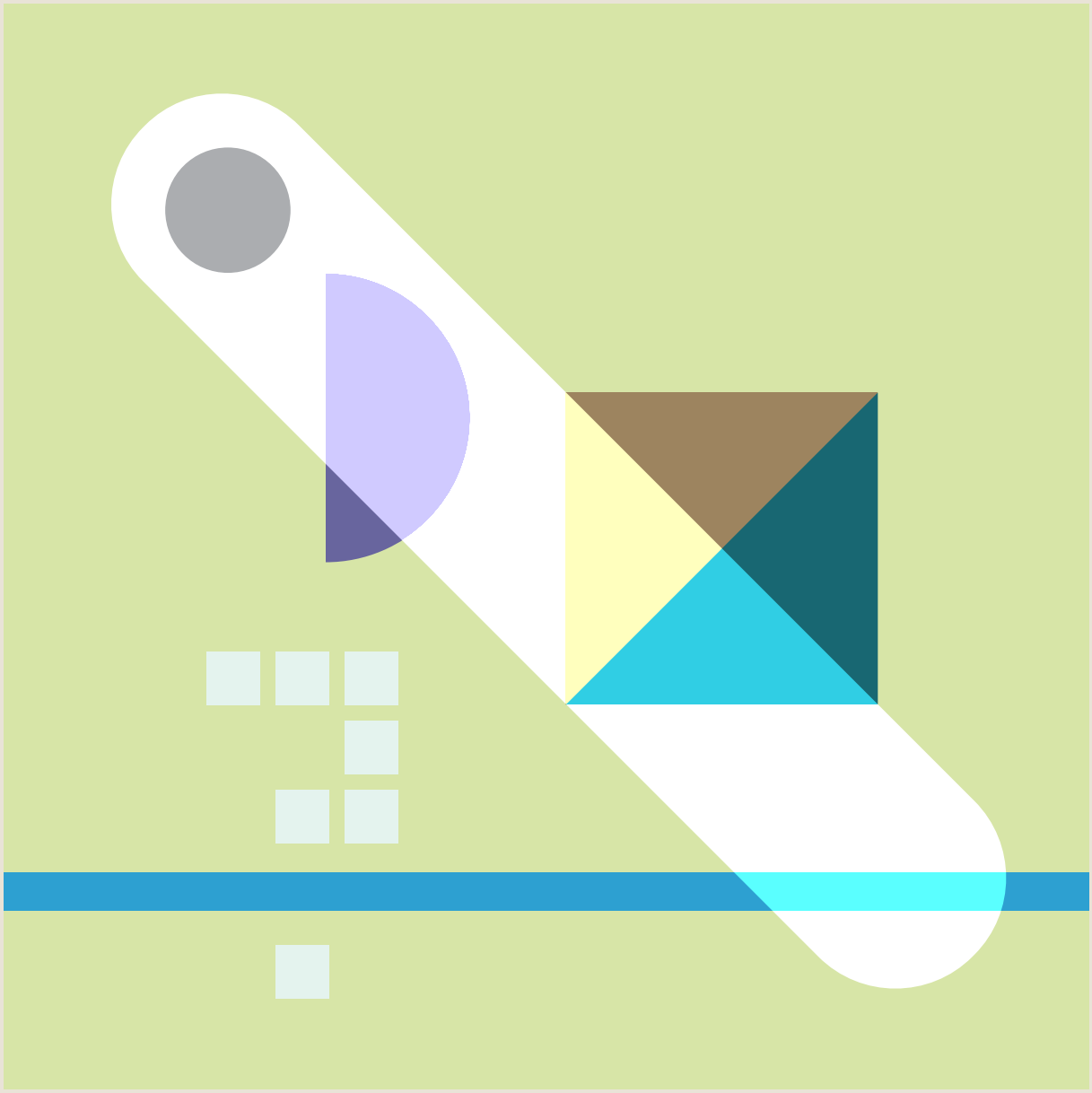
高中數學領域

雙語教學資源手冊 英語授課用語

A Reference Handbook for Senior High School Bilingual Teachers in the Domain of Mathematics: Instructional Language in English

〔 高一下學期 〕



臺師大雙語教學研究中心 發行

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單元一 數列
Sequence

國立新竹科學園區實驗高級中等學校　周慧蓮老師

■ 前言 Introduction

此單元延伸國中有限數列的概念，主要內容偏重無限數列與遞迴數列。進一步教導學生數學歸納法的原理。利用數學歸納法證明的特殊級數的求和公式，可以在下一個章節用來求和。教師在介紹英文數學詞彙時，可以有效地利用中英文詞彙的對應關係來幫助學生加深記憶。例如，介紹「公差」這個概念時，可以指出其英文為 “common difference”，並強調英文縮寫 *d* 與中文中「差」首字的對應關係。介紹「公比」這個概念時，可以指出其英文為 common ratio，並強調英文縮寫 “*r*”與中文中「比」首字的對應關係。將數學詞彙與其英文表示法連結的教學策略，有助於學生更全面地理解數學概念，並在不同語境中靈活應用。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
sequence	**數列**	**term**	**項**
finite	有限的	**explicit**	**明確的**
infinite	無限的	**general**	**一般的**
consecutive	連續的	**recursive**	**遞迴的**
arithmetic	**算術的**	**geometric**	**幾何的**

common difference	公差	**common ratio**	公比
induction	**歸納**	domino	骨牌
assume	**假設**	statement	陳述

The Principle of Mathematical Induction
Let P_n be a statement involving the positive integers n. If 1. P_1 is true 2. P_k being true implies that P_{k+1} is true. Then P_n must be true for all positive integers n.

