。

❺ ____ can be associated.

例句：There are three different relationships in which a line and a circle **can be associated.**

一條直線及一圓有關的位置關係有三種。

❻ We can conclude that _________.

例句：So, **we can conclude that** the central angle of the sector is 90°.

因此，我們能推論此扇形的圓心角為 90 度。

❼ Assume that ________.

例句：**Assume that** △ ABC is an equilateral triangle.

假設三角形 ABC 是等邊三角形。

❽ There is a trick to ________.

例句：**There's a trick to** get the answer quickly if you add the three equations and then divide it by two.

有一個小訣竅：如果你將三個式子相加，再除以二將能較快得到答案。

❾ ____________, respectively.

例句：The inscribed circle of the right △ ABC is a tangent to $\overline{AB}$ and $\overline{BC}$ at points D and E, **respectively**.

直角三角形 ABC 的兩邊$\overline{AB}$與$\overline{BC}$與其內切圓 O 分別相切於 D、E 兩點。

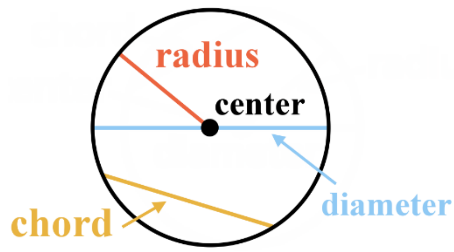
■ 問題講解 Explanation of Problems

☙ 說明 ❧

We have learned some plane figures (two-dimensional shapes) and their properties, such as triangles, parallelograms, trapezoids, and so on. In this chapter, we will discuss another plane figure – the circle.

In this unit, we will learn the positional relation between points, straight lines, and circles. A **circle** is the set of all points in a plane that are equidistant from a given point, and the given point is called the **center** of the circle. Here are some terms that we need to know:

Radius – A line segment extending from the center of a circle to the circumference.

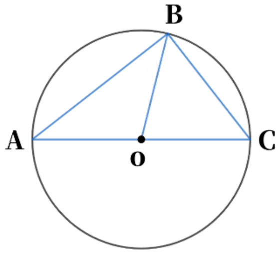


Chord – A chord is a line segment that has its endpoints on a circle. The **diameter** is a chord that contains the center of the circle. The diameter is the longest chord of a circle.

Central Angle – A central angle of a circle is an angle where its vertex is the center of the circle.

In the diagram, $\angle AOB$ and $\angle BOC$ are the central angles of circle O.

Arc – A circular arc is a portion of a circle. In the diagram, the diameter $\overline{AC}$ divides the circle into two equal arcs (semicircles). The chord $\overline{AB}$ divides the circle into two unequal arcs. If $m\angle AOB < 180°$, then the circular arc is called "a minor arc" and is written as: $\overset{\frown}{AB}$ (read as "arc AB"). The bigger arc is called "the major arc" and is usually written as: $\overset{\frown}{ACB}$.

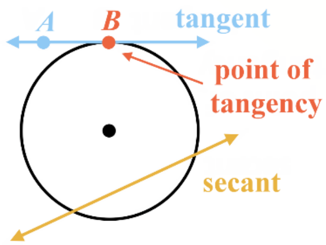


Segment – A segment of a circle is a region bounded by an arc and a chord.

Sector – A sector of a circle is a pie-shaped part enclosed by two radii and an arc.

Secant – A secant is a line that intersects a circle in two points.

Tangent – A tangent is a line in the plane of a circle that intersects the circle at exactly one point, the point of tangency.

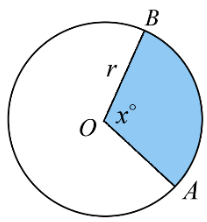


This section contains a lot. We can divide it into three core concepts.

Core Concept 1 How do we calculate the arc length and the area of a sector?

We learned that the circumference of a circle is equal to π times the diameter. (C = $\pi \cdot d$ or C $=2\pi r$) and the area of a circle is equal to $\pi \cdot r^2$.

If the measure of a central angle of an arc or sector is $x°$, then we divide $x°$ by 360°, which is $\frac{x}{360}$. The arc length is a part of a circumference and is equivalent to $\frac{x}{360}$ of the circumference. Similarly, the area of a sector is equal to $\frac{x}{360}$ of the area of the circle.



The arc length and the area of a sector can be calculated with the radius r and the central angle $x°$ as follows:

Arc length = $\frac{x}{360} \cdot 2\pi r$

Sector area = $\frac{x}{360} \cdot \pi r^2$

Core Concept 2 **The relationships between the positions of points, lines, and circles.**

The relationships between the positions of points and circles. A circle divides into three parts the plane it lies on. They are:

(1) inside the circle, which is also called the interior of the circle

(2) the circle and,

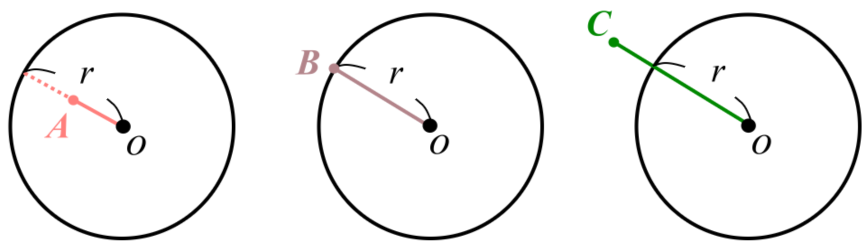
(3) outside the circle, this is also called the exterior of the circle.

Refer to the diagrams below,

(1) Point A is inside circle O with radius r: $\overline{OA} < r$

(2) Point B is on circle O with radius r: $\overline{OB} = r$

(3) Point C is outside circle O with radius r: $\overline{OC} > r$

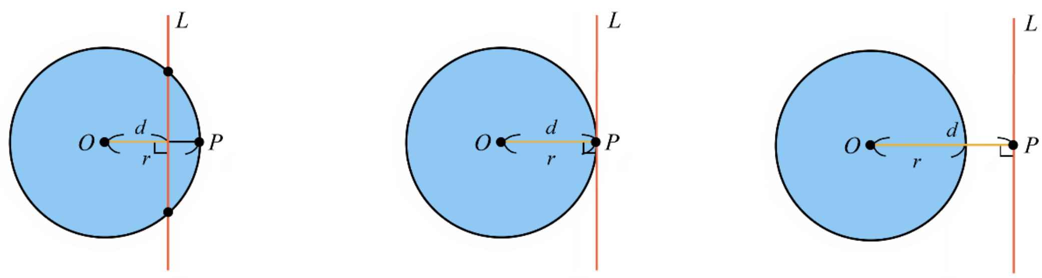


The relationships between the positions of lines and circles. There are three different ways that a line and a circle can be associated:

(1) The line cuts the circle at two different points. (L is a secant line.) The distance from the center O to line L is less than the radius (d $< r$).

(2) The line is tangent to the circle. (L is a tangent line.) The distance from the center O to line L is equal to the radius (d $= r$).

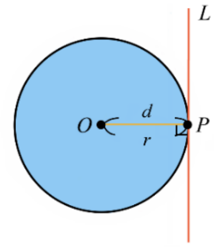
(3) The line and the circle have no intersection. The distance from the center O to line L is greater than the radius (d $>r$).



Theorems Using Properties of Tangents

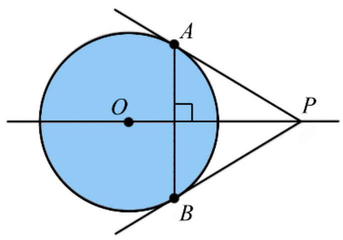
1. **Tangent Line to Circle Theorem:** On a plane, a line is tangent to a circle if and only if the line is perpendicular to a radius of the circle at the endpoint on the circle.

In the graph to the right, line L is tangent to circle O if and only if $L \perp \overline{OP}$ where P is the endpoint of radius $\overline{OP}$.



2. **External Tangent Congruence Theorem:** Tangent segments from a common external point are congruent.

In the graph to the right, if $\overline{PA}$ and $\overline{PB}$ are tangent segments, and both points A and B are points of tangency, then



(1) $\overline{PA} \cong \overline{PB}$

(segment PA is congruent to segment PB)

(2) $\overline{PO}$ bisects $\angle APB$

(3) $\overline{PO}$ is the perpendicular bisector of $\overline{AB}$

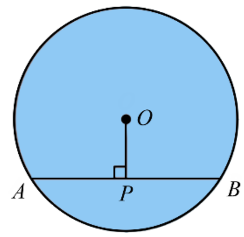
(proof)

1. Connect $\overline{OA}$, $\overline{OB}$ and $\overline{OP}$.
2. From the given information, $\overline{PA}$ and $\overline{PB}$ are tangent segments where A and B are the points of tangency. We know that
   (1) $\overline{OA} \cong \overline{OB}$ (Both $\overline{OA}$ and $\overline{OB}$ are radii of circle O.)
   (2) $\angle OAP \cong \angle OBP$ (Both angles are right angles.)
   (3) $\overline{OP} \cong \overline{OP}$ (Reflexive property)
3. From the RHS Congruence Rule, we get $\triangle OAP \cong \triangle OBP$
4. Because the corresponding parts of congruent triangles are congruent, we can conclude: **(i)** $\boldsymbol{\overline{PA} \cong \overline{PB}}$
   (ii) $\boldsymbol{\angle APO \cong \angle BPO}$**;** $\boldsymbol{\overline{PO}}$ **bisects** $\boldsymbol{\angle APB}$
5. Connect $\overline{AB}$ and intersect $\overline{OP}$ at point M. Since $\overline{PA} \cong \overline{PB}$ and $\overline{PO}$ bisects $\angle APB$, we can conclude that
   (iii) $\boldsymbol{\overline{PO} \perp \overline{AB}}$ **and** $\boldsymbol{\overline{AM} \cong \overline{BM}}$
   from the Isosceles Triangle Theorem "In an isosceles triangle, the angle bisector of the vertex angle is also the perpendicular bisector of the base."

Core Concept 3 The perpendicular from the center of a circle to a chord bisects the chord.

In the diagram, $\overline{AB}$ is a chord in circle O and $\overline{OP}$ is perpendicular to $\overline{AB}$.

Prove: $\overline{OP}$ bisects the chord $\overline{AB}$



(proof)

By the given information, $\overline{OP}$ is perpendicular to $\overline{AB}$,

then we get

$\angle APO = 90° = \angle BPO$

In $\triangle OAP$ and $\triangle OBP$,

$\overline{OA} \cong \overline{OB}$ (radii of a circle)

$\overline{OP} \cong \overline{OP}$ (Reflexive Property).

From the RHS Congruence Rule, we get $\triangle OAP \cong \triangle OBP$

Because the corresponding parts of congruent triangles are congruent, we can conclude:

$\overline{AP} \cong \overline{BP}$; $\overline{OP}$ bisects the chord $\overline{AB}$.

運算問題的講解

例題一

說明：了解學生是否知道與圓有關的半徑，直徑，弦，切線及割線等數學名詞。

（英文）Write whether the line, ray, or segment best describes a radius, chord, diameter, secant, or tangent of circle *O*.

（中文）在下列空格中填入適當的名詞（半徑，直徑，弦，切線及割線）：

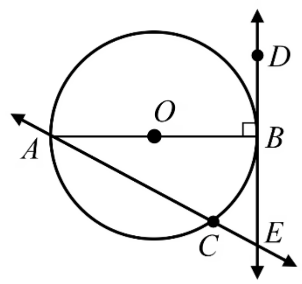
(1) $\overline{OA}$ ______________

(2) $\overline{AB}$ ______________

(3) $\overleftrightarrow{DE}$ ______________

(4) $\overleftrightarrow{AE}$ ______________

(5) $\overline{AC}$ ______________



Teacher: The first example is to identify special segments and lines of a circle. What is segment *OA* ($\overline{OA}$)?

Student: The radius.

Teacher: Correct. Now fill in the correct answers for (2) to (5). Tell me the answers of (2) to (5).

Student: $\overline{AB}$ is the diameter, $\overleftrightarrow{DE}$ (line *DE*) is a tangent line, $\overleftrightarrow{AE}$ is a secant line, and $\overline{AC}$ is a chord.

Teacher: Good. Let's look at the next question.

老師： 第一題是在空格中填入適當名稱。線段 $\overline{OA}$ 是什麼？

學生： 半徑。

老師： 答對了。現在填入 (2) 到 (5) 的答案。回答第(2)至(5)題的答案。

學生： 線段 $\overline{AB}$ 是直徑，直線 $\overleftrightarrow{DE}$ 是切線，直線 $\overleftrightarrow{AE}$ 是割線，而 $\overline{AC}$ 是弦。

老師： 很好。現在我們來看下個例題。

例題二

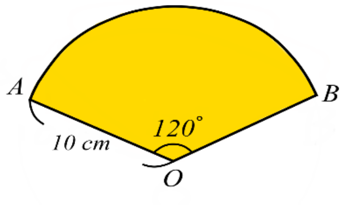
說明：已知圓心角的度數，運用弧長及扇形面積公式求扇形周長和面積。

（英文）Look at the graph on the right. The radius of circle *O* is 10 cm and the measure of the central angle($\angle AOB$) is 120°. Find the answer:

(1) the length of arc $\widehat{AB}$

(2) the perimeter of sector *AOB*

(3) the area of sector *AOB*



（中文）如右圖，圓 *O* 的半徑為 10 公分，圓心角 $\angle AOB$=120°，則

(1) $\widehat{AB}$ 的長度為多少公分？

(2) 扇形 *AOB* 的周長為多少公分？

(3) 扇形 *AOB* 的面積為多少平方公分？

Teacher: The arc length and the area of a sector formulas are:

$$\textbf{Arc length} = \frac{x}{360} \cdot 2\pi r$$

$$\textbf{Sector area} = \frac{x}{360} \cdot \pi r^2$$

From the given information, *r* is 10 cm and the measure of $\angle$AOB is 120°.

So, the arc length is equal to $\frac{120}{360} \cdot 2\pi \cdot 10$ which is $\frac{20}{3}\pi$ cm.

In part 2, the perimeter of the sector AOB is two radii plus the arc length of $\widehat{AB}$. Now find the answer.

Student: $(\frac{20}{3}\pi + 20)$ cm.

Teacher: Correct. In part (3), the area of sector AOB is $\frac{120}{360} \cdot \pi \cdot 10^2$ which is $\frac{100}{3}\pi$ cm^2 ("one hundred pi over three square centimeters").

老師：弧長和扇形面積的公式分別是：

弧長 $= \frac{x}{360} \cdot 2\pi r$；扇形面積 $= \frac{x}{360} \cdot \pi r^2$

根據題目所給的條件，半徑 r 為 10 公分，$\angle AOB$ 的度數為 120°。因此，弧長等於 $\frac{120}{360} \cdot 2\pi \cdot 10$，即 $\frac{20}{3}\pi$ 公分。

接著第(2)小題，扇形 AOB 的周長是兩個半徑加上 $\widehat{AB}$ 的弧長。請找出答案。

學生：$(\frac{20}{3}\pi + 20)$ 公分。

老師：答對了。在第(3)小題，扇形 AOB 的面積為 $\frac{120}{360} \cdot \pi \cdot 10^2$，即 $\frac{100}{3}\pi$ 平方公分。

例題三

說明：求弓形面積與周長。

（英文）Look at the graph on the right. The radius of a circle O is 12 cm and the measure of the central angle is 60°. Find the answer.

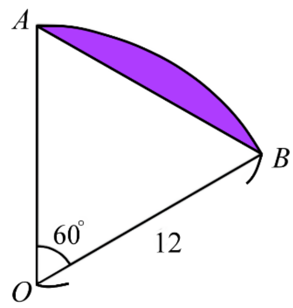
(1) The area of the segment (the purple shaded region).

(2) The perimeter of the segment.