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ Write the first ___ terms of _____.

例句(1)：**Write the first** four **terms of** $a_n = 2n - 3$.
寫出數列$< 2n - 3 >$的前四項。

例句(2)：Given the general terms of a sequence are $a_n = 2^n - 1$, **write the first** five **terms of** this sequence.
若數列 a_n 的一般項為 $a_n = 2^n - 1$，請寫出此數列的前五項。

❷ A(n) _____ rule for a sequence gives _____.

例句(1)：**A** recursive **rule for a sequence gives** the beginning term(s) of the sequence and then an equation relating the nth term (a_n) to one or more preceding terms.
遞迴定義式提供數列的初始項和一個連結第 n 項和前（幾）項的方程式。

例句(2)：**An** explicit **rule for a sequence gives** the nth term (a_n) as a function of the term's position number n in the sequence.
數列一般式是將數列的第 n 項視為數列位置值(n)的函數。

❸ Write the _____ rule for the sequence.

例句(1)：**Write the** recursive **rule for this arithmetic sequence.**
將這個等差數列寫成遞迴定義式。

例句(2)：**Write the** general **rule for the sequence.**
寫出這個數列的一般式。

❹ Recognize/Describe the (number) pattern and _____.

例句(1)：**Recognize the number pattern and** find a rule for the *n*th term.
識別數字規律，並寫出第 *n* 項的式子。

例句(2)：**Describe the pattern and** write the next term.
描述規律，並寫出下一項。

❺ Substitute *A* for *B*.

例句(1)：Double-check the solution by **substituting** the solution **for** the variable in the equation.
將解答代入方程式裡的變項進行驗算。

例句(2)：To find the first term, **substitute** 1 **for** *n*.
將 1 代入 *n* 可找到第一項。

❻ This sequence is defined recursively/explicitly by _____.

例句(1)：**The Fibonacci sequence can be defined recursively by** $\begin{cases} a_1 = 1 \\ a_2 = 1 \\ a_n = a_{n\text{-}1} + a_{n\text{-}2} \end{cases}$ for all positive integers $n \geq 3$.
費氏數列可以遞迴定義成首項和第二項皆等於 1。自第三項起，第 *n* 項等於前兩項的和。

例句(2)：**This sequence is defined explicitly by** $a_n = 2 + 3n$.
這個數列被明確的定義為第 *n* 項是 2 加 $3n$。

❼ Prove that _____ for all positive integers $n \geq 1$.

例句(1)：**Prove that** $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$ is true **for all positive integers**.

證明 1 加到 n 的正整數和等於 $\frac{n(n+1)}{2}$ 對任意正整數 n 恆成立。

例句(2)：**Prove that** $4^n - 1$ is divisible by 3 **for all positive integers** n.

證明對於所有的正整數 n，$4^n - 1$ 可以被 3 整除。

❽ Assume that _____ is true.

例句(1)：**Assume that** the formula **is true** for $n = k$.

假設該公式在 $n = k$ 時成立。

例句(2)：**Assume that** $3n \leq 3^n$ **is true** for $n = k$.

假設 $3n \leq 3^n$ 在 $n = k$ 時為真。

❾ The next term on the left-hand side is _____.

例句(1)：The next term on the left-hand side is $k + 1$.

等式左邊的下一項是 $k + 1$。

例句(2)：The next term on the left-hand side is $\frac{(k+1)\cdot][(k+1)+1]\cdot[2(k+1)+1]}{6}$.

等式左邊的下一項是 $\frac{(k+1)\cdot][(k+1)+1]\cdot[2(k+1)+1]}{6}$。

（註：此句型用於解說數學歸納法公式證明中的下一項）

■ 問題講解 Explanation of Problems

☙ 說明 ❧

In this section, we will focus mostly on arithmetic and geometric sequences. We will learn the general and recursive rules of these sequences to solve real-life problems. In addition to the sequences, we learn to apply mathematical induction to prove any theorems involving positive integers. This process can be visualized as a series of dominos toppling over, where each domino represents a case, and the induction hypothesis is used to knock over the next domino in the sequence until all cases are proven true. Any kth domino will knock down the $(k+1)$th domino as long as the first domino is knocked down. Eventually, the infinite sequence of the dominos will be knocked down.

❧ 運算問題的講解 ☙

例題一

說明：此題為數列的基本題，主要讓學生熟悉如何應用數列的一般項找到第 n 項的值。

（英文）Write the first five terms of the sequence $< 4^n - 1 >$.

（中文）寫出數列$< 4^n - 1 >$的前五項。

Teacher: $4^n - 1$ represents the general term of the sequence. Who can tell me how to find the first term?

Student: Substitute 1 for n. We will have four minus one, which is 3.

Teacher: Great. We apply the same logic to find the other 4 terms. Plug in the numbers from 2 to 5. What are the results?

Student: They are 15, 63, 255, and 1023.

Teacher: Do you notice the pattern in these numbers?