（中文）如右圖，圓的半徑為 12 公分，圓心角為 60°，則：

(1) 紫色弓形面積為多少平方公分？

(2) 紫色弓形的周長為多少公分？



Teacher: The segment of a circle (the purple region) is bounded by a chord and an arc. So we can find the area of the segment by getting the difference between the area of the sector AOB and the area of $\triangle AOB$. So, what is the area of the sector?

Student: The sector's area is $24\pi\ \text{cm}^2$.

Teacher: Correct. The sector's area is $\frac{60}{360}\cdot\pi\cdot 12^2$ that is equal to $24\pi\ \text{cm}^2$. So how do you find the area of △AOB? It is not hard for you to find that $\triangle AOB$ is an equilateral triangle because $\angle AOB = 60°$ and $\overline{OA} = \overline{OB}$. What is the formula for the area of an equilateral triangle?

Student: $\frac{\sqrt{3}}{4}\cdot s^2$ (Square root of three over four times s squared.)

Teacher: You are right. Now, find the area of $\triangle AOB$. Then find the area of the segment which is the difference between the area of the sector and the area of the triangle.

Student: $(24\pi - 36\sqrt{3})\ \text{cm}^2$

Teacher: Excellent. The second part is to find the perimeter, which is the sum of the arc length AB and the chord AB. The arc length is equal to $\frac{60}{360}$ or $\frac{1}{6}$ of the circumference of a circle. So, what is the answer of part (2)?

Student: The answer is $(12 + 4\pi)$ cm.

老師： 這個弓形（紫色區域）是由弦和弧所圍成的。因此可以透過計算扇形 AOB 的面積和$\triangle AOB$ 的面積差來求得該部分的面積。那麼，扇形的面積是多少呢？

學生： 扇形的面積是 24π 平方公分。

老師： 正確。扇形的面積是 $\frac{60}{360}\cdot\pi\cdot 12^2$ ，即 24π 平方公分。那麼，如何計算$\triangle AOB$ 的面積呢？因為$\angle AOB = 60°$且 $\overline{OA} = \overline{OB}$，所以不難可以發現$\triangle AOB$ 是一個等邊三角形。

那麼，等邊三角形的面積公式為何？

學生： $\frac{\sqrt{3}}{4}\cdot s^2$ 。

老師： 沒錯。現在，找出 $\triangle AOB$ 的面積。然後找出紫色弓形的面積，為扇形面積和

三角形面積的差。

學生： $(24\pi - 36\sqrt{3})$平方公分。

老師： 太好了。第 2 小題是求周長，即弦 AB 和弧 AB 長度之和。弧長等於圓周的 $\frac{60}{360}$（或 $\frac{1}{6}$）。那麼，答案是什麼？

學生： 答案是 $(12+4\pi)$ 公分。

例題四

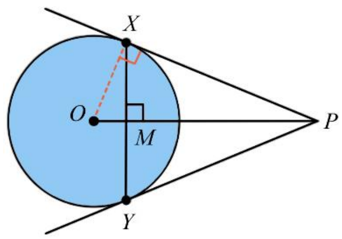
說明：運用圓的切線性質求解(圓心與切點的連線必垂直過此切點的切線)。

（英文）In the diagram, $\overleftrightarrow{PX}$ and $\overleftrightarrow{PY}$ are tangent to circle O at the points X and Y. If the radius of circle O is 8 cm and $\overline{PX}$ =15 cm, then find:

(1) $\overline{OP}$ (2) $\overline{XY}$

（中文）如右圖，直線 $\overleftrightarrow{PX}$ 和 $\overleftrightarrow{PY}$ 為圓 O 的切線，X、Y 為切點。若圓 O 的半徑為 8 公分，$\overline{PX}$ =15 公分，則：

(1) $\overline{OP}$ 為多少公分？ (2) $\overline{XY}$ 為多少公分？



Teacher: We know that $\triangle PXO$ is a right triangle because $\overleftrightarrow{PX}$ is tangent to circle O and $\angle OXP$ is a right angle.

From the given information, the two sides $\overline{OX}=8$ and $\overline{PX}=$ 15. Find $\overline{OP}$ by using the Pythagorean Theorem. You have two minutes.

(After 2 minutes) Time's up. What is the length of $\overline{OP}$?

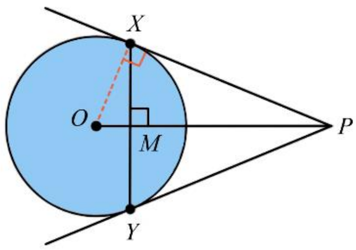
Student: 17 cm.

Teacher: Good. Since $\overline{OP}^2 = 15^2 + 8^2 = 225 + 64 = 289$ we get $\overline{OP} =$ 17 cm.

For part 2, we know that $\overline{OP}$ is the perpendicular bisector of $\overline{XY}$.

So, $\overline{XM} = \overline{YM} = \frac{1}{2}\overline{XY}$.

So how can we find $\overline{XM}$?



We know that $\triangle OXP$ is a right triangle, so the area of $\triangle OXP$ is equal to $\frac{1}{2}\times\overline{PX}\times\overline{OX}=\frac{1}{2}\times\overline{PO}\times$

$\overline{XM}$.

The area is the same when using different bases and their corresponding heights.

Now, it's your turn to find $\overline{XM}$ and $\overline{XY}$.

Student: $\overline{XM} = \frac{120}{17}$ and $\overline{XY} = \frac{240}{17}$.

Teacher: Great job. Let's do the next question.

老師： 我們知道$\triangle PXO$ 是一個直角三角形，因為 $\overleftrightarrow{PX}$ 是切線且 $\angle OXP$ 是一個直角。根據題目給的條件，兩邊為 $\overline{OX} = 8$ 和$\overline{PX} = 15$，請使用勾股定理求解 $\overline{OP}$ 的長度。你有兩分鐘的時間。

（2 分鐘後）時間到。$\overline{OP}$ 的長度是多少？

學生： 17 公分。

老師： 很好！因為 $\overline{OP}^2 = 15^2 + 8^2 = 225 + 64 = 289$，算出 $\overline{OP} = 17$ 公分。

老師： 第 2 小題，我們知道 $\overline{OP}$ 是 $\overline{XY}$的垂直平分線。所以， $\overline{XM} = \overline{YM} = \frac{1}{2}\overline{XY}$。

那麼要怎樣才能找到 $\overline{XM}$ 呢？

我們知道 $\triangle OXP$ 是一個直角三角形，因此 $\triangle OXP$ 的面積等於$\frac{1}{2} \times \overline{PX} \times \overline{OX}$

$= \frac{1}{2} \times \overline{PO} \times \overline{XM}$。

當使用不同的底和相應的高時，面積是相同的。現在換你們來求$\overline{XM}$ 和 $\overline{XY}$。

學生： $\overline{XM} = \frac{120}{17}$，$\overline{XY} = \frac{240}{17}$。

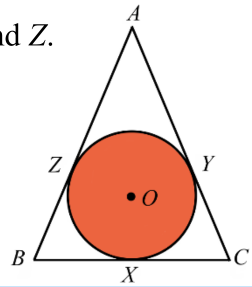
老師： 很好，我們接著來做下一個例題。

例題五

說明：運用圓的切線性質（圓外一點到圓的切線段相等）求解。

（英文）In the diagram, $\triangle ABC$ is tangent to circle O at the points X, Y, and Z. If $\overline{AB} = \overline{AC} = 13$ and $\overline{BC} = 10$, find $\overline{AZ}$, $\overline{BX}$, and $\overline{CY}$.

（中文）如右圖，$\triangle ABC$ 三邊分別與圓 O 相切於 X、Y、Z 三點，已知 $\overline{AB} = \overline{AC} = 13$，$\overline{BC} = 10$，求 $\overline{AZ}$，$\overline{BX}$及$\overline{CF}$。



Teacher: $\triangle ABC$ is tangent to circle O at the points X, Y, and Z. Using this we can get $\overline{AZ} = \overline{AY}$; $\overline{BZ} = \overline{BX}$; $\overline{CX} = \overline{CY}$ from the property that tangents from an external point have equal lengths.
Assume that $\overline{AZ} = \overline{AY} = x$; $\overline{BZ} = \overline{BX} = y$; $\overline{CX} = \overline{CY} = z$, then we can get $x + y = 13$, $x + z = 13$, and $y + z = 10$.
Now, find the values of x, y, and z.

Student: $x = 8$, $y = 5$, and $z = 5$.

Teacher: Good. There is a trick to get the answer quickly if you add the three equations first.
You will get $2x + 2y + 2z = 36$.
Then divide the equation by 2 and get $x + y + z = 18$.
Use the equation $x + y + z = 18$ to subtract the first equation $x + y = 13$; you will get $z = 5$.
Similarly, you can get $y = 5$ and $x = 8$ easily.
So what are the values of $\overline{AZ}$, $\overline{BX}$, and $\overline{CY}$?

Student: $\overline{AZ} = 8$, $\overline{BX} = 5$, and $\overline{CY} = 5$.

Teacher: Yes, you are right.

老師：已知$\triangle ABC$ 在圓 O 上的切點分別為 X、Y 和 Z，由此可得性質：圓外一點到圓的切線段長度相等，因此$\overline{AZ} = \overline{AY}$；$\overline{BZ} = \overline{BX}$；$\overline{CX} = \overline{CY}$。
假設 $\overline{AZ} = \overline{AY} = x$、$\overline{BZ} = \overline{BX} = y$、$\overline{CX} = \overline{CY} = z$，則可得 $x + y = 13$、$x + z = 13$ 及 $y + z = 10$。
現在，請找出 x、y 和 z 的值。

學生：$x = 8$、$y = 5$ 及 $z = 5$。

老師：答對了。有一個小技巧可以較快速得到答案，就是把三個方程式相加，得到 $2x + 2y + 2z = 36$。然後把方程式除以 2，得到 $x + y + z = 18$。

用 $x+y+\ z=18$ 減第一個方程式 $x+y=13$，會得到 $z=5$。用相同的做法，可以輕鬆得到 $y=5$ 和 $x=8$。

那麼 $\overline{AZ}$、$\overline{BX}$ 和 $\overline{CY}$ 的值是多少？

學生： $\overline{AZ}=8$、$\overline{BX}=5$ 和 $\overline{CY}=5$。

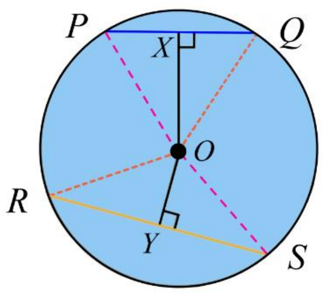
老師： 答對了。

例題六

說明：弦心距的應用：通過圓心且垂直於弦的線段會平分此弦。

（英文）$\overline{OX}$ and $\overline{OY}$ are perpendicular to the chords $\overline{PQ}$ and $\overline{RS}$ at the points X and Y, respectively. If $\overline{OX}$ =20, $\overline{PQ}=30$, and $\overline{RS}=48$, find the length of $\overline{OY}$.

（中文）$\overline{OX}$ 及 $\overline{OY}$ 分別垂直於兩弦 $\overline{PQ}$ 和 $\overline{RS}$ 於 X、Y 兩點。已知 $\overline{OX}=20$ 公分、$\overline{PQ}=30$ 公分、 $\overline{RS}=48$ 公分，求 $\overline{OY}$ 的長度為多少公分？



Teacher: From the given information, we know that $\overline{OX}$ and $\overline{OY}$ also bisect $\overline{PQ}$ and $\overline{RS}$ from core concept 3.

Hence, $\overline{PX}=\overline{QX}=\frac{1}{2}\overline{PQ}$ =15 and $\overline{RY}=\overline{SY}=\frac{1}{2}\overline{RS}$ =24.

In the right $\triangle OAM$, $\overline{OP}$ is the hypotenuse. By using the Pythagorean Theorem, we get $\overline{OP}^2=\overline{PX}^2+\overline{OX}^2$

Now find $\overline{OP}$ by yourselves.

Student: 25.

Teacher: Yes, you are right. Because $\overline{OP}^2=\overline{PX}^2+\overline{OX}^2$ =400 + 225 = 625; $\overline{OP}$ =25 cm.

Also, radius $\overline{OR}=\overline{OP}=$ 25 cm

Let's find $\overline{OY}$ in the right $\triangle ORY$, given that $\overline{RY}=$ 24 cm.

Student: $\overline{OY}$ = 7 cm.

Teacher: You did a great job.

老師： 由題目及前面提到的核心概念 3，我們知道$\overline{OX}$ 和 $\overline{OY}$ 分別平分 $\overline{PQ}$ 和 $\overline{RS}$。

因此，$\overline{PX} = \overline{QX} = \frac{1}{2}\overline{PQ} = 15$，且 $\overline{RY} = \overline{SY} = \frac{1}{2}\overline{RS} = 24$。

在直角三角形 $\triangle OPX$ 中，$\overline{OP}$ 是斜邊。運用勾股定理，我們可以列出 $\overline{OP}^2 = \overline{PX}^2 + \overline{OX}^2$。

現在請找出 $\overline{OP}$ 的值。

學生： 25。

老師： 沒錯。因為 $\overline{OP}^2 = \overline{PX}^2 + \overline{OX}^2 = 400 + 225 = 625$；$\overline{OP} = 25$ 公分。此外，半徑 $\overline{OR} = \overline{OP} = 25$ 公分。

已知 $\overline{RY} = 24$ 公分，請找出直角 $\triangle ORY$ 中$\overline{OY}$的值，。

學生： $\overline{OY} = 7$ 公分。

老師： 很好。

ᘓ 應用問題 / 會考素養題 ᘐ

例題一

說明：運用圓的切線性質（圓外一點到圓的切線段相等）求解。

（英文）In the diagram, the inscribed circle of the right $\triangle ABC$ is tangent to $\overline{AB}$ and $\overline{BC}$ at points D and E, respectively. Find the length of $\overline{AD}$.

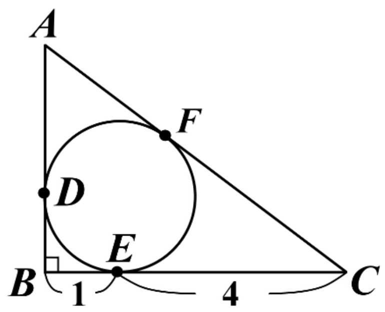
（中文）如圖，直角三角形 ABC 的內切圓分別於 $\overline{AB}$、$\overline{BC}$ 相切於 D 點、E 點。根據圖中標示的長度與角度，求 $\overline{AD}$ 的長度為何？

(A) $\frac{3}{2}$

(B) $\frac{5}{2}$

(C) $\frac{4}{3}$

(D) $\frac{5}{3}$



（108 年國中會考 19）

Teacher: Let's assume that the inscribed circle is also tangent to $\overline{AC}$ at point F and $\overline{AF} = x$. From the given information, the inscribed circle of the right $\triangle ABC$ is tangent to $\overline{AB}$ and $\overline{BC}$ at points D and E.

We get $\overline{BD}=\overline{BE}=1$ and $\overline{CF}=\overline{CE}=4$ and $\overline{AD}=\overline{AF}=x$ from the Two-Tangent Theorem. What are the contents of the Two-Tangent theorem?

Student: Tangent segments from a common external point are congruent.

Teacher: Correct. Then we can use the Pythagorean Theorem to find the value of x by the given information that $\triangle ABC$ is right. Please do it now.

(After a few minutes…) Did you get the value of x?

Student: Yes, x is $\frac{5}{3}$. (five over three)

Teacher: Great. In the right $\triangle ABC$, $\overline{AC}$ is the hypotenuse. We can get $\overline{AC}^2=\overline{AB}^2+\overline{BC}^2$ (The square of $\overline{AC}$ is equal to the sum of the square of $\overline{AB}$ and the square of $\overline{BC}$.)

or $(x+4)^2=(x+1)^2+5^2$

You can get the value of x after you solve the equation.