Student: I know, they are all multiples of 3

Teacher Glad that you were able to find out the pattern. Well done! Studying the pattern of numbers is usually a good start when analyzing a sequence.

老師： $4^n - 1$ 是這個數列的一般項公式。請問如何找出第一項？
學生： 將 n 替換為 1，得到 4−1，也就是 3。
老師： 很好。我們可以套用相同的邏輯來找出其它 4 項。將數字由 2 到 5 代入公式，結果是什麼？
學生： 分別是 15、63、255 和 1023。
老師： 看到規律了嗎？
學生： 我知道，都是 3 的倍數。
老師： 很棒，找出規律！研究數列中規律，就是分析的開始。

例題二

說明：此題為等差的基本運算，除了找出一般項、也需要求前 n 項的和。

（英文）Three is the first term of an arithmetic sequence. Its common difference is 2.
(1) Find the first four terms of the sequence and its general term.
(2) Find the sum of the first 20 terms of the sequence.
（中文）等差數列首項為 3，公差為 2。
(1) 試求出數列前四項和一般項公式。
(2) 試求其前 20 項的和。

Teacher: For an arithmetic sequence, each term is found by adding the common difference to the previous term. According to the description, $a_1 = 3$ and $d = 2$, what are the second, third, and fourth terms?
Student: If we keep adding 2, they will be 5, 7, and 9.
Teacher: Correct.
How about the general term of this sequence?
Student: We can apply the formula $a_n = a_1 + (n-1)d$ to find the general term.
It will be: $a_n = 3 + (n-1) \times 2$.
After simplification it will be: $a_n = 1 + 2n$
Teacher: Very good.
To find the nth term, you have to add the common difference for $n-1$ times. Therefore, we have $(n-1) \times 2$ in the formula.
Then how can we find the sum of the first twenty terms?

Student: I remember that there is a formula similar to finding the trapezoid area.
Multiply the height by the sum of base one and base two. Then dividing it by two will lead to the trapezoid area.
Similarly, multiply the number of terms by the sum of the first and the last term. Then dividing it by two will lead to the partial sum.

Teacher: Correct.
To apply this formula, we have to find the twentieth term first.
What is it?

Student: It is 41.

Teacher: Since we have the values of the first and last terms, what is the partial sum of the first twenty terms?

Student: One plus forty-one is forty-two.
Twenty divided by two is ten.
Forty-two multiplied by ten equals four hundred twenty.

Teacher: You explain the solution clearly. Thank you.

老師： 等差數列的每一項，都是前項加上公差。根據題目敘述，$a_1 = 3$，$d = 2$，那麼這個數列的第二、第三和第四項分別是多少？

學生： 持續加 2，可以得出 5、7 和 9。

老師： 正確。那這個數列的一般項公式怎麼表示？

學生： 我們可以套用公式 $a_n = a_1 + (n-1)d$，求一般項。
$a_n = 3 + (n-1) \times 2$。化簡變成 $a_n = 1 + 2n$。

老師： 非常好。
要找到第 n 項，加 $(n-1)$ 個公差。因此，從公式中可以看到 $(n-1) \times 2$。
那要怎樣找出前 20 項的和呢？

學生： 我記得有個類似於求梯形面積的公式，上底加下底乘以高除以二就可以得到梯形面積。同樣地，將項數乘以首項和末項的和，然後除以二就可以得到和。

老師： 正確。應用這個公式之前，我們必須先找到第二十項的值。

學生： 是 41。

老師： 既然我們已經知道了第一項和最後一項的值，那前 20 項的和是多少？

學生： 1+41=42
20÷2=10
42×10=420

老師： 謝謝這位同學！過程很清楚。

例題三

說明：此題為等比數列的基本運算，藉由已知的特定項找出第一項、公比和一般項。

（英文）Given a geometric sequence $< a_n >$, the second term is 6, and the fifth term is 162.

(1) Find the general rule for this sequence.

(2) Find the recursive rule for this sequence.

（中文）等比數列 $< a_n >$中，已知第二項是 6，第五項是 162。

(1) 求其一般項。

(2) 求其遞迴關係。

Teacher: For a geometric sequence, each term is found by multiplying the common ratio by the previous term. According to the description, $a_2 = 6$ and $a_5 = 162$. Can you figure out the common ratio?

Student: Multiplying the second term by r cubed will lead to the fifth term.
Therefore, we have the equation $6 \times r^3 = 162$
r cubed is 27. *r* should be 3.

Teacher: Thank you for the clear explanation.
Since we know the common ratio, how do we find the first term?

Student: The second term equals the first term multiplied by 3.
Therefore, the first term is 2.

Teacher: With the first term and the common ratio, we can find the general rule for this sequence. What is it?

Student: It will be $a_n = a_1 \times r^{n-1} = 2 \times 3^{n-1}$

Teacher: How do you find the recursive rule?

Student: First of all, we write down $a_1 = 2$.
We then need an equation to relate the *n*th term to its previous term.
By the definition of a geometric sequence, the *n*th term is the $(n-1)$th term multiplied by 3.
$a_n = 3 \times a_{n\text{-}1}$

Teacher: Great.
The recursive rule of a geometric sequence is related to its definition.
Also, whenever you write the recursive rule, don't forget to write the beginning term.

老師：　等比數列的每一項，都由前一項乘以公比得出。根據描述，$a_2 = 6$，$a_5 = 162$，公比推得出來嗎？

學生：　將第二項乘以 r 的三次方得到第五項，所以可以列出 $6 \times r^3 = 162$。$r^3 = 27$，r=3。

老師：　很好，很清楚。既然知道公比了，那麼如何求第一項？

學生：　第二項等於第一項乘以 3，所以第一項是 2。

老師：　有了第一項和公比，就可以找出一般項，怎麼表示呢？

學生：　為 $a_n = a_1 \times r^{n-1} = 2 \times 3^{n-1}$。

老師：　遞迴關係式怎麼求？

學生：　首先寫下 $a_1 = 2$。需要寫出一個方程來連結第 n 項和它的前一項。根據等比數列的定義，第 n 項是第（$n-1$）項乘以 3。答案是 $a_n = 3 \times a_{n-1}$。

老師：　很棒。等比數列的遞迴關係跟定義有關。不過在寫遞迴關係式時，首項也要記得寫出來喔。

例題四

說明：此題為數學歸納法的基本題，教師可以藉由此題的講解教導學生如何用英文撰寫數學歸納法。

（英文）Prove that $1^2+2^2+3^3+...+n^2=\frac{n(n+1)(2n+1)}{6}$ is true for all positive integers n.

（中文）利用數學歸納法證明：$1^2+2^2+3^3+...+n^2=\frac{n(n+1)(2n+1)}{6}$ 對所有正整數 n 均成立。

Teacher:　Mathematical induction is a powerful tool for proving the theorems about positive integers.

Usually, we start the proof by establishing the truth of the theorem for $n = 1$, and then we assume the statement is true when $n = k$. Next, we try to prove that "the truth of the statement for $n = k$" implies "the truth of the statement for $n = k + 1$."

To check whether this statement is true for $n = 1$, we substitute 1 for the variable n. What are the values on each side?

Student:　Both sides have a value of 1.

This statement is true for $n = 1$.

Teacher:　Great.

Who knows the second step of the mathematical induction?

Student: I know. It's pretty easy.

Substitute k for n into the formula.

Replace "prove" with "assume" because the second step is an assumption.

Assume that $1^2 + 2^2 + 3^3 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$

Teacher: Very good.

If you have a problem with mathematical induction, remember to write the first two steps even if you do not know how to do the third step.

The next term on the left-hand side of the given formula is $(k+1)^2$.

Combining the assumption and $(k+1)^2$, we can rewrite the left side as $\frac{k(k+1)(2k+1)}{6} + (k+1)^2$.

Who can simplify this expression?

Student: We can factor out: $(k+1)$.

We then get: $(k+1)(\frac{2k^2+k}{6} + k + 1)$.

It becomes: $\frac{1}{6}(k+1)(2k^2 + 7k + 6)$

Factor the quadratic expression, and you will get: $\frac{1}{6}(k+1)(k+2)(2k+3)$

Teacher: We can think of this expression as: $\frac{1}{6}(k+1)[(k+1)+1][2(k+1)+2]$

It is the same as the right side of the formula when n is replaced by $k+1$.

What is the last step?

Student: Rewrite the statement, indicating that it is true for all positive integers n.

Teacher: Well done!

老師： 有關正整數定理的證明，常用到數學歸納法。

要開始證明的時候，會先確定 $n = 1$ 時敘述成立，並假設當$n = k$ 時敘述也成立。

我們接著證明「敘述對於 $n = k$ 成立，代表$n = k + 1$也成立」。

老師： 為確定敘述在$n = 1$時是否成立，我們將未知數 n 換成 1 並判斷等號兩邊的值是多少？

學生： 兩邊的值都是1。當$n = 1$算式成立。

老師： 好。數學歸納法的第二步是什麼？

學生： 我知道，很簡單。把 n 換為 k 代入公式中，然後把「證明」換為「假設」，因為第二步是一個假設。

假設 $1^2+2^2+3^3+\ldots+k^2=\frac{k(k+1)(2k+1)}{6}$。

老師： 非常好。

不太會數學歸納法的人，如果不知道如何進行第三步，前兩個步驟一定要寫。

下一步，等號左邊的下一項是$(k+1)^2$。

將第二步的假設與$(k+1)^2$ 結合，左邊可以寫成 $\frac{k(k+1)(2k+1)}{6}+(k+1)^2$。

接下來怎麼化簡呢？

學生： 提出公因式 $(k+1)$，變成 $(k+1)(\frac{2k^2+k}{6}+k+1)$。

接著簡化並因式分解一元二次式： $\frac{1}{6}(k+1)(2k^2+7k+6)$

$=\frac{1}{6}(k+1)(k+2)(2k+3)$。

老師： 我們可以將這個算式看成：$\frac{1}{6}(k+1)[(k+1)+1][2(k+1)+2]$。

當 n 被 $(k+1)$取代時，這和等號右邊是相同的。

最後一步是什麼呢？

學生： 重新陳述題目的敘述並指出原式對所有正整數 n 均成立

老師： 很好！

∞ 應用問題 / 指考素養題 ∞

例題一

說明：此題題目相當複雜，包含等比級數和對數的基本運算。教師須引導學生一步一步拆解後進行計算。必要時，需複習對數的基本觀念。

（英文）A geometric sequence has both the first term and the common ratio as 10.

a_1, a_2, a_3, a_4 stand for the first, second, third, and fourth term respectively.