老師：假設內切圓也與 $\overline{AC}$ 相切於 F 點且 $\overline{AF}=x$。已知，△ABC 的內切圓與 $\overline{AB}$ 和 $\overline{BC}$ 相切於在 D 點和 E 點。我們從圓的切線段性質可以得知 $\overline{BD}=\overline{BE}=1$ 和 $\overline{CF}=\overline{CE}=4$ 以及 $\overline{AD}=\overline{AF}=x$。記得圓的切線段性質中的相關內容是什麼嗎？

學生：圓外一點到圓的切線段等長。

老師：正確。接下來因為$\triangle ABC$ 是直角三角形，我們可以用勾股定理求解 x 的值。現在請找出 x 的值。

（幾分鐘後…）算出 x 了嗎？

學生：x 等於 $\frac{5}{3}$。

老師：很好。在直角 $\triangle ABC$ 中，$\overline{AC}$ 是斜邊。我們可以列出 $\overline{AC}^2=\overline{AB}^2+\overline{BC}^2$ 也就是 $(x+4)^2=(x+1)^2+5^2$

之後解方程式算出 x 的值。

例題二

說明：運用切線性質求解。

（英文）In the diagram, circle P intersects the x-axis and the y-axis at points A and B, respectively. Line L passes through the center of the circle P and is perpendicular to $\overline{AB}$. Also, line L intersects the y-axis at the point C. What is the value of a ($a<0$) if the coordinates of A, B, and C are $(a , 0)$, $(0 , 4)$, and $(0 , -5)$?

(A) $-2\sqrt{14}$　(B) $-2\sqrt{5}$　(C) -8　(D) -7

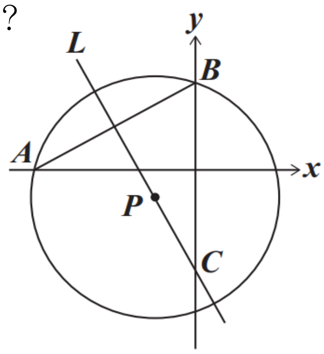
（中文）如圖，坐標平面上，A、B 兩點分別為圓 P 與 x 軸、y 軸的交點，有一直線 L 通過 P 點且與 $\overline{AB}$ 垂直，C 點為 L 與 y 軸的交點。若 A、B、C 的坐標分別為 $(a , 0)$、$(0 , 4)$、$(0 , -5)$，其中 $a<0$，則 a 的值為何？

(A) $-2\sqrt{14}$

(B) $-2\sqrt{5}$

(C) -8

(D) -7



（107 年國中會考 26）

Teacher: From the given information, L passes through the center P and is perpendicular to $\overline{AB}$. We know that L also bisects $\overline{AB}$ because the perpendicular from the center of a circle to a chord bisects the chord (core concept 3).

Connect $\overline{CA}$ and we can get $\overline{CA}=\overline{CB}$ by the perpendicular bisector theorem.

Do you remember the Perpendicular Bisector Theorem?

Student: Any point on the perpendicular bisector is equidistant from both the endpoints of the line segment.

Teacher: Very good. So, $\overline{CA}=\overline{CB}=9$.

If O is the origin of the x-y coordinate system, then $\overline{AO}^2+\overline{OC}^2=\overline{AC}^2$ (by the Pythagorean theorem).

You can find the value of a from the equation $a^2+25=81$.

Then, $a^2=81-25=56$.

So, $a = \pm 2\sqrt{14}$. What is the value of a if $a < 0$?

Student: (A) $-2\sqrt{14}$.

Teacher: Yes, you are right. That's all for today. Class dismissed.

老師： 已知，L 經過圓心 P 點並垂直於$\overline{AB}$。且因為圓心到弦的垂直線將弦二等分（核心概念 3），所以直線 L 平分$\overline{AB}$，。連接$\overline{CA}$，由垂直平分線定理得知 $\overline{CA} = \overline{CB}$。大家記得垂直平分線定理嗎？

學生： 垂直平分線上的任一點與線段的兩個端點的距離相等。

老師： 很好。因此，$\overline{CA} = \overline{CB} = 9$。如果 O 是 x-y 座標系統的原點，則 $\overline{AO}^2 + \overline{OC}^2 = \overline{AC}^2$（勾股定理）。你可以列出 $a^2 + 25 = 81$ 找到 a 的值；$a^2 = 81 - 25 = 56$。所以，$a = \pm 2\sqrt{14}$。如果 $a < 0$，a 的值是多少？

學生： (A) $-2\sqrt{14}$。

老師： 是的，沒錯。今天就上到這裡，下課。

單元六 圓心角、圓周角與弧的關係
The Relationships Between Central Angles, Inscribed Angles, and Arcs

國立新竹科學園區實驗高級中等學校　印娟娟老師

■ 前言 Introduction

本章節討論圓心角、圓周角與弧的關係，介紹圓周角與圓心三種不同位置關係來證明圓周角公式，並討論圓內接四邊形的性質等三個部分。建議老師每介紹完一個部分內容後，以圖形來口頭測試學生是否清楚瞭解本單元新介紹的字彙及定義，並配合例題練習以加深學習內容。

■ 詞彙 Vocabulary

單字	中文	單字	中文
central angle	圓心角	measurement	測量
inscribed angle	圓周角	measure	(度)量
intercepted arc	截斷弧	semicircle	半圓
intercept (v)	截斷(動詞)	pentagon	五邊形
inscribed	內接的	inscribed polygon	內接多邊形
circumscribed	外接的	circumscribed circle	外接圓
concentric circles	同心圓	supplementary	互補的

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ _______ if and only if _________.

例句：A triangle is an equilateral triangle **if and only if** it is an equiangular triangle.

一個三角形為等邊三角形若且唯若此三角形為等角三角形。

❷ ______ is read as _____________.

例句：The expression $m\overset{\frown}{AB}$ **is read as** "the measure of arc *AB*"

$m\overset{\frown}{AB}$的表示法讀作「弧 *AB* 的度量」。

❸ ______ is inscribed in _____________.

例句：Triangle *ABC* (△ *ABC*) **is inscribed in** circle *O*.

三角形 *ABC* 是圓 *O* 的內接三角形。

❹ ______ is circumscribed about _____________.

例句：Circle *O* **is circumscribed about** the pentagon *ABCDE*.

圓 *O* 是五邊形 *ABCDE* 的外接圓。

❺ Note that ______.

例句：**Note that** the measure of a semicircle is 180°.

注意半圓的度數是 180°。

❻ Given that __________.

例句：**Given that** triangle *ABC* is a right triangle, select a set of possible side lengths.

假設三角形 *ABC* 是直角三角形，哪一組是可能的邊長？

❼ ________ so far

例句：Do you have any question **so far**?

到目前為止，有任何問題嗎？

❽ **core concept____________.**

例句：A **core concept** is a **fundamental, central concept that is essential**.
核心概念是必要的基本中心概念。

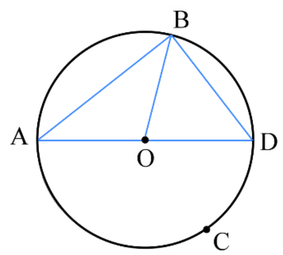
❾ **The more/(adj + er)_________ the more/(adj + er) _______.**

例句：**The greater angle has the longer side** opposite to it.
角的度數越大，對應的邊越長（大角對大邊）。

■ 問題講解 Explanation of Problems

☙ 說明 ❧

In the previous unit, we learned how to find a sector's arc length and area. In this section, we will discuss the arc measure in a degree measurement.



Core Concept 1 How are circular arcs measured?

The measure of an arc is the measure of its central angle.

If $m\angle AOB < 180°$, the **measure of a minor arc** is equal to $m\angle AOB$.

The expression $m\widehat{AB}$ is read as "the measure of arc AB."

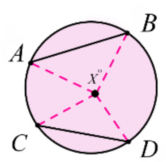
The **measure of a major arc** is equal to the difference between the measure of the entire circle 360° and the measure of the related minor arc.

In the diagram, $m\widehat{ACB} = 360° - m\widehat{AB}$.

The measure of a semicircle is 180° ($m\widehat{ACD} = m\widehat{ABD} = 180°$).

Theorem-In the same circle, or in congruent circles, two minor arcs are congruent if and only if their corresponding central angles are congruent.

We know the length of arc AB is equal to $\frac{x}{360} \times$ circumference when x is the measure of the central angle, and the measure of an arc is the measure of its central angle.



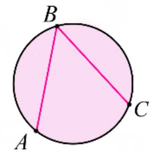
We can conclude that:

In the same circle or congruent circles:

(i) two arcs have equal measures if and only if two arcs have equal arc lengths;

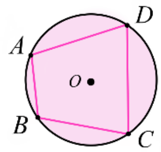
(ii) an arc has greater length if and only if the arc has a greater measure.

Core Concept 2



Inscribed Angle - An inscribed angle is an angle where the vertex is on a circle and the sides contain chords of the circle. ($\angle ABC$ is an inscribed angle.)

Intercepted Arc - An arc that lies between two lines, rays, or segments is called an intercepted arc. (Arc $\overset{\frown}{AC}$ is intercepted by $\angle ABC$.)



Inscribed Polygon - A polygon is an inscribed polygon when all its vertices lie on a circle. (Quad. *ABCD* is inscribed in the circle *O*.)

Measure of an Inscribed Angle Theorem -

The measure of an inscribed angle is one-half the measure of its intercepted arc.

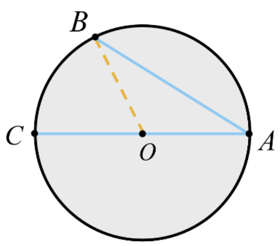
The proof of the measure of an inscribed angle theorem involves three cases:

(1) Center *O* is on a side of the inscribed angle.

(2) Center *O* is inside the inscribed angle.

(3) Center *O* is outside the inscribed angle.

The proof of case 1: (1) Connect $\overline{OB}$; $\overline{OA} \cong \overline{OB}$ (the radii of a circle are congruent)



(2) ΔAOB is an isosceles triangle, $m\angle ABO = m\angle BAO$

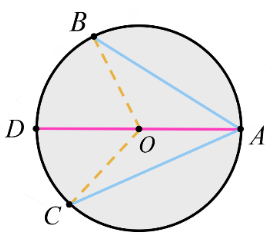
(3) From the triangle exterior angle theorem:

$m\angle BOC = m\angle ABO + m\angle BAO$

and we get $m\angle BOC = 2m\angle BAO$; $m\angle BAO = \frac{1}{2}m\angle BOC$

(4) Since $m\angle BOC = m\overset{\frown}{BC}$, we get $m\angle BAC(m\angle BAO) = \frac{1}{2}m\overset{\frown}{BC}$.

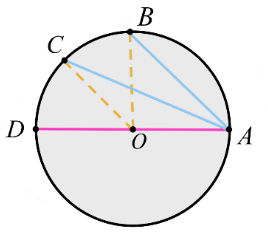
The proof of case 2: (1) Connect $\overline{OB}$ and $\overline{OC}$. Then, draw $\overrightarrow{AO}$ which intersects circle *O* at point *D*.



(2) From case 1, we get $m\angle BAD = \frac{1}{2}m\overset{\frown}{BD}$, $m\angle DAC = \frac{1}{2}m\overset{\frown}{DC}$

(3) $m\angle BAC = m\angle BAD + m\angle DAC = \frac{1}{2}m\overset{\frown}{BD} + \frac{1}{2}m\overset{\frown}{DC} = \frac{1}{2}m\overset{\frown}{BC}$

The proof of case 3: (1) Connect $\overline{OB}$ and $\overline{OC}$. Then draw $\overrightarrow{AO}$ which intersects circle O at point D.

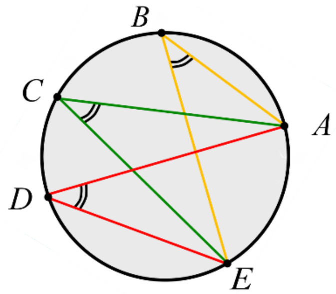


(2) From case 1, we get $m\angle BAD = \frac{1}{2} m\,\widehat{BD}$, $m\angle CAD = \frac{1}{2} m\,\widehat{CD}$

(3) $m\angle BAC = m\angle BAD - m\angle CAD = \frac{1}{2} m\,\widehat{BD} - \frac{1}{2} m\,\widehat{CD} = \frac{1}{2} m\,\widehat{BC}$

The inscribed angles that intercept the same arc are congruent in a circle.

In the diagram, $\angle B = \angle C = \angle D = \frac{1}{2} m\,\widehat{AE}$.



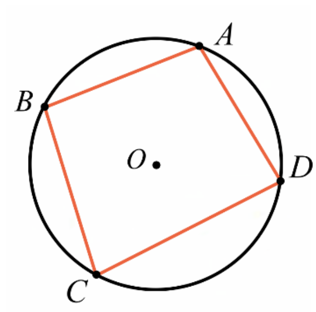
Core Concept 3

The opposite angles of an inscribed quadrilateral in a circle are supplementary.

In the diagram, Quad $ABCD$ is inscribed in circle O. Show that: $\angle B + \angle D = 180°$

(proof) We know $\angle B = \frac{1}{2}\widehat{ADC}$ and $\angle D = \frac{1}{2}\widehat{ABC}$, then we can get

$\angle B + \angle D = \frac{1}{2}(\widehat{ADC} + \widehat{ABC}) = \frac{1}{2} \cdot 360° = 180°$



ꕥ 運算問題的講解 ꕥ

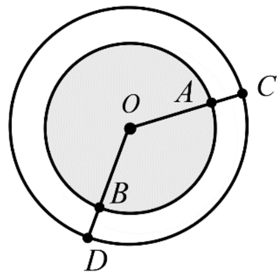
例題一

說明：已知圓心角求弧的度數與長度。

（英文）In the diagram, the radii of two concentric circles with the center O are 8 cm and 10 cm. If $m\angle AOB = m\angle COD = 120°$, find:

(1) $m\overset{\frown}{AB}$ and $m\overset{\frown}{CD}$

(2) the arc lengths of $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$



（中文）如右圖，兩個以 O 點為圓心的同心圓半徑分別 8 公分及 10 公分，且 $\angle AOB = \angle COD = 120°$，求：

(1) $\overset{\frown}{AB}$ 與 $\overset{\frown}{CD}$ 的度數。

(2) $\overset{\frown}{AB}$ 與$\overset{\frown}{CD}$的長度。

Teacher: Note that $m\angle AOB = m\angle COD = 120°$. So, what are the measures of arc AB and arc CD($\overset{\frown}{AB}$ and $\overset{\frown}{CD}$)?

Student: 120°.

Teacher: Correct. Also, the measure of an arc is the measure of its central angle, and the arc length is equal to $\frac{x}{360}\times$ circumference where x is the measure of the central angle.

So, we can get the arc length of $\overset{\frown}{AB}$ is $\frac{120}{360}\times 16\pi = \frac{16}{3}\pi$.

What is the arc length of $\overset{\frown}{CD}$?

Student: $\frac{20}{3}\pi$.

Teacher: Good. Let’s look at the next question.

老師： 已知 $m\angle AOB = m\angle COD = 120°$。那麼 $\overset{\frown}{AB}$ 弧 以及 $\overset{\frown}{CD}$ 的度數是多少呢？

學生： 120°。

老師： 正確。且一弧的度數等於其圓心角的度數，而弧長則等於 $\frac{x}{360}\times$ 圓周長，其中 x 為圓心角的度數。因此，我們可以得到 $\overset{\frown}{AB}$ 的長度為 $\frac{120}{360}\times 16\pi = \frac{16}{3}\pi$。

那麼$\overset{\frown}{CD}$ 的弧長是多少？

學生： $\frac{20}{3}\pi$.