Assume that $b = log_{a_1} a_2 + log_{a_2} a_3 + log_{a_3} a_4$, which of the following is correct?

(1) $2 < b \leq 3$　(2) $3 < b \leq 4$　(3) $4 < b \leq 5$

(4) $5 < b \leq 6$　(5) $6 < b \leq 7$

（中譯）設 a_1, a_2, a_3, a_4 是首項為 10、公比是 10 的等比數列。

令 $b = log_{a_1} a_2 + log_{a_2} a_3 + log_{a_3} a_4$，，試選出正確的選項。

(1) $2 < b \leq 3$　(2) $3 < b \leq 4$　(3) $4 < b \leq 5$

(4) $5 < b \leq 6$　(5) $6 < b \leq 7$

（改編自 111 年指考數甲）

Teacher: This problem seems difficult. Let's list what we know first.

Student: $a_1 = 10, a_2 = 100, a_3 = 1000$, and $a_4 = 10{,}000$

Teacher: We can replace these terms by the corresponding values.

Then we have $b = log_{10} 100 + log_{100} 1000 + log_{1000} 10000$.

If you cannot tell the answer directly, you can rewrite them as the exponents of ten.

Then you will have $log_{10} 10^2 + log_{10^2} 10^3 + log_{10^3} 10^4$.

What is the answer?

Student: Two plus one point five plus four-thirds.

That equals twenty-nine over six.

It is four point something.

Therefore, (3) is the correct choice.

Teacher: Great.

老師：這題看起來很難。我們先列出已知的條件。

學生：$a_1 = 10$、$a_2 = 100$、$a_3 = 1000$，以及 $a_4 = 10{,}000$

老師：我們可以將這幾項代換成對應的值，得到 $log_{10}100 + log_{100}1000 + log_{1000}10000$。

如果無法直接算出答案，可以改寫真數為底數為 10 的指數：

$log_{10}10^2 + log_{10^2}10^3 + log_{10^3}10^4$

這樣答案是多少？

學生：$2+1.5+\frac{4}{3}=\frac{29}{6}$，等於 4 點多，所以答案是選項(3)。

老師：很棒。

例題二

說明：本題是遞迴數列的延伸，同時涉及等差和等比數列的理解，屬於綜合題。

（英文）The recursive rule for a sequence is given as $a_1 = 1$ and $a_{n+1} = \frac{2n+1}{2n-1}a_n$. n is a natural number. Please select all that are correct.

（中文）已知一實數數列$< a_n >$滿足 $a_1 = 1$，$a_{n+1} = \frac{2n+1}{2n-1}a_n$，$n$ 為正整數。試選出正確的選項。

(1) $a_2 = 3$　　(2) $a_4 = 9$　　(3) $< a_n >$為等比數列

（改編自 110 數乙指考補考）

Teacher: Substitute 1 for the variable n; we will have the second term.

$$a_2 = \frac{2+1}{2-1}a_1 = 3a_1 = 3.$$

Any questions about this?

Student: No.

Teacher: We can use similar logic to find the third and the fourth terms.

Do any of you want to give it a try?

Student: Let me try to find the third term.

Substitute 2 for the variable n; we will get the third term.

$$a_3 = \frac{4+1}{4-1}a_2 = \frac{5}{3}a_2 = 5$$

Teacher: Very good. How about the fourth term?

Student: Substitute 3 for the variable n; we will get the fourth term.

$$a_4 = \frac{6+1}{6-1}a_3 = \frac{7}{5}a_3 = 7$$

Teacher: According to the values of the first 4 terms, do you think this sequence is geometric?

Student: The first term is 1, and the second term is 3.
If it is geometric, the third term should be 9.
Obviously, this is not a geometric sequence.

Teacher: Therefore, the correct answers for this problem are (1),

老師： 將變數 n 換為 1，會得到第二項。

$$a_2 = \frac{2+1}{2-1}a_1 = 3a_1 = 3.$$

到這邊有問題嗎？

學生： 沒有。

老師： 可以用相似的邏輯來找第三和第四項。
誰來試一下？

學生： 我想試看看第三項。
把 2 帶進去 n，會得到第三項。

$$a_3 = \frac{4+1}{4-1}a_2 = \frac{5}{3}a_2 = 5$$

老師： 非常好。那第四項呢？

學生： 把 3 帶進去 n，會得到第四項。

$$a_4 = \frac{6+1}{6-1}a_3 = \frac{7}{5}a_3 = 7$$

老師： 根據前四項，這個數列是否為等比數列？

學生： 第一項是 1，第二項是 3，如果是等比數列，第三項應該是 9。
顯然不是等比數列。

老師： 因此本題的正確答案是(1)。

例題三

說明：本題為美國數學競賽題，需要判斷數列的規律，再進一步求解。

參考網址：https://artofproblemsolving.com/wiki/index.php/2021_AMC_10B_Problems/Problem_8

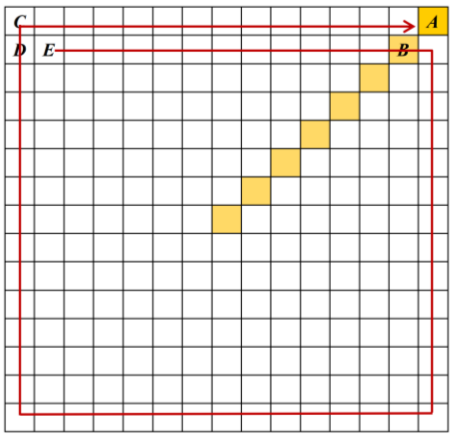
（英文）Mr. Zhou places all the integers from 1 to 225 into a 15 by 15 grid. He places 1 in the middle square (eighth row and eighth column) and places other numbers one by one clockwise, as shown in part in the diagram below. What is the sum of the greatest number and the least number that appear in the second row from the top?

⋯	⋯	⋯	⋯	⋯	⋯	⋯
⋯	21	22	23	24	25	⋯
⋯	20	7	8	9	10	⋯
⋯	19	6	1	2	11	⋯
⋯	18	5	4	3	12	⋯
⋯	17	16	15	14	13	⋯
⋯	⋯	⋯	⋯	⋯	⋯	⋯

（中文）周老師將 1 至 225 的正整數置放於15 × 15的方格中。正中間位於第八行第八列的位置，放入整數 1。如圖所示，以逆時針旋轉方式放入接續的正整數。試問，由上而下數來的第二列中，最大和最小的數字和是多少？

（2021 AMC 10A）

Teacher: If you have no idea of how to start, look for the pattern.
(Circle 1, 9 and 25 before giving explanations.)
Perfect squares of odd integers appear on the right upper diagonal, as shown in the yellow blocks.
What is the value of A?
What is the value of B?



Student: A is 15^2 and B is 13^2.

Teacher: The numbers go clockwise from the least number. In other words, the numbers go counter-clockwise from the greatest number. Check the red route, D is the greatest number in the second row, and E is the least in the second row.
What are the values of D and E?
Student: E is $169-(15-3)=157$.
D is $225-14-1=210$.
Teacher: Very good.
Next, find the sum.
Student: $157+210=367$.

老師：如果不知道從哪裡開始，可以找規律。
（老師圈選「1、9、25」後再進行講解。）奇數的平方數出現在右上角的對角線上，如圖所示的黃色區域。
A 的值為何？B 的值為何？
學生：A 為15^2，而 B 為13^2。
老師：這些數字由小到大按照順時針方向排列；換句話說，由大到小則逆時針方向排列。觀察紅色路線，D 是第二行中最大的數字，而 E 是第二行中最小的數字。
D 和 E 的值是多少？
學生：E 為 $169-(15-3)=157$。
D 為$225-14-1=210$。
老師：很好。最後求他們的和。
學生：$157+210=367$。

單元二 級數
Series

國立新竹科學園區實驗高級中等學校　周慧蓮老師

■ 前言 Introduction

此單元延伸數列的概念到級數。將數列依序用加號連結起來的表示式，便是級數。此章節會介紹常用的級數求和公式。除了計算等差級數與等比級數的和，級數在生活中的應用，如複利孳息問題也是本章重點。財務問題是學生在準備出國時參加英文標準測驗的常見口說考題，我們建議教師在教學中可以強調與財務相關的英文詞彙和表達方式。這包括學生必須熟悉的投資與利率計算等相關術語。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
series	**級數**	**interest**	**利息**
period	**週期**	**interest rate**	**利率**
simple interest	**單利**	**compound interest**	**複利**
principal	**本金**	**deposit**	**存款**
denote	表示	partial	部份
forward	向前	backward	向後

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ ____ denoted by _____.

例句(1)：The sum of just the first *n* terms of a sequence is called the *n*th partial sum and is **denoted by** $S_n = a_1 + a_2 + a_3 + \ldots + a_n$

數列前 *n* 項的和稱為 *n* 項部份和，以 $S_n = a_1 + a_2 + a_3 + \ldots + a_n$ 表示。

例句(2)：The absolute value of a real number *a* is **denoted by** $|a|$.

實數 *a* 的絕對值由$|a|$表示。

❷ Find the _____ partial sum of the sequence _____.

例句(1)：**Find the** 4^{th} **partial sum of** $a_n = 2n - 3$.

寫出數列$< 2n - 3 >$的前四項和

例句(2)：**Find the** fifth **partial sum of** the series $0.2 + 0.02 + 0.002 + 0.0002 + \ldots$.

寫出此級數的前五項和。

❸ Construct the sequence forward/backward by _____.

例句(1)：We can **construct the sequence forward by** starting with 2 and adding 3 each time.

我們可以以 2 為首項，依序加 3，依序寫下數列。

例句(2)：We can **construct the sequence backward by** starting with 2 and subtracting 3 each time.

我們可以以 2 為首項，依序減 3，反向寫下數列。

❹ This formula works only for _____.

例句(1)：**This formula works only for** arithmetic sequences.

這個公式只適用於等差數列。

例句(2)：**This formula works only for** geometric sequences.

這個公式只適用於等比數列。

❺ The sum of the first _____ terms

例句(1)：**The sum of the first** 20 **terms** of the sequence is 210.
這個數列前 20 項的總和是 210。

例句(2)：This formula can be used to find **the sum of the first** *n* **terms** of an arithmetic sequence.
這個公式可以用來求等差數列前 *n* 項的和。

❻ A principal of P is invested/borrowed at r % annual/monthly interest.