老師： 很好！繼續看下個例題。

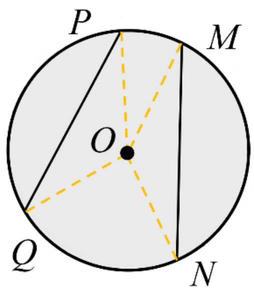
例題二

說明：等弧對等弦。

（英文）In the same circle or congruent circles, two chords are congruent if and only if their corresponding arcs are congruent.

In the diagram, $\widehat{MN} \cong \widehat{PQ}$ if and only if $\overline{MN} \cong \overline{PQ}$.

（中文）在同圓或等圓中，度數相等的兩弧（等弧）其所對的弦長也相等；長度相等的兩弦（等弦）其所對的弧長也相等。



Teacher: This statement has two parts to prove:

(1) If two arcs are congruent, then their corresponding chords are congruent.

(2) If two chords are congruent, then their corresponding arcs are congruent.

In part (1), we know that $m\angle MON = m\widehat{MN} = m\widehat{PQ} = m\angle POQ$ because $\widehat{MN} \cong \widehat{PQ}$ implies $m\widehat{MN} = m\widehat{PQ}$.

Connect $\overline{OM}, \overline{ON}, \overline{OP}$ and $\overline{OQ}$. We know $\overline{OM} = \overline{ON} = \overline{OP} = \overline{OQ}$ (the radii of the same circle).

We get $\triangle MON = \triangle POQ$ from the Side-Angle-side (SAS) Congruence Postulate.

So $\overline{MN} \cong \overline{PQ}$.

In part 2, we can use $\overline{MN} = \overline{PQ}$ and $\overline{OM} = \overline{ON} = \overline{OP} = \overline{OQ}$ to easily get $\angle MON = \angle POQ$. Which Congruence Postulate can be applied here?

Student: The Side-side-side (*SSS*) Congruence Postulate.

Teacher: Correct. From $\angle MON = \angle POQ$, we get $m\widehat{MN} = m\widehat{PQ}$ or $\widehat{MN} \cong \widehat{PQ}$.

老師： 這個性質需要分兩部分證明：

(1) 度數相等的兩弧（等弧），其所對的弦長相等。

(2) 長度相等的兩弦（等弦），其所對的弧長相等。

第(1)部分，我們知道 $m\angle MON = m\widehat{MN} = m\widehat{PQ} = m\angle POQ$，因為 $\widehat{MN} \cong \widehat{PQ}$ 代表 $m\widehat{MN} = m\widehat{PQ}$。

連接 $\overline{OM}$、$\overline{ON}$、$\overline{OP}$ 和 $\overline{OQ}$。已知 $\overline{OM}=\overline{ON}=\overline{OP}=\overline{OQ}$（皆為此圓的半徑）。由 SAS 全等原理，得出 △MON = △POQ。因此 $\overline{MN}\cong\overline{PQ}$。

第(2)部份，我們可以利用 $\overline{MN}=\overline{PQ}$ 和 $\overline{OM}=\overline{ON}=\overline{OP}=\overline{OQ}$ 輕易得出 $\angle MON=\angle POQ$。這裡可以應用哪一個全等原理？

學生：SSS 全等原理。

老師：正確。由 $\angle MON=\angle POQ$，得出 $m\widehat{MN}=m\widehat{PQ}$ 或 $\widehat{MN}\cong\widehat{PQ}$。

例題三

說明：運用圓周角的度數等於其所對弧度數的一半的性質求解。

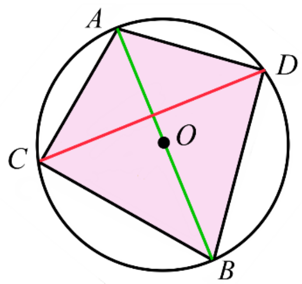
（英文）In the diagram, $\overline{AB}$ is the diameter of circle O.

If $\angle ADC=46°$, $m\widehat{BD}=110°$, then find

(1) $m\widehat{AC}$, $m\widehat{BC}$ and $m\widehat{AD}$.

(2) the measures of $\angle ABC$, $\angle ABD$, and $\angle ACD$

(3) the measures of $\angle ACB$



（中文）如右圖，$\overline{AB}$ 為圓 O 的直徑。已知$\angle ADC=46°$、$\widehat{BD}=110°$，求

(1) $\widehat{AC}$、$\widehat{BC}$ 及 $\widehat{AD}$ 的度數。

(2) $\angle ABC$、$\angle ABD$ 及 $\angle ACD$ 的度數。

(3) $\angle ACB$的度數。

Teacher: We just learned that the measure of an inscribed angle is half the measure of the intercepted arc. What is the measure of $\widehat{AC}$, if $\angle ADC=46°$?

Student: 92°.

Teacher: Correct. Note that both $\widehat{ACB}$ and $\widehat{ADB}$ are semicircles where the measures are both 180°. Find the measures of the arcs $\widehat{AD}$ and $\widehat{BC}$ if $m\widehat{BD}=110°$.

Student: $m\widehat{BC}=88°$ and $m\widehat{AD}=70°$.

Teacher: Excellent. So what are the measures of $\angle ABC$, $\angle ABD$, and $\angle ACD$?

Student: $\angle ABC=46°$, $\angle ABD=35°$, and $\angle ACD=35°$

Teacher: Good. Actually, we can find that $\angle ABC$ and $\angle ADC$ have the same measures because they intercept the same arc. Similarly, we get $\angle ABD=\angle ACD$.

So what is the measure of $\angle ACB$?

Student: 90°.

Teacher: You are correct. An inscribed angle is always right if it intercepts a semicircle because the measure of a semicircle is 180°.

老師：剛才學到：圓周角的度數是其對應弧度的一半。如果$\angle ADC = 46°$，$\widehat{AC}$ 的度數是多少呢？

學生：92°。

老師：答對了。請注意，$\widehat{ACB}$ 和 $\widehat{ADB}$ 都是半圓，其度數都是 180°。已知 $m\widehat{BD} = 110°$，請算出 $\widehat{AD}$ 和 $\widehat{BC}$ 的弧度數。

學生：$m\widehat{BC} = 88°$，$m\widehat{AD} = 70°$。

老師：很好。那麼$\angle ABC$、$\angle ABD$ 和$\angle ACD$ 的度數分別是多少？

學生：$\angle ABC = 46°$、$\angle ABD = 35°$、$\angle ACD = 35°$。

老師：很好。實際上，由於它們所對的弧相同，因此我們可以發現 $\angle ABC$ 和$\angle ADC$ 的度數相同。同樣地，$\angle ABD = \angle ACD$。

學生：那麼$\angle ACB$ 的度數是多少？

老師：90°。

學生：答對了。如果一個圓周角所截弧為半圓，那麼它一定是直角。因為一個半圓的度數為 180°。

例題四

說明：運用圓周角的度數等於其所對弧度數的一半的性質求解。

（英文）In the diagram, $\angle COD = 80°$, $m\,\widehat{AD} = 70°$, $\angle DAB = 110°$.

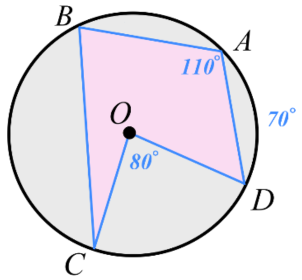
Find (1) the measure of $\angle ABC$.

(2) the measure of $\widehat{BC}$.

（中文）如右圖，已知 $\angle COD = 80°$、$\widehat{AD} = 70°$ 且 $\angle DAB = 110°$，求：

(1) $\angle ABC$的度數。

(2) $\widehat{BC}$ 的度數。



Teacher: Given that $\angle COD = 80°$ and $m\,\widehat{AD} = 70°$, we can get $m\,\widehat{CD} = 80°$ and $m\,\widehat{ADC} = m\,\widehat{AD} + m\,\widehat{CD} = 150°$.

So what is the measure of $\angle ABC$?

Student: $\angle ABC = \frac{1}{2} m\,\widehat{ADC} = 75°$.

Teacher: That's correct. What can you get if $\angle DAB = 110°$?

Student: $\widehat{BCD} = 220°$.

Teacher: Correct. So now find the measure of $\widehat{BC}$ by using $\widehat{BCD} = \widehat{BC} + \widehat{CD}$.

Student: $\widehat{BC} = 140°$.

Teacher: Excellent!

老師： 已知 $\angle COD = 80°$和$m\,\widehat{AD} = 70°$，我們可以得到 $\widehat{CD} = 80°$ 以及$m\,\widehat{ADC} = m\,\widehat{AD} + m\,\widehat{CD} = 150°$。

那麼 $\angle ABC$ 的度數是多少呢？

學生： $\angle ABC = \frac{1}{2} m\,\widehat{ADC} = 75°$。

老師： 答對了。那由已知 $\angle DAB = 110°$，你能得到什麼？

學生： $\widehat{BCD} = 220°$。

老師： 正確。現在請用 $\widehat{BCD} = \widehat{BC} + \widehat{CD}$來找出 $\widehat{BC}$ 的度數。

學生： $\widehat{BC} = 140°$。

老師： 讚啦！

例題五

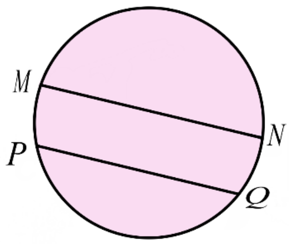
說明：一圓中兩平行弦所截弧相等。

（英文）In a circle, two chords $\overline{MN}$ and $\overline{PQ}$ are parallel.

Show that $\widehat{MP}=\widehat{NQ}$

（中文）如右圖，若弦 $\overline{MN}$ 和弦 $\overline{PQ}$ 互相平行。

試說明 $\widehat{MP}$ 和 $\widehat{NQ}$ 會相等。

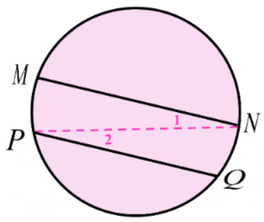


Teacher: Draw $\overline{NP}$. Since $\overline{MN}\,//\,\overline{PQ}$, we can find $\angle 1=\angle 2$ from the property that alternate interior angles of parallel lines are congruent.

Besides, $\angle 1=\frac{1}{2}\widehat{MP}$ and $\angle 2=\frac{1}{2}\widehat{NQ}$ (Inscribed angles $=\frac{1}{2}$ intercepted arc).

So $\frac{1}{2}\widehat{MP}=\frac{1}{2}\widehat{NQ}$; $\widehat{MP}=\widehat{NQ}$.

Do you have any question so far?



Student: No.

Teacher: Good. Now let's go on to the next part.

老師： 首先畫出 $\overline{NP}$。由於 $\overline{MN}\,//\,\overline{PQ}$，我們可以從平行線內錯角相等的性質得出 $\angle 1=\angle 2$。此外，$\angle 1=\frac{1}{2}\widehat{MP}$ 且 $\angle 2=\frac{1}{2}\widehat{NQ}$（圓周角 $=\frac{1}{2}$ 截弧）。因此，$\frac{1}{2}\widehat{MP}=\frac{1}{2}\widehat{NQ}$；$\widehat{MP}=\widehat{NQ}$。

目前為止，有任何問題嗎？

學生： 沒有。

老師： 很好，那就繼續下一個例題囉。

應用問題 / 會考素養題

例題一

說明：運用圓周角的度數等於其所對弧度數的一半，等弦對等弧及三角形的邊角關係（大角對大邊）求解。

（英文）In the circle the two chords $\overline{AC}$ and $\overline{BD}$ intersect at E, and $\overline{AB}=\overline{BC}$. Which of the following segments is the longest?

(A) $\overline{AE}$　(B) $\overline{BE}$　(C) $\overline{CE}$　(D) $\overline{DE}$

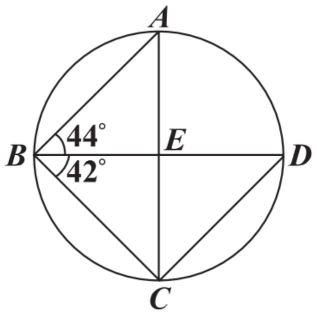
（中文）圓上有 A、B、C、D 四點，其位置如圖所示，其中 $\overline{AC}$ 與 $\overline{BD}$ 相交於 E 點，且 $\overline{AB}=\overline{BC}$。根據圖中標示的角度，判斷下列四條線段何者長度最長？

(A) $\overline{AE}$

(B) $\overline{BE}$

(C) $\overline{CE}$

(D) $\overline{DE}$



（109 年國中會考 19）

Teacher: In the diagram, $\angle ABE=44°$, $\angle CBD=42°$ and $\angle ABE=\frac{1}{2}\overset{\frown}{AD}$, $\angle DBC=\frac{1}{2}\overset{\frown}{CD}$.

We get $\overset{\frown}{AD}=88°$ and $\overset{\frown}{CD}=84°$.

Besides, we can get $\overset{\frown}{AB}=\overset{\frown}{BC}$ given that $\overline{AB}=\overline{BC}$.

In a circle, $\overset{\frown}{AB}+\overset{\frown}{BC}+\overset{\frown}{CD}+\overset{\frown}{AD}=360°$.

So we get $\overset{\frown}{AB}=\overset{\frown}{BC}=\frac{1}{2}(360°-88°-84°)=94°$.

Now, find the measures of the angles $\angle BAE$ and $\angle BCE$.

Student: $\angle BAE=\angle BCE=47°$.

Teacher: Correct. You can find $\angle EDC=47°$ because both $\angle EDC$ and $\angle BAE$ intercept the same arc $\overset{\frown}{BC}$. Similarly, you can also find $\angle ECD=44°$.

In ΔABE, $\angle BAE>\angle ABE$.

Can you tell which of the segments is greater?

Student: $\overline{BE}$ is greater.

Teacher: Great. In a triangle, the **greater angle has** the **longer side** opposite to it. Similarly, compare the sides $\overline{BE}$ and $\overline{CE}$ in $\triangle CDE$. Which one is greater?

Also, compare the sides $\overline{CE}$ and $\overline{DE}$ in $\triangle CDE$. Which one is greater?

Student: $\overline{BE} > \overline{CE}$ and $\overline{CE} > \overline{DE}$.

Teacher: Very good. So the longest side is (B) $\overline{BE}$.

老師：由圖得知，$\angle ABE = 44°$、$\angle CBD = 42°$，且$\angle ABE = \frac{1}{2}\widehat{AD}$，$\angle DBC = \frac{1}{2}\widehat{CD}$。

可以得出 $\widehat{AD} = 88°$ 和 $\widehat{CD} = 84°$。此外，因為$\overline{AB} = \overline{BC}$，所以 $\widehat{AB} = \widehat{BC}$。

任一圓中，$\widehat{AB} + \widehat{BC} + \widehat{CD} + \widehat{AD} = 360°$。

所以我們得到 $\widehat{AB} = \widehat{BC} = \frac{1}{2}(360° - 88° - 84°) = 94°$。

現在，找出 $\angle BAE$ 和 $\angle ABE$ 的度數。

學生：$\angle BAE = \angle BCE = 47°$。

老師：對，你可以得出 $\angle EDC = 47°$，因為$\angle EDC$ 和$\angle BAE$ 都截自同一弧 $\widehat{BC}$。同樣地，你也可以找出$\angle ECD = 44°$。

在$\triangle ABE$ 中，$\angle BAE > \angle ABE$。你能比較出哪條邊比較長嗎？

學生：$\overline{BE}$ 比較長。

老師：很好。在三角形中，大角對大邊。同樣地，比較$\triangle CDE$ 中的$\overline{BE}$ 和 $\overline{CE}$，哪一邊比較長？再比較$\overline{CE}$ 和$\overline{DE}$，哪一個更長？

學生：$\overline{BE} > \overline{CE}$ 且$\overline{CE} > \overline{DE}$。

老師：非常好。所以最長的邊是(B) $\overline{BE}$。

例題二

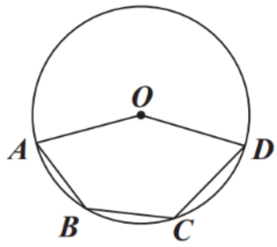
說明：運用圓周角的度數等於其所對弧度數的一半的性質求解。

（英文）In the diagram, circle O passes through the four vertices A, B, C, and D of a pentagon $OABCD$. If $\overparen{ABD} = 150°$, $\angle A = 65°$, and $\angle D = 60°$, find the measure of $\overparen{BC}$.

(A) 25 (B) 40 (C) 50 (D) 55

（中文）如圖(七)，圓 O 通過五邊形 $OABCD$ 的四個頂點。若 $\overparen{ABD} = 150°$，$\angle A = 65°$，且 $\angle D = 60°$，求 $\overparen{BC}$ 的度數。

(A) 25 (B) 40 (C) 50 (D) 55



圖（七）

（105 年國中會考 14）

Teacher: Connect $\overline{OB}$ where $\overline{OA} = \overline{OB}$ (radii of the same circle). Then we can get $\angle OBA = \angle A = 65°$. Connect $\overline{OC}$. What is $\angle OCD$ by the same method?

Student: $\angle OCD = \angle D = 60°$.

Teacher: Good. By using the sum of the three angles of a triangle is 180°, we can get $\angle AOB = 50°$ and $\angle COD = 60°$. What are the measures of $\overparen{AB}$ and $\overparen{CD}$?

Student: $\overparen{AB} = 50°$ and $\overparen{CD} = 60°$.

Teacher: Yes, you are correct. Now, find the measure of $\overparen{BC}$ by given $\overparen{ABD} = 150°$. Who knows the answer?

Student: The answer is (B) 40.

Teacher: Correct. You did a good job!

老師：連接 $\overline{OB}$，其中 $\overline{OA} = \overline{OB}$（皆為圓的半徑）。然後，我們可以用相同的方法得到 $\angle OBA = \angle A = 65°$。連接 $\overline{OC}$，用相同的方法求 $\angle OCD$ 是多少？

學生：$\angle OCD = \angle D = 60°$。

老師：很好。因三角形內角和為 180°，我們可以得到 $\angle AOB = 50°$ 和 $\angle COD = 60°$。那麼 $\overparen{AB}$ 和 $\overparen{CD}$ 的度數為何？

學生：$\overparen{AB} = 50°$、$\overparen{CD} = 60°$。

老師：　很好，沒錯。現在，透過已知 $\overset{\frown}{ABD}$ =150° 來找出 $\overset{\frown}{BC}$ 的度數。誰知道答案？

學生：　答案是(B) 40。

老師：　答對了！

例題三

說明：運用圓周角的度數等於其所對弧度數的一半的性質求解（但 108 課綱已刪除弦切角的內容，老師可看學生學習程度來決定是否補充。）

（英文）In the diagram, $ABCD$ is a trapezoid with $\overline{AD} // \overline{BC}$ and circle O passes through the three points A, B, and C. If $\angle B = 58°$ and $\overline{AD}$ is tangent to circle O at point A, find the measure of $\overset{\frown}{BC}$.

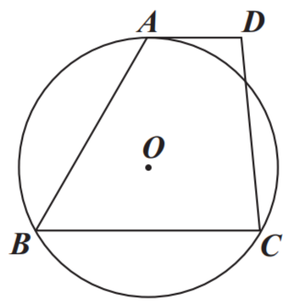
（中文）如圖，梯形 $ABCD$ 中，$\overline{AD} // \overline{BC}$，有一圓 O 通過 A、B、C 三點，且 $\overline{AD}$ 與圓 O 相切於 A 點。若 $\angle B = 58°$，則 $\overset{\frown}{BC}$ 的度數為何？

(A) 116

(B) 120

(C) 122

(D) 128



（110 年國中會考 17）

Teacher:　In the trapezoid $ABCD$, note that $\angle B$ and $\angle BAD$ are supplementary, given that $\overline{AD} // \overline{BC}$.

Given that $\angle B = 58°$, you can get $\angle BAD = 180° - 58° = 122°$.

Then $m\overset{\frown}{AC} = 2\angle B = 116°$ because $\angle B$ is an inscribed angle of $\overset{\frown}{AC}$.

Besides, $\angle BAD$ is an angle formed by a chord and a tangent which is equal to $\frac{1}{2}m\overset{\frown}{ACB}$. So $m\overset{\frown}{ACB} = 2\angle BAD = 244°$.

What is the measure of $\overset{\frown}{BC}$? Explain how you get it.

Student:　(D) 128.

Because $m\overset{\frown}{BC} = m\overset{\frown}{ACB} - m\overset{\frown}{AC} = 244° - 116° = 128°$.

Teacher:　Yes, you are correct.

老師：　在梯形 $ABCD$ 中，已知 $\overline{AD} // \overline{BC}$，所以 $\angle B$ 和 $\angle BAD$ 互補。

已知 $\angle B = 58°$，因此可以得到 $\angle BAD = 180° - 58° = 122°$。

然後 $m\widehat{AC} = 2\angle B = 116°$，因為 $\angle B$ 是 $\widehat{AC}$ 的圓周角。此外，$\angle BAD$ 是由弦和切線形成的角，其度數等於 $\frac{1}{2}\, m\widehat{ACB}$。

因此 $m\widehat{ACB} = 2\angle BAD = 244°$。

你能找出 $\widehat{BC}$ 的度數並如何得到的嗎？

學生：　答案是 (D)128。

因為 $m\widehat{BC} = m\widehat{ACB} - m\widehat{AC} = 244° - 116° = 128°$。

老師：　答對了。

單元七 證明與推理
Deductive Reasoning

國立新竹科學園區實驗高級中等學校　陳立業老師

■ 前言 Introduction

在歐幾里得的幾何原本中，從許多的定義及公設的基礎之上，延伸出了許多的定理，也同時發現了在不同的圖形中的一些幾何性質。在本單元，如在教學常用例子 3 到 6 是在證明中常會用到的句型，但因為句型較長且用到的專用名詞也比較多，可以一節課先將專有名詞練習完後，再去熟練寫出整個句子。

■ 詞彙 Vocabulary

單字	中文	單字	中文
theorem	定理	if-then statement	條件陳述
definition	定義	hypothesis	假設
prove (v.)	證明	counterexample	反例
conclusion	結論	biconditional	雙條件
statement	途述	parallel	平行
postulate	公設	corresponding angle	同位角
proof (n.)	證明	alternative interior angle	內錯角
given	已知	isosceles triangle	等腰三角形
angle bisector	角平分線	height	高

congruence	全等	midpoint	中點
congruent	全等的	segment	線段
similarity	相似	square	正方形
similar	相似的	quadrilateral	四邊形
odd number	奇數	equiangular triangle	等角三角形(正三角形)
even number	偶數	equilateral triangle	等邊三角形(正三角形)

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ If________, then ________.

例句：**If** two sides of a triangle are congruent, **then** the angles opposite those sides are congruent.

如果在一個三角形中有兩邊全等，那麼這兩個邊的相對應的角也會全等。

❷ Given ________, prove__________.

例句：**Given**: $\overline{AB} = \overline{AC}$, **prove**: A is on the perpendicular bisector of $\overline{BC}$.

已知：$\overline{AB} = \overline{AC}$，求證：$A$ 在線段 $\overline{BC}$ 的中垂線上。

❸ Corresponding ________ of congruent triangles are congruent.

例句：a. **Corresponding** sides **of congruent triangles are congruent.**

b. **Corresponding** angles **of congruent triangles are congruent.**

全等三角形的對應邊及對應角相等。

❹ △ _____ ≅ △_________ by the ___________ congruence property.

例句：$\triangle ABC \cong \triangle DEF$ **by the** *SAS* **(side-angle-side) congruence property.**

(read as "triangle *ABC* is congruent to triangle *DEF* by the *SAS* congruence property.")

由 *SAS* 全等性質得出三角形 *ABC* 與三角形 *DEF* 為全等。

❺ If two parallel lines are cut by a transversal, then _____ are______.

例句：a. **If two parallel lines are cut by a transversal, then** the corresponding angles **are** congruent**.**

b. **If two parallel lines are cut by a transversal, then** the alternate interior angles **are** congruent**.**

c. **If two parallel lines are cut by a transversal, then** the interior angles on the same side **are** supplementary**.**

兩平行線被一直線所截的同位角相等、內錯角相等、同側內角互補。

❻ Corresponding __________ of similar triangles are _________.

例句(1)：**Corresponding** sides **of similar triangles are** proportional**.**

相似三角形的對應邊成比例。

例句(2)：**Corresponding** angles **of similar triangles are** congruent**.**

相似三角形的對應角相等。

❼ If *a* is _________, then we can assume *a* equals_______.

例句(1)：**If** a **is** even, **then we can assume** a **equals** $2m$, where m is an integer.

如果 a 是偶數，我們可以假設 a 等於 $2m$，其中 m 為一整數。

例句(2)：**If** a **is** odd, **then we can assume** a **equals** $2m+1$, where m is an integer.

如果 a 是奇數，我們可以假設 a 等於 $2m+1$，其中 m 為一整數。

■ 問題講解 Explanation of Problems

說明

When we prove the answer to a geometry question, we can use the theorems and properties we have learned, such as the Pythagorean theorem, the congruence properties of triangles, and the properties of angle bisectors, to be the reasoning to support our conclusions.

我們能利用已知的定理和性質作為幾何推理的理由。

Conclusions based on accepted statements (properties, definition, previous theorems, and given information) must be true if the hypotheses are true.

由已知的定理，性質或定義而推導出來的結論為真。

運算問題的講解

例題一

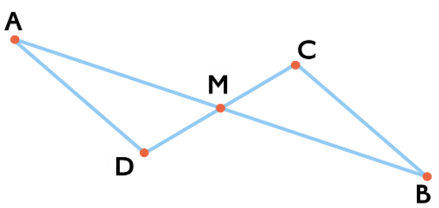
說明：可以先找到兩個全等的三角形，並利用全等三角形對應角相等的性質，接下來發現這組對應角也可視作線被一截線所截後的一組內錯角；當內錯角相等時，則兩線平行。

（英文）Given: $\overline{AB}$ and $\overline{CD}$ bisect each other at point M.

Prove: $\overline{AD} \parallel \overline{BC}$

（中文）已知：$\overline{AB}$與$\overline{CD}$ 兩線段互相平分於 M 點，

求證：$\overline{AD} \parallel \overline{BC}$。



Teacher: There are several ways to show that two lines (or segments) are parallel. Please identify some of these ways.

Student: I can prove two lines are parallel by showing that the corresponding angles are congruent, that the alternate interior angles are congruent, or that the interior angles on the same side are supplementary.

Teacher: Assume $\overline{CD}$ is one of the transversals that cuts through $\overline{AD}$ and $\overline{BC}$. If we can show $\angle C \cong \angle D$, then we can prove $\overline{AD} \parallel \overline{BC}$.

Student: I see. So how can we show $\angle C \cong \angle D$?

Teacher: For this question, we need to find the relationship between $\triangle ADM$ and $\triangle BCM$. Are they congruent with each other?

Now, we try to show that $\triangle ADM$ and $\triangle BCM$ are congruent. What is given in this question?

Student: $\overline{AB}$ and $\overline{CD}$ bisect each other at point M .

Teacher: Yes, by using the property of segment bisectors, we can say $\overline{AM} \cong \overline{BM}$ and $\overline{DM} \cong \overline{CM}$. We still need one more pair of corresponding sides or angles which are congruent. We want to show that they are congruent.

Student: I found that $\angle AMD \cong \angle BMC$ because vertical angles are congruent.

Teacher: Great! You have found the right one to complete the proof . Which property can we use to show those two triangles are congruent?

Student: The side-angle-side (SAS) theorem.

Teacher: Since $\triangle ADM \cong \triangle BCM$, we can also conclude that $\angle C \cong \angle D$. Why can we get this conclusion?

Student: Because corresponding angles of congruent triangles are congruent.

Teacher: Well done! Since $\angle C \cong \angle D$, we can deduce that $\overline{AD} \,//\, \overline{BC}$. If two lines are cut by a transversal, and alternate interior angles are congruent, then the lines are parallel. So far, this is the plan of the proof.

Let's complete the proof now.

Statements	Reasoning
1. $\overline{AB}$ and $\overline{CD}$ bisect each other at M.	1. Given
2. $\overline{AM} \cong \overline{BM}$ and $\overline{DM} \cong \overline{CM}$.	2. Property of segment bisector
3. $\angle AMD \cong \angle BMC$	3. Vertical angles are congruent.
4. $\triangle ADM \cong \triangle BCM$	4. SAS (side-angle-side) theorem
5. $\angle C \cong \angle D$	5. Corresponding angles of congruent triangles are congruent.
6. $\overline{AD} \,//\, \overline{BC}$	6. If two lines are cut by a transversal, and alternate interior angles are congruent, then the lines are parallel.

老師：　有幾種方法可以證明兩條線（或線段）是平行的。請列舉。

學生：　我可以用證明同位角相等、內錯角相等或同側內角相等來證明兩條線是平行的。

老師：　假設 $\overline{CD}$ 是一條切過 $\overline{AD}$ 和 $\overline{BC}$ 的直線。如果我們能夠證明 $\angle C \cong \angle D$ ，那麼我們就可以證明 $\overline{AD} // \overline{BC}$。

學生：　了解。那要怎樣證明 $\angle C \cong \angle D$ 呢？

老師：　我們需要找到 $\triangle ADM$和$\triangle BCM$ 之間的關係。此二個三角形是否相等？現在要證明$\triangle ADM$ 和 $\triangle BCM$是相等的。題目給出哪些資訊？

學生：　$\overline{AB}$與$\overline{CD}$ 兩線段互相平分於 M 點。

老師：　是的，使用線段平分線的性質，我們可以說 $\overline{AM} \cong \overline{BM}$ 以及 $\overline{DM} \cong \overline{CM}$。我們仍需要一對相對應的邊或角相等。

學生：　我發現$\angle AMD \cong \angle BMC$ 因為對頂角是相等的。

老師：　太好了！你找到了第三個條件來完成這個證明。哪個性質可以證明這兩個三角形是相等的？

學生：　SAS 定理。

老師：　因為 $\triangle ADM \cong \triangle BCM$, 我們可以得出$\angle C \cong \angle D$. 我們是如何得到這個結論的？

學生：　因為兩全等三角形的對應角全等。

老師：　很好。因為$\angle C \cong \angle D$我們接著可以得出 $\overline{AD} // \overline{BC}$。如果兩條直線被另一直線所截，且內錯角相等，則兩條直線平行。這就是證明的過程。現在將證明寫下來吧。

敘述	原因
1. $\overline{AB}$與$\overline{CD}$ 兩線段互相平分於 M 點	已知
2. $\overline{AM} \cong \overline{BM}$ 且$\overline{DM} \cong \overline{CM}$.	線段平分性質
3. $\angle AMD \cong \angle BMC$	對頂角相等
4. $\triangle ADM \cong \triangle BCM$	SAS 定理
5. $\angle C \cong \angle D$	兩個全等三角型對應角相同
6. $\overline{AD} // \overline{BC}$	如果兩條直線被一條直線所截，內錯角相等，則兩條直線平行。

例題二

說明：本題是關於代數證明。

（英文）Proof: If x is an odd integer, show that x^2 is an odd integer.
（中文）證明：任何奇數的平方也是奇數。

Teacher: Let x be an odd integer. We can assume that x is an even integer plus one or minus one, written as $x = 2k + 1$ (or $x = 2k - 1$) for some integer k. What will you get if you substitute x with $2k + 1$?