例句(1)：**A principal of** $1000 **is invested at** 5% **annual interest.**
一千元本金以年利率 5%進行投資。

例句(2)：**If a principal of** 1000 dollars **is borrowed for** a period of *t* years **at an annual interest** rate of 1%, the simple interest charged is $1000 \times 1\% \times t$.
如果一千元本金以年利率 1% 被借出，單利孳息 *t* 年後的利息是$1000 \times 1\% \times t$。

❼ A deposit of P is made/invested _____.

例句(1)：**A deposit of** $1000 **is made** at the beginning of each month.
每個月初存款一千元。

例句(2)：**A deposit of** $1,000 **is invested** at compound interest.
投資一千元並以複利孳息。

■ 問題講解 Explanation of Problems

說明

In this section, we will work on a series. When the terms of a sequence are added together, the resulting expression is a series. We will develop the formulas for finding the sum for an arithmetic sequence, a geometric sequence, and several special series. Additionally, daily life applications like compound interest problems will be instructed.

∞ 運算問題的講解 ∞

例題一

說明：此題為級數的基本題，主要讓學生熟悉如何應用數列的一般項找到前 n 項的和。

（英文）(a.) Write the 5^{th} partial sum of the sequence $< 4^n - 1 >$ in terms of a series.

(b.) Find the 5^{th} partial sum of the sequence $< 4^n - 1 >$

（中文）(a.) 以級數表示，寫出數列$< 4^n - 1 >$的前五項。

(b.) 求數列$< 4^n - 1 >$前五項的和。

Teacher: $4^n - 1$ represents the general term of the sequence. Who can tell me how to find the first term?

Student: Substitute 1 for n. We will get four minus one, which is 3.

Teacher: Great. We apply the same logic to find the other 4 terms. Plug in the numbers from 2 to 5. What are the results?

Student: They are 15, 63, 255, and 1023.

Teacher: To write this sequence as a series, we add a plus sign between the consecutive terms. It becomes $3 + 15 + 63 + 255 + 1023$.
What is the sum?

Student: It is 1359.

老師： $4^n - 1$ 代表這個數列的一般項。誰知道怎麼求第一項？

學生： 將 n 替換為 1，得到 4−1，也就是 3。

老師： 很好。同理，其它 4 項也能這樣找出來。將數字由 2 到 5 代入公式，結果為何？

學生： 分別是 15、63、255 和 1023。

老師： 為了將這個數列寫成一個級數，我們要在相鄰項之間加上加號。
第(a.)小題答案就是 $3 + 15 + 63 + 255 + 1023$。
那總和是多少呢？

學生： 1359。

例題二

說明：此題為等差級數求和的基本運算，除了找出公差和項數、也需要求前 n 項的和。

$100 + 97 + 94 + \ldots + 31$

（英文）(1.) Find the number of terms of this arithmetic series.

(2.) Find the sum of this arithmetic series.

（中文）(1.) 求等差數列的項數。

(2.) 求等差數列的和。

Teacher: For an arithmetic sequence, each term is found by adding the common difference to the previous term. According to the description, $a_1 = 100$ and $d = -3$, what is the general term of the arithmetic sequence?

Student: $a_n = 100 + (n-1)\cdot(-3) = 103 - 3n$

Teacher: By using the explicit formula, can you tell me the number of terms in this series?

Student: We can solve the equation $31 = 103 - 3n$ to find the value of n.
It will be $3n = 72$.
After simplification, $n = 24$. There are 24 terms in this series.

Teacher: Well done.
Who can remember the formula of the sum of an arithmetic sequence? It is similar to finding the trapezoid area.

Student: Multiply the number of terms by the sum of the first and the last term. Subsequently, dividing it by two will lead to the sum.
One hundred plus thirty-one is one hundred and thirty-one.
Multiply one hundred and thirty-one by twenty-four, then divide it by two. It is the same as multiplying one hundred and thirty-one by twelve. The result is 1572 (one thousand five hundred and seventy-two).

Teacher: That's a very clear explanation.

老師：等差數列的每一項都是前一項加上公差。從題目可以看出 $a_1 = 100$且 $d = -3$，一般項要如何表示？

學生：$a_n = 100 + (n-1)\cdot(-3) = 103 - 3n$。

老師：可以用一般項公式知道數列有多少項嗎？

學生：解出方程式 $31 = 103 - 3n$來找到 n 的值。
常數項移到右邊，$3n = 72$

進一步化簡得到 $n = 24$，此數列有 24 項。

老師： 做得很好。

有誰還記得等差數列求和的公式？跟梯形面積的公式很類似。

學生： 首項和末項相加乘以項數，然後除以二就可以得到總和。

$100 + 31 = 131$

131×24，再除以 2，也就等於是131×12，答案是 1572。

老師： 很好，解釋得很清楚。

例題三

說明：此題為等比級數求和的基本運算，藉由已知的特定項找出第一項、公比和一般項進一步可以利用等比級數的求和公式。

（英文）Given a geometric sequence $< 2^n >$,

(1) Find the 10^{th} partial sum.