Student: Then x^2 is replaced by $(2k + 1)^2$

Teacher: Try to expand $(2k + 1)^2$. Apply the formula of the square of a sum. What is $(a + b)^2$? (What is the square of a plus b?)

Student: $(a + b)^2 = a^2 + 2ab + b^2$.

Teacher: Substitute a with $2k$ and b with 1, and expand $(2k + 1)^2$.

Student: $(2k + 1)^2$
$= (2k)^2 + 2 \times 2k \times 1 + 1^2$
$= 4k^2 + 4k + 1$

Teacher: What could you tell from the sum of the first two terms?

Student: $4k^2$ is even since it can be expressed as $2 \times 2k^2$, and $4k$ is also expressed as $2 \times 2k$. So, both $4k^2$ and $4k$ are even. The sum of two even integers is also even. Note that the last term, “1”, is an odd integer.

Teacher: What can you conclude about the sum of an even integer and an odd integer?

Student: The sum is odd.

Teacher: Great! So the square of any odd integer is also an odd integer.

老師： 假設x是一個奇數。我們可以設 x 是一個偶數加一或減一，寫成 $x = 2k + 1$ （或 $x = 2k - 1$），其中 k 是一個整數。如果你將題目的 x 替換為 $2k + 1$，會得到什麼？

學生： $(2k + 1)^2$。

老師： 請以和的平方公式展開 $(2k + 1)^2$。$(a + b)^2$ 等於？

學生： $(a + b)^2 = a^2 + 2ab + b^2$。

老師： 將 a 換為 $2k$；b 替換為 1，然後展開。

學生： $(2k + 1)^2$

$= (2k)^2 + 2 \times 2k \times 1 + 1^2$

$= 4k^2 + 4k + 1$

老師： 從前兩項的和中可以得出什麼結論？

學生： $4k^2$ 是偶數，因為它可以表示為 $2 \times 2k^2$，$4k$ 也可以表示為 $2 \times 2k$。因此，$4k^2$ 和 $4k$ 都是偶數。兩個偶數的和也是偶數。偶數加上最後一項 1 是奇數。

老師： 你可以得出有關偶數和奇數之和的結論嗎？

學生： 偶數加奇數會等於奇數。

老師： 沒錯！因此，任何奇數的平方也是奇數。

應用問題 / 會考素養題

例題一

（英文）There are $\triangle ABC$ and $\triangle ADE$ with point C and point E on $\overline{AD}$ and $\overline{AB}$, respectively. $\overline{BC}$ and $\overline{DE}$ intersect at F. If $\angle A = 90^\circ$, $\angle B = \angle D = 30^\circ$ and $\overline{AC} = \overline{AE} = 1$, find the perimeter of quadrilateral $AEFC$.

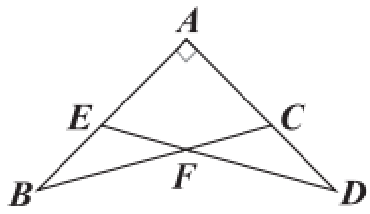
（中文）$\triangle ABC$、$\triangle ADE$ 中，C、E 兩點分別在 $\overline{AD}$、$\overline{AB}$ 上，且 $\overline{BC}$ 與 $\overline{DE}$ 相交於 F 點。若$\angle A = 90^\circ$，$\angle B = \angle D = 30^\circ$，$\overline{AC} = \overline{AE} = 1$，則四邊形 $AEFC$ 的周長為何？

(A) $2\sqrt{2}$

(B) $2\sqrt{3}$

(C) $2 + \sqrt{2}$

(D) $2 + \sqrt{3}$



（106 年國中會考 21）

Teacher: This is a question from the Comprehensive Assessment Program for Junior High School Students in 2017. First, let's read the question.

Student: There are $\triangle ABC$ and $\triangle ADE$ with point C and point E on $\overline{AD}$ and $\overline{AB}$, respectively. $\overline{BC}$ and $\overline{DE}$ intersect at F. If $\angle A = 90^\circ$, $\angle B = \angle D = 30^\circ$, and $\overline{AC} = \overline{AE} = 1$, find the perimeter of quadrilateral $AEFC$.

Teacher: Thank you. Let's find the perimeter of quadrilateral $AEFC$ Given $\angle A = 90^\circ$ and

$\angle B = 30^{\circ}$, please find the measure of $\angle ACB$.

Student: I can do that. Since the sum of the measure of the three interior angles equals 180, $\angle ACB = 180^{\circ} - 90^{\circ} - 30^{\circ} = 60^{\circ}$.

Teacher: Correct. Now, you may use the same method to find the measure of $\angle AED$.

Student: I got it! $\angle AED$ also equals 60° because I also know the other two interior angles in $\triangle AED$ are 90° and 30°.

Teacher: I need you to focus on $\triangle CDF$ now. Can you tell me the measure of $\angle CFD$?

Here is a hint: you can think of $\angle ACB$ as one of the exterior angles of $\triangle CDF$.

Student: Yes, since the measure of an exterior angle equals the sum of its two remote interior angles, we know the measure of $\angle CFD$ is equal to $60^{\circ} - 30^{\circ} = 30^{\circ}$.

Teacher: What kind of triangle is $\triangle CDF$?

Student: $\triangle CDF$ is an isosceles triangle because $\angle CFD$ is congruent to $\angle D$.

Teacher: We can also conclude that $\overline{CF}$ is congruent to $\overline{CD}$ because the sides opposite to base angles in an isosceles triangle are congruent.

Student: It looks like we can use the same method to analyze $\triangle BEF$. We can get the conclusion that $\overline{BE}$ is congruent to $\overline{FE}$.

Teacher: What kind of triangle is $\triangle ABC$.

Student: ΔABC is a right triangle with three interior angles equal to 30°, 60°, and 90°.

Teacher: What is the ratio of the three sides of a triangle with three interior angles equal to 30°, 60°, and 90°?

Student: The ratio is $1 : \sqrt{3} : 2$.(one to two to square root of 3). Since $\overline{AC} = 1$, so $\overline{AB} = \sqrt{3}$.

We can get $\overline{AD} = \sqrt{3}$ by the same reasoning.

Teacher: We are ready to solve the perimeter of $AEFC$ now.

The perimeter of $AEFC$

$= \overline{AE} + \overline{EF} + \overline{FC} + \overline{CA}$

$= \overline{AE} + \overline{EB} + \overline{DC} + \overline{CA}$ (because $\overline{EF} \cong \overline{EB}$ and $\overline{FC} \cong \overline{DC}$)

$= \overline{AB} + \overline{AD}$

$= 2\sqrt{3}$

老師： 這是 2017 年國中會考題目。首先，請同學念一下題目。

學生： $\triangle ABC$、$\triangle ADE$ 中，C、E 兩點分別在 $\overline{AD}$、$\overline{AB}$ 上，且 $\overline{BC}$ 與 $\overline{DE}$ 相交於 F 點。若 $\angle A = 90^\circ$，$\angle B = \angle D = 30^\circ$，$\overline{AC} = \overline{AE} = 1$，則四邊形 $AEFC$ 的周長為何？

老師： 謝謝。現在來求四邊形 AEFC 的周長。已知 $\angle A = 90^\circ$ 和 $\angle B = 30^\circ$，請求出 $\angle ACB$。

學生： 我會。三角形內角和等於 180，所以 $\angle ACB = 180^\circ - 90^\circ - 30^\circ = 60^\circ$。

老師： 正確。現在可以用同樣的方法來求出 $\angle AED$ 的度數。

學生： 我知道！$\angle AED$ 也等於 60°，因為我知道 $\triangle AED$ 中的另外兩個內角分別為 90° 和 30°。

老師： 現在來看 $\triangle CDF$。你能告訴我 $\angle CFD$ 的度數嗎？

這裡提示：可以把 $\angle ACB$ 看作 $\triangle CDF$ 的一個外角。

學生： 好，因為外角的度數等於其兩個內對角的度數之和，所以我們知道 $\angle CFD$ 的度數等於 $60^\circ - 30^\circ = 30^\circ$。

老師： $\triangle CDF$ 是什麼三角形？

學生： $\triangle CDF$ 是等腰三角形，因為 $\angle CFD$ 與 $\angle D$ 相等。

老師： 我們還可以得出 $\overline{CF} \cong \overline{CD}$，因為在等腰三角形中，底角的兩邊是相等的。

學生： 看起來我們可以使用相同的方法分析 $\triangle BEF$。我們可以得出結論，即 $\overline{BE}$ 與 $\overline{FE}$ 全等。

老師： $\triangle ABC$ 是什麼三角形？

學生： 它是一個三個內角分別為 30°，60° 和 90° 的直角三角形。

老師： 三個內角分別為 30°，60° 和 90° 的三角形的邊長比是多少？

學生： $1:\sqrt{3}:2$。由於 $\overline{AC} = 1$，所以 $\overline{AB} = \sqrt{3}$。
我們可以通過相同的推理得出 $\overline{AD} = \sqrt{3}$。

老師： 我們現在來算 $AEFC$ 的周長。

$AEFC$ 的周長

$= \overline{AE} + \overline{EF} + \overline{FC} + \overline{CA}$

$= \overline{AE} + \overline{EB} + \overline{DC} + \overline{CA}$ （因為 $\overline{EF} \cong \overline{EB}$、$\overline{FC} \cong \overline{DC}$）

$= \overline{AB} + \overline{AD}$

$= 2\sqrt{3}$

例題二

說明： 本題是用讓學生觀察一個式子在代入不同的整數時，是否會都得到質數，再用反例來跟學生說明，歸納出來的結果不一定為真。

（英文）a. Substitute each of the integers from 1 to 9 for n in the expression n^2+n+11.

b. Use inductive reasoning to guess what kind of number you will get when you substitute any integer for n in the expression n^2+n+11.

c. Test your guess by substituting 10 and 11 for n.

（中文）a. 將數字 1 到 9 代換到 n^2+n+11 的 n。

b. 觀察最後的結果，請問你有發現他們都是怎樣的整數？

c. 試著再將 10 及 11 代換到 n^2+n+11 的 n 來檢驗你在 b. 所下的結論。

Teacher: Substitute 1 for n in the expression n^2+n+11. What do you get?

Student: $1^2+1+11=13$

Teacher: Good! Continue to substitute 2 to 9 for n in the expression n^2+n+11.

Student: $2^2+2+11=17$, $3^2+3+11=23$, $4^2+4+11=31$, $5^2+5+11=41$, $6^2+6+11=53$, $7^2+7+11=67$, $8^2+8+11=83$, and $9^2+9+11=101$.

Teacher: Do all those numbers have anything in common?

Student: They are all odd numbers. Since $n^2+n=n(n+1)$ is the product of two consecutive numbers, it is even. Eleven added to an even number always becomes odd.

Teacher: Besides them all being odd numbers, what else can you find?

Student: They are all prime numbers, too.

Teacher: Substitute 10 for n in the expression n^2+n+11.

Student: $10^2+10+11=121$. 121 is the square of 11, which is not a prime number. $11^2+11+11=143$. 143 is a multiple of 11, which is not a prime number either.

Teacher: If there is a statement which says, "For any integer n, n^2+n+11 is a prime number," is this statement true or false?

Student: It is false. 10 or 11 can be counterexamples to show it is false.

Teacher: Wonderful! From this question we know that the conclusion from inductive reasoning might not always be true.

老師： 將 1 代入n^2+n+11中的 n，會得到什麼？

學生： $1^2+1+11=13$

老師： 很好！再繼續用 $n=2$ 到 9 代入 n^2+n+11。

學生： $2^2+2+11=17$、$3^2+3+11=23$、$4^2+4+11=31$、$5^2+5+11=41$、$6^2+6+11=53$、$7^2+7+11=67$、$8^2+8+11=83$、$9^2+9+11=101$

老師： 這些數字有什麼共同之處嗎？

學生： 它們都是奇數。因為 $n^2+n=n(n+1)$ 是兩個相鄰數的乘積，所以是偶數。再加上 11 會變成奇數。

老師： 除了它們都是奇數外，你們還有發現什麼嗎？

學生： 它們也都是質數。

老師： 那接著再試著將 10 和 11 代換到 n^2+n+11 的 n

學生： $10^2+10+11=121$。121 是 11 的平方，不是質數。

$11^2+11+11=143$。143 是 11 的倍數，也不是質數。

老師： 所以「對於任何整數 n，n^2+n+11是必為質數」這個敘述是真還是偽？

學生： 假的。$n=10$ 或 11 可以當作反例來證明此敘述錯誤。

老師： 很好！從問題中，我們知道歸納法的結論並不一定都是真的。

單元八 三角形的外心、內心與重心
Circumcenter, Incenter, and Centroid of a Triangle

國立新竹科學園區實驗高級中等學校 陳立業老師

■ 前言 Introduction

本節在介紹三角形的三心，包含了外心，重心及內心。三角形的外心到三角形的三頂點等距，內心則是到三角形的三邊等距，而重心則是一個可以跟學生介紹物體質量中心的一個媒介；在認識完這三心的性質之後，除了會找到三角形三心的位置，我們也可以看看生活中有哪些問題是可以用這三心的哪一個心來解決。

■ 詞彙 Vocabulary

單字	中文	單字	中文
circumcenter	外心	perpendicular bisector	中垂線
acute triangle	銳角三角形	circumscribed circle	外接圓
obtuse triangle	鈍角三角形	angle bisector	角平分線
right triangle	直角三角形	inscribed circle	內切圓
incenter	內心	centroid	重心
median	中線	equidistant	等距

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ The________ of a triangle is the intersection of the three _________ of the triangle.

例句(1)：**The** circumcenter **of a triangle is the intersection of the three** perpendicular bisectors of the three sides **of the triangle.**

外心是三角形三邊中垂線的交點。

例句(2)：**The** incenter **of a triangle is the intersection of the** three angle bisectors of the interior angles **of the triangle**.

內心是三角形三內角角平分線的交點。

例句(3)：**The** centroid **of a triangle is the intersection of** the three medians **of the triangle**.

重心是三角形三中線的交點。

❷ The circumcenter of ____________ falls___________.

例句(1)：**The circumcenter of** an acute triangle **falls** in the interior of the triangle.

銳角三角形的外心在三角形內部。

例句(2)：**The circumcenter of** an obtuse triangle **falls** outside of the triangle.

鈍角三角形的外心在三角形外部。

例句(3)：**The circumcenter of** a right triangle **falls** on the midpoint of its hypotenuse.

直角三角形的外心在三角形的斜邊中點上。

❸ The ________ of a triangle is equidistant from ____________.

例句(1)：**The** circumcenter **of a triangle is equidistant from** its three vertices.

外心到三角形的三頂點等距。

例句(2)：**The** incenter **of a triangle is equidistant from** its three sides.

內心到三角形的三邊等距。

❹ **The distance from ________ of a triangle to the centroid is two-thirds of the distance from ___________ to __________.**

例句：**The distance from** a vertex **of a triangle to the centroid is two-thirds of the distance from** that vertex **to** its opposite side.

重心到一頂點的距離等於過該頂點之中線長的 $\frac{2}{3}$。

❺ **If _______ is the centroid of ________, then ________ = ________ =_________.**

例句：**If** G **is the centroid of** $\triangle ABC$**, then** area of $\triangle ABG$ = area of $\triangle BCG$ = area of $\triangle ACG$.

G 為三角形 ABC 的重心，則面積$\triangle ABG$ = 面積$\triangle BCG$ = 面積$\triangle ACG$。

■ 問題講解 Explanation of Problems

說明

Where can we build a library that is equidistant from three different schools? The circumcenter is the point that can help us solve this problem.

如何建立一個圖書館與三學校等距呢？這個問題就可以用三角形的外心來解答了。

Where can we build a playground in a triangular park, if the playground needs to be equidistant from the three sides of this park? The incenter plays an important role in choosing the location of the playground.

如何建立一個遊戲場與一個三角形公園的三邊等距呢？這個問題就換由找三角形的內心來解答了。

Have you ever seen buskers (street artists) performing on the street? One of these artists might use a long stick to balance a plate in the air for a long time without it falling down. You can imagine that they must be good at finding the centroid of the objects. Imagine that there is a triangular piece of cardboard on your index finger. How could you balance it without tilting to one side? Before making it happen, you must first find this triangle's centroid.

有些街頭藝人擅於將盤子用長長的桿子撐在空中，也許他們精於找到這些物品重心的所在吧？我們也來試試如何將一個三角形的紙板撐在空中許久而不來它掉落，首先要先找到這個三角形的重心。

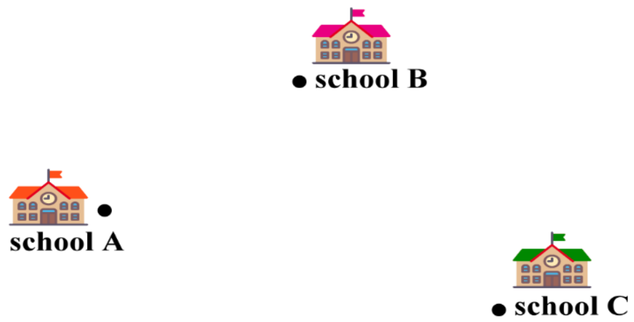
❧ 運算問題講解 ☙

例題一

說明：利用三角形的外心性質來找滿足題目條件的地點。

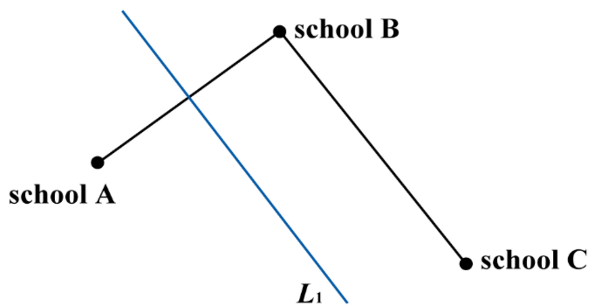
（英文）If we want to build a library that is equidistant from three different schools A, B, and C, where should this library be located?

（中文）如果我們想要建一座圖書館離三間學校(A, B 和 C)一樣近，我們將圖書館設在哪裡呢？



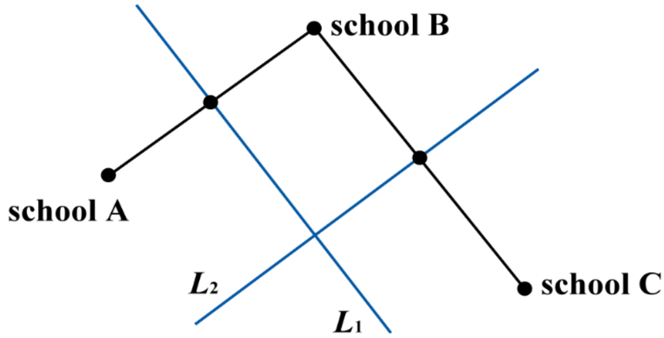
Teacher: Let's connect $\overline{AB}$ and $\overline{BC}$. Construct the perpendicular bisector of $\overline{AB}$ and label it as L_1.

Student: I am done drawing L_1.

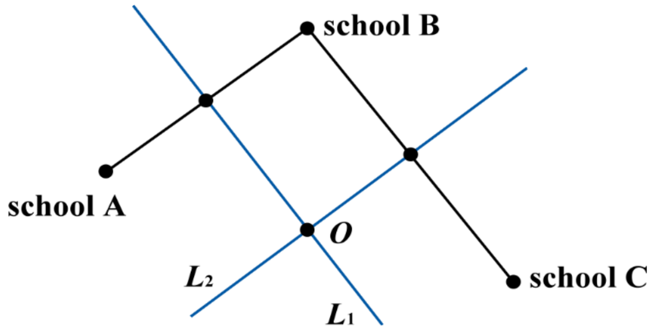


Teacher: Well done! Construct the perpendicular bisector of $\overline{BC}$ and label it as L_2

Student: I am done drawing L_2



Teacher: Label the intersection between L_1 and L_2 as O. Since O is on the perpendicular bisector of $\overline{AB}$, so $\overline{OA} = \overline{OB}$. And O is on the perpendicular bisector of $\overline{BC}$, so $\overline{OB} = \overline{OC}$.



Student: Therefore, I can conclude that $\overline{OA} = \overline{OB} = \overline{OC}$. O is the location of this library.

老師： 請連接 $\overline{AB}$ 和$\overline{BC}$。畫出 $\overline{AB}$ 的垂直平分線，標記為L_1。

學生： 畫好 L_1了。

老師： 做得好！畫出 $\overline{BC}$ 的垂直平分線，標記為 L_2。

學生： 完成。

老師： 將L_1和L_2的交點標記為 O。由於 O 在 $\overline{AB}$ 的垂直平分線上，所以 $\overline{OA} = \overline{OB}$。同理，$O$ 在$\overline{OB}$的垂直平分線上，所以 $\overline{OB} = \overline{OC}$。

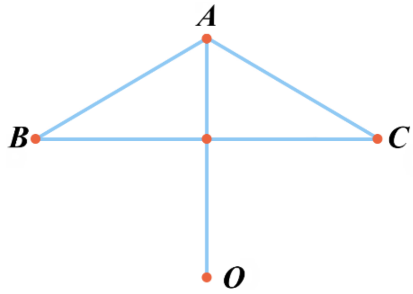
學生： 因此，我可以得出結論：$\overline{OA} = \overline{OB} = \overline{OC}$。$O$ 是這座圖書館的位置。

例題二

說明：利用三角形的外心性質來找外接圓的半徑長度。

（英文）O is the circumcenter of an obtuse $\triangle ABC$. If $\overline{AB} = \overline{AC} = 13$, and $\overline{BC} = 24$, find the radius of the circumscribed circle of $\triangle ABC$.

（中文）O 是$\triangle ABC$ 是三角形外接圓的圓心，如果$\overline{AB} = \overline{AC} = 13$，且$\overline{BC} = 24$，試求出三角形外接圓的半徑長。



Teacher: If we know O is the circumcenter, what can we deduce from that?

Student: We can say $\overline{OB} = \overline{OC}$ because the circumcenter is equal distance from the three vertices of a triangle.

Teacher: The question also mentions $\overline{AB} = \overline{AC}$. What kind of quadrilateral is $OABC$?

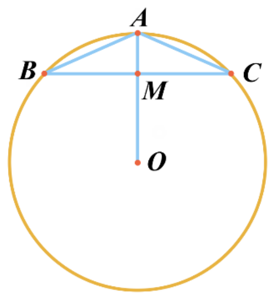
Student: Quadrilateral $OABC$ is a kite.

Teacher: We know that the diagonals of a kite are perpendicular to each other. So $\Delta ABM \cong \Delta ACM\,(HL)$, and $\overline{BM} \cong \overline{CM}$ (Corresponding parts of congruent triangles are congruent.)

That makes $\overline{OA}$ a perpendicular bisector of $\overline{BC}$.

Assume $\overline{OA}$ intersects $\overline{BC}$ at M.

Because $\overline{AB} = 13$, $\overline{BM} = \frac{1}{2}\overline{BC} = 12$, can you find $\overline{AM}$?



Student: Yes, I can use the Pythagorean theorem to find $\overline{AM} = \sqrt{13^2 - 12^2} = 5$

Teacher: Great. Now, let $\overline{OA} = \overline{OB} = \overline{OC} = x$, express $\overline{OM}$ in terms of x

Student: $\overline{\text{OM}} = x - 5$.

Teacher: Awesome! Now focus on $\triangle BOM$. It is also a right triangle with the sides of x, 12, and $x - 5$. Use the Pythagorean theorem to solve for x again.

Student: $x^2 = (x-5)^2 + 12^2$,

$x = \frac{169}{10}$

(Use the Pythagorean theorem to write out the equation. Expand the right side, cancel out x^2 from both sides and divide both sides by 10.)

Teacher: Great! You have found the radius of the circumscribed circle.

老師： 如果我們知道 O 是外心，那我們可以推論出什麼別的條件呢？

學生： 我們可以說 $\overline{OB} = \overline{OC}$，因為外心到三角形的三個頂點的距離相等。

老師： 問題還提到 $\overline{AB} = \overline{AC}$。那麼 $OABC$ 是什麼樣的四邊形呢？

學生： $OABC$ 是個菱形。

老師： 我們知道菱形的對角線互相垂直，我們可以得到 $\triangle ABM \cong \triangle ACM(HL)$，且 $\overline{BM} \cong \overline{CM}$ (兩全等三角形的對應邊全等)，可得到 $\overline{OA}$ 為 $\overline{BC}$ 的垂直平分線。

假設 $\overline{OA}$與 $\overline{BC}$ 相交於 M 點。

因為 $\overline{AB}$=13，$\overline{BM} = \frac{1}{2}\overline{BC} = 12$，試求出 $\overline{AM}$？

學生： 有，用勾股定理可以求得 $\overline{AM} = \sqrt{13^2 - 12^2} = 5$。

老師： 很好。現在，令 $\overline{OA} = \overline{OB} = \overline{OC} = x,$，用 x 表示 $\overline{OM}$ 的長度。

學生： $\overline{\mathrm{OM}} = x - 5$。

老師： 讚啦！現在看三角形 $\triangle BOM$。它也是一個直角三角形，邊長為 x、12、和 $x - 5$。再次使用勾股定理求解。

學生： $x^2 = (x-5)^2 + 12^2,$

$x^2 = x^2 - 10x + 25 + 144$

$10x = 169$

$x = \frac{169}{10}$

用勾股定理列出方程式。展開右側，從兩側消去 x^2，然後將兩側除以 10。

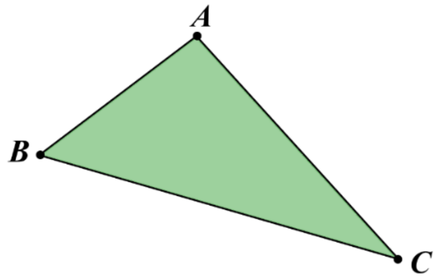
老師： 太強！外接圓的半徑找到了。

例題三

說明：利用三角形的內心性質來找滿足題目條件的地點。

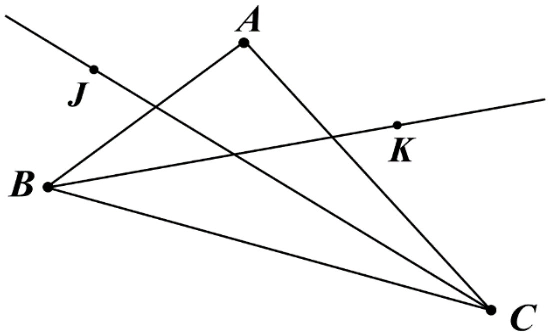
（英文）If we want to build a water fountain that is equidistant from the three sides of a triangular park ABC, where should the water fountain be located?

（中文）如果我們想要設置一個飲水機離三角形公園的三邊一樣近，我們該將飲水機設在哪裡呢？



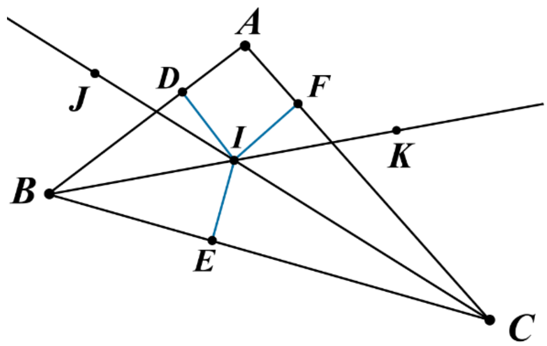
Teacher: Draw $\overrightarrow{BK}$ and $\overrightarrow{CJ}$ as the angle bisectors of $\angle B$ and $\angle C$.

Student: I am done.



Teacher: Label the intersection as I.

Draw $\overline{ID} \perp \overline{AB}$, $\overline{IE} \perp \overline{BC}$, and $\overline{IF} \perp \overline{AC}$.



Teacher: Since I is on the angle bisector of $\angle B$, therefore, $\overline{ID} = \overline{IE}$. And I is on the angle bisector of $\angle C$, therefore, $\overline{IE} = \overline{IF}$.

Student: I get it now. So, $\overline{ID} = \overline{IE} = \overline{IF}$ and I is the location of the fountain.

老師：畫出 $\angle B$ 和 $\angle C$ 的角平分線，分別為 $\overrightarrow{BK}$ 和 $\overrightarrow{CJ}$。

學生：好了。

老師：標出交點 I。

畫出 $\overline{ID}$、$\overline{IE}$、$\overline{IF}$.分別和 $\overline{AB}$、$\overline{BC}$、$\overline{AC}$ 垂直

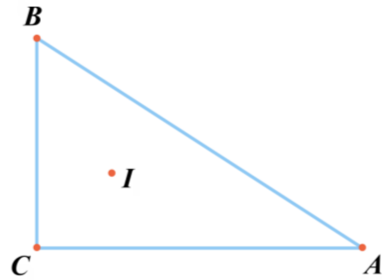
老師：因為點 I 在 $\angle B$ 的角平分線上，所以 $\overline{ID}=\overline{IE}$。並且點 I 也在 $\angle C$ 的角平分線上，所以 $\overline{IE}=\overline{IF}$。

學生：了解，所以 $\overline{ID}=\overline{IE}=\overline{IF}$，點 I 就是飲水機的位置。

例題四

說明：利用三角形的內心性質來證直角三角形的兩股和等於斜邊長加上兩倍內切圓的半徑長。

（英文）I is the incenter of a right triangle $\triangle ABC$ with $\angle C=90^\circ$. Let r be the radius of the inscribed circle of $\triangle ABC$, and show that $\overline{AC}+\overline{BC}=\overline{AB}+2r$.

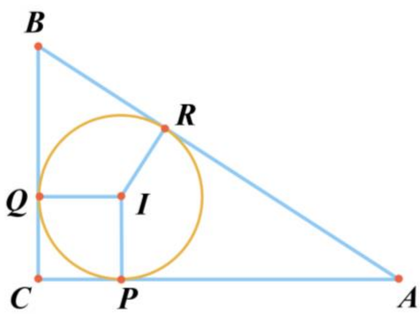


（中文）I 為直角三角形 $\triangle ABC$ 內接圓的圓心，且 $\angle C=90^\circ$。若 r 為角三角形 $\triangle ABC$ 內接圓的半徑，求證 $\overline{AC}+\overline{BC}=\overline{AB}+2r$。

Teacher: Draw an inscribed circle of $\triangle ABC$ with center I. Let P, Q and R be the points of tangency and assume the radius of the inscribed circle equal to r.

Connect $\overline{IR}$, $\overline{IP}$and $\overline{IQ}$.

What kind of quadrilateral is $APIR$?



Student: It looks like a kite to me.

Teacher: Why is it a kite?

Student: First, $\overline{IP} = \overline{IR}$ because they are the radius of the same circle. Second, $\overline{AP} = \overline{AR}$ because tangents to a circle from a point are congruent. By using the same reasoning, we can also conclude that quadrilateral $BQIR$ is also a kite with $\overline{BQ} = \overline{BR}$.

Teacher: Wonderful! Let's figure out why $\overline{AC} + \overline{BC} = \overline{AB} + 2r$.

$\overline{AC} + \overline{BC}$
$= \overline{AP} + \overline{PC} + \overline{BQ} + \overline{QC}$
$= \overline{AR} + r + \overline{BR} + r$
$= \overline{AB} + 2r$

(Substitute AC with AP+PC and BC with BQ+QC, using the segment addition property.)

(Substitute AP with AR and replace PC by r. Substitute BQ by BR and replace QC by r.)

(Combine AR+BR into AB and we complete our proof.)

老師： 畫一個以 I 為圓心的 $\triangle ABC$ 內切圓。設該內切圓的半徑為 r，其切點分別為 P、Q 和 R。

然後連接 $\overline{IR}$、$\overline{IP}$ 和 $\overline{IQ}$。$APIR$ 是什麼四邊形？

學生： 看起來像是菱形。

老師： 為什麼是菱形呢？

學生： 首先，$\overline{IP} = \overline{IR}$，因為都是同一個圓的半徑。其次 $\overline{AP} = \overline{AR}$，因為從點 A 到圓的切線長度相等。同理，還可以得出四邊形 $BQIR$ 也是菱形，其中 $\overline{BQ} = \overline{BR}$。

老師： 很好！那來看看為什麼 $\overline{AC} + \overline{BC} = \overline{AB} + 2r$

$\overline{AC} + \overline{BC}$
$= \overline{AP} + \overline{PC} + \overline{BQ} + \overline{QC}$
$= \overline{AR} + r + \overline{BR} + r$
$= \overline{AB} + 2r$

（將 $\overline{AC}$ 換為 $\overline{AP} + \overline{PC}$，$\overline{BC}$ 換為 $\overline{BQ} + \overline{QC}$。）

（將 $\overline{AP}$ 換為 $\overline{AR}$，並將 $\overline{PC}$ 換為 r。$\overline{BQ}$ 換為 $\overline{BR}$，$\overline{QC}$ 換為 r。）

（$\overline{AR}$ 及 $\overline{BR}$ 的和為 $\overline{AB}$，由此得證。）

應用問題 / 會考素養題

例題一

（英文）I is the incenter of $\triangle ABC$. There is a line passing through I, intersecting $\overline{AB}$ at D, and intersecting $\overline{AC}$ at E. If $\overline{AD} = \overline{DE} = 5$, $\overline{AE} = 6$, find the distance from I to $\overline{BC}$.

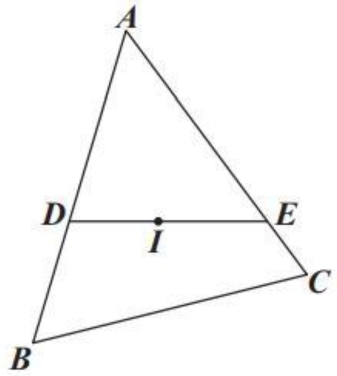
（中文）如圖（十二），I 為 $\triangle ABC$ 的內心，有一直線通過 I 點且分別與相交於 D 點、E 點。若 $\overline{AD} = \overline{DE} = 5$，$\overline{AE} = 6$，則 I 點到 $\overline{BC}$ 的距離為何？

(A) $\frac{24}{11}$

(B) $\frac{30}{11}$

(C) 2

(D) 3

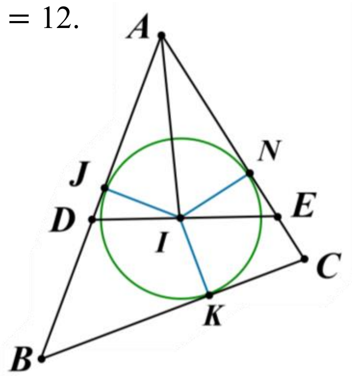


圖（十二）　　（110 年國中會考 26）

Teacher: Connect $\overline{AI}$. Let's draw $\overline{IJ} \perp \overline{BC}$, $\overline{IK} \perp \overline{AB}$, and $\overline{IN} \perp \overline{AC}$. From the property of incenter, we know $\overline{IJ} = \overline{IK} = \overline{IN}$. Assume $\overline{IJ} = \overline{IK} = \overline{IN} = r$.

Next, let's find the area of $\triangle ADE$. $\triangle ADE$ is an isosceles triangle with sides 5, 5, and 6. We draw an altitude $\overline{DP}$ from D, then $\overline{DP}$ will bisect $\overline{AE}$. Use the Pythagorean theorem to find $\overline{DP}$.

Student: $\overline{DP} = \sqrt{5^2 - 3^2} = 4$, so the area of $\triangle ADE = \frac{1}{2} \times 6 \times 4 = 12$.



Teacher: We can also think of the area of $\triangle ADE$ as the sum of the area of $\triangle ADI$ and the area of $\triangle AEI$.

What is the area of $\triangle ADI$ in terms of r?

Student: The area of $\triangle ADI = \frac{r\times 5}{2}$

Teacher: What is the area of $\triangle AEI$ in terms of r?

Student: The area of $\triangle AEI = \frac{r\times 6}{2} = 3r$

Teacher: Therefore,

$$\frac{5r}{2} + 3r = 12$$

$$\frac{11r}{2} = 12$$

What is r?

Student: $r = \frac{24}{11}$, the answer is (A).

Teacher: Good job!

老師：連接$\overline{AI}$。畫 $\overline{IK}$ ，$\overline{IJ}$，和 $\overline{IN}$ 分別垂直 $\overline{BC}$, $\overline{AB}$ 和 $\overline{AC}$。
根據內心性質，知道 $\overline{IJ} = \overline{IK} = \overline{IN}$。設$\overline{IJ} = \overline{IK} = \overline{IN} = r$。
接下來我們求 $\triangle ADE$ 面積。$\triangle ADE$ 是等腰三角形，其邊長為 5.5 和 6。我們從 D 畫一條高 $\overline{DP}$，然後將平分$\overline{AE}$，可使用勾股定理找到 $\overline{DP}$。

學生：$\overline{DP} = \sqrt{5^2 - 3^2} = 4$，所以 $\triangle ADE = \frac{1}{2} \times 6 \times 4 = 12$。

老師：我們也可以把$\triangle ADE$ 的面積看作是$\triangle ADI$ 的面積和$\triangle AEI$ 的面積之和。
請以 r 表示 $\triangle ADI$ 的面積。

學生：$\triangle ADI = \frac{r\times 5}{2}$。

老師：以 r 表示 $\triangle AEI$ 的面積。

學生：$\triangle AEI = \frac{r\times 6}{2} = 3r$

老師：所以我們可以得到：$\frac{5r}{2} + 3r = 12$

$$\frac{11r}{2} = 12$$

r 是多少？

學生：$r = \frac{24}{11}$，答案是(A)。

老師：答對了！

例題二

說明：此題是三角形的重心的應用題。

（英文）$ABCD$ is a parallelogram with M the midpoint of $\overline{BC}$. If $\overline{BD}$ intersects $\overline{AC}$ at O and $\overline{AM}$ intersects $\overline{BD}$ at P,

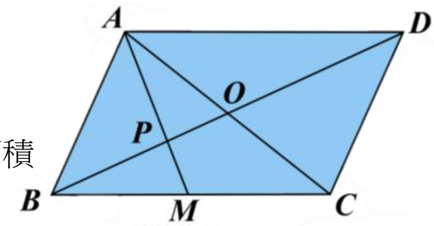
(1) $\overline{BP}=6$, find $\overline{BD}=$ ______

(2) If the area of $\Delta ABP=12$, find the area of the parallelogram $ABCD$.

（中文）如圖，在 ▱$ABCD$ 中，兩對角線 $\overline{AC}$、$\overline{BD}$ 交於 O 點，M 點是 $\overline{BC}$ 的中點，$\overline{AM}$ 與 $\overline{BD}$ 交於 P 點。

(1) 若 $\overline{BP}=6$，求 $\overline{BD}$ 的長度

(2) 若 $\triangle ABP$ 的面積為 12，求 ▱$ABCD$ 的面積



Teacher: We know that diagonals bisect each other in a parallelogram, so $\overline{AO}=\overline{CO}$. Is P a circumcenter, incenter, or centroid of $\triangle ABC$?

Student: Because $\overline{AO}=\overline{CO}$, that makes $\overline{BO}$ a median of $\triangle ABC$. From the question, M is the midpoint of $\overline{BC}$ which makes $\overline{AM}$ the median of $\triangle ABC$. Hence, P is the intersection of medians and P is the centroid of $\triangle ABC$.

Teacher: Correct! From the property of centroid, how many times as long as $\overline{BO}$ is $\overline{BP}$?

Student: $\overline{BP}=\frac{2}{3}\overline{BO}$. ($\overline{BP}$ is two-thirds as long as $\overline{BO}$.)

Teacher: Great! We also know $\overline{BO}=\frac{1}{2}\overline{BD}$. Then $\overline{BP}=\frac{2}{3}\overline{BO}=\frac{2}{3}(\frac{1}{2}\overline{BD})=\frac{1}{3}\overline{BD}$.

Student: I get it now. $\overline{BD}=3\overline{BP}=18$.

Teacher: From the property of centroid, how many times as large as area of $\triangle ABC$ of is area of $\triangle ABP$?

Student: The area of $\triangle ABC$ 3= the area of $\triangle ABP$ (the area of $\triangle ABC$ is three times as large as the area of $\triangle ABP$.So the area of $\triangle ABC$ is 36. The parallelogram is twice as big as $\triangle ABC$, so the area of the parallelogram is 72.

Teacher: Well done!

老師： 已知平行四邊形的的對角線互相平分，因此 $\overline{AO}=\overline{CO}$。那 P 是△ABC的外心、

內心，還是的重心？

學生： 因為 $\overline{AO}=\overline{CO}$，使 $\overline{BO}$ 成為 $\triangle ABC$ 的中線。題目說 M 是 $\overline{BC}$ 的中點，使得 $\overline{AM}$ 成為 $\triangle ABC$ 的中線。因此，P 是三中線的交點，也是 $\triangle ABC$ 的重心。

老師： 沒錯，根據重心性質，$\overline{BP}$ 的長度是 $\overline{BO}$ 的幾倍？

學生： $\overline{BP}=\frac{2}{3}\overline{BO}$。

老師： 很棒！那我們知道 $\overline{BO}=\frac{1}{2}\overline{BD}$，所以 $\overline{BP}=\frac{2}{3}\overline{BO}=\frac{2}{3}(\frac{1}{2}\overline{BD})=\frac{1}{3}\overline{BD}$。

學生： 明白了，那就是 $\overline{BD}=3\overline{BP}=18$.

老師： 根據重心的性質，$\triangle ABC$ 的面積是 $\triangle ABP$ 面積的幾倍？

學生： $\triangle ABC$ 的面積是 $\triangle ABP$ 面積的三倍。因此，$\triangle ABC$ 的面積為 36。平行四邊形的面積是 $\triangle ABC$ 的兩倍，因此平行四邊形的面積為 72。

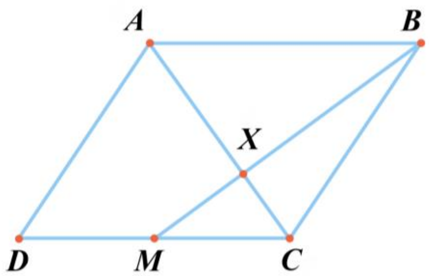
老師： 讚啦！

例題三

說明： 此題是三角形的重心的應用題。

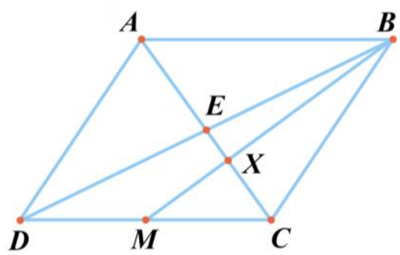
（英文）$ABCD$ is a parallelogram with M the midpoint of $\overline{CD}$. If $\overline{BM}$ intersects $\overline{AC}$ at X, prove that $\overline{CX}=\frac{1}{3}\overline{AC}$.

(Hint: draw $\overline{BD}$.)



（中文）在平行四邊形 $ABCD$ 中，M 是 $\overline{CD}$ 中點。如果 $\overline{BM}$ 與 $\overline{AC}$ 交於 X，求證：$\overline{CX}=\frac{1}{3}\overline{AC}$。（提示：連接 $\overline{BD}$。）

Teacher: Let's follow the hint by drawing $\overline{BD}$.



Student: I have drawn $\overline{BD}$.

Teacher: Assume that $\overline{BD}$ and $\overline{AC}$ intersects at point E

Student: E is labeled.

Teacher: What conclusion can you make for $\overline{BE}$ and $\overline{DE}$?

Student: $\overline{BE}=\overline{DE}$, because in a parallelogram, the diagonals bisect each other.

Teacher: If $\overline{BE}=\overline{DE}$, then $\overline{CE}$ is a median of $\triangle BCD$. $\overline{BM}$ is also a median of $\triangle BCD$ since M is the midpoint of $\overline{CD}$. What can we say about point X?

Student: X is the centroid of $\triangle BCD$ because X is the intersection of two medians of a triangle.

Teacher: Great! What is the ratio of $\overline{CX}$ to $\overline{XE}$?

Student: $\overline{CX}$ to $\overline{XE}$ is 2 to 1. That makes $\overline{CX}$ equals two thirds of $\overline{CE}$. Write $\overline{CX}=\frac{2}{3}\overline{CE}$.

We also know $\overline{CE}$ is equal to half of $\overline{AC}$. Substituting $\overline{CE}$ with $\frac{1}{2}\overline{AC}$, we get

$$
\begin{aligned}
\overline{CX}&=\frac{2}{3}\overline{CE}\\
&=\frac{2}{3}\times\frac{1}{2}\overline{AC}\\
&=\frac{1}{3}\overline{AC}
\end{aligned}
$$

Teacher: Fantastic!

老師：先照題目畫出 $\overline{BD}$.

學生：畫好 $\overline{BD}$ 了。

老師：假設 $\overline{BD}$ 和 $\overline{AC}$ 交於 E 點

學生：標好 E 點了。

老師：看得出來$\overline{BE}$ 和 $\overline{DE}$ 的關係嗎？

學生：因為在平行四邊形中，對角線相互平分，所以 $\overline{BE}=\overline{DE}$

老師：如果 $\overline{BE}=\overline{DE}$，$\overline{CE}$ 就是 $\triangle BCD$ 的中線。M 點是 $\overline{CD}$ 的中點，所以 $\overline{BM}$ 也是 $\triangle BCD$ 的中線。關於點 X 能得出什麼結論呢？

學生：因為 X 是三角形 $\triangle BCD$ 兩條中線的交點，所以 X 點是 $\triangle BCD$ 的重心。

老師：$\overline{CX}$ 跟 $\overline{XE}$的比例是多少？

學生：$\overline{CX}$ 跟 $\overline{XE}$ 的比例為 2:1，也就是 $\overline{CX}=\overline{CE}$ 的三分之二。

寫成數學式 $\overline{CX}=\frac{2}{3}\overline{CE}$。我們也知道 $\overline{CE}$ 等於 $\overline{AC}$ 的一半，將 $\overline{CE}$ 用 $\frac{1}{2}$ $\overline{AC}$ 替換，得到

$$\begin{aligned}\overline{CX}&=\frac{2}{3}\overline{CE}\\&=\frac{2}{3}\times\frac{1}{2}\overline{AC}\\&=\frac{1}{3}\overline{AC}\end{aligned}$$

老師： 非常好！

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查詢學科詞彙 https://terms.naer.edu.tw/search/	
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Twinkl	
國外教學及影音資源，除了數學領域還有其他科目，多為小學及學齡前內容 https://www.twinkl.com.tw/	

Khan Academy	
可汗學院，有分年級數學教學影片及問題的討論。 https://www.khanacademy.org/	
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MATH is FUN	
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國高中數學雙語教學相關教材 https://math.ntnu.edu.tw/~jschen/index.php?menu=Teaching_Worksheets
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科學專業英文相關教材，除了數學領域，還有其他領域。 https://sites.google.com/view/ntseccompetition/%E5%B0%88%E6%A5%AD%E8%8B%B1%E6%96%87%E5%AD%B8%E7%BF%92%E8%B3%87%E6%BA%90/%E7%9B%B8%E9%97%9C%E6%95%99%E6%9D%90?authuser=0

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