(2) Find the nth partial sum.

（中文）等比數列 $< 2^n >$，

(1) 求前十項的和。

(2) 求前 n 項的和。

Teacher: For a geometric sequence, each term is found by multiplying the common ratio by the previous term. According to the description, $a_1 = 2$ and $r = 2$. Can you remember the formula for the sum of a geometric series?

Student: It is: $a_1 \times \frac{(1-r^n)}{1-r}$. (Read as A one times the quantity one minus r to the nth power divided by the quantity one minus r).

Teacher: Substitute $a_1 = 2$, $r = 2$, and $n = 10$ into this formula. What do you get?

Student: $2 \times \frac{1-2^{10}}{1-2} = 2 \times 1023 = 2046$

Teacher: As you can see here, the common ratio is greater than one. So it will result in a negative denominator. I suggest that you rewrite the formula as: $a_1 \times \frac{r^n-1}{r-1}$ to find the sum of the first n terms. Who knows the answer to part (2)?

Student: It is $2 \times (2^n - 1)$.

老師：　等比數列的每一項，會是前一項乘上公比。
由題目敘述可以得知，$a_1 = 2$ 且 $r = 2$。大家還記得等比數列和的公式嗎？

學生：　公式是：$a_1 \times \frac{(1-r^n)}{1-r}$。

老師：　將 $a_1 = 2$、$r = 2$ 和 $n = 10$ 代入算式，算出答案。

學生：　$2 \times \frac{1-2^{10}}{1-2} = 2 \times 1023 = 2046$。

老師：　如你所見，公比大於 1，所以分母是負數。建議重新寫為$a_1 \times \frac{r^n-1}{r-1}$ 以求出前 n 項的和。
那麼誰知道第(2)小題的答案呢？

學生：　答案是：$2 \times (2^n - 1)$。

例題四

說明：此題為前 n 個連續正整數的“和”、“平方和”與“立方和”的級數求和公式應用。

（英文）Find the sum of the following series.

（中文）求下列級數的和。

$$1^2 + 2^2 + 3^2 + \ldots + 9^2 + 10^2$$

$$1^3 + 2^3 + 3^3 + \ldots + 9^3 + 10^3$$

Teacher: The formula of the sum of perfect squares and the sum of cubes of the first n positive integers are proved with the mathematical induction in the previous section. Using the formula listed in this section, we can find the sum of perfect squares of the first ten integers.

$\frac{10(10+1)(20+1)}{6} = 385.$

Student: Can I add these numbers one by one?

Teacher: If there are not too many terms, you can find or check your answer by adding the terms one by one. Who wants to solve the second question?

Student: I know. It's pretty easy.

We start from finding $\frac{10(10+1)}{2} = 55$

We then square 55. That equals: 3025.

Teacher: Well done. $\frac{10(10+1)}{2}$ is the sum of the first ten positive integers. When this sum is squared, it becomes equal to the sum of the cubes of the first ten positive integers.

老師：在上一章節中，數學歸納法證明了前 n 個正整數的完全平方和公式以及立方和公式。本章節中列出的公式可以找到前十個整數的完全平方和。

$$\frac{10(10+1)(20+1)}{6}=385$$

學生：可以將這些數字一個一個加起來嗎？

老師：如果數字不多，可以一個一個相加驗算答案。怎麼算第(2)小題？

學生：我會，這很簡單。

我們從找到 $\frac{10(10+1)}{2}=55$ 開始，然後我們將55平方等於 3025。

老師：好。$\frac{10(10+1)}{2}$ 是前十個正整數的和。

因此，它的平方就是前十個正整數的立方和。

∞ 應用問題 / 學測指考題 ∞

例題一

說明：此題題目相當複雜，包含等比級數、和(sigma)的符號和對數的基本運算。教師須引導學生一步一步拆解後進行計算。必要時，需複習正三角形的面積公式。

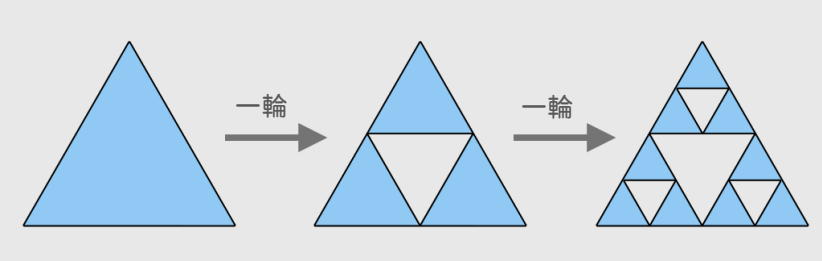
（英文）There is an equilateral triangle with a side of length 1. First, divide this triangle into four congruent equilaterals by connecting the midpoint of the side of the original equilateral. Remove the central equilateral. Repeat the steps of each smaller equilateral and remove the central equilateral again. What is shown below involves two stages.

(a) How many equilaterals are removed in the first, second, and third stages?

(b) Find the area of all the remaining equilaterals in the third stage.

(c) Write a general term for the total number of removed equilateral after the nth stage.

（中文）一個朝上著色的正三角形邊長為 1 公分，在每一輪中，連接著色的正三角形之三邊中點形成四個小的正三角形，並移除正中間的三角形形成新的圖形。下圖為進行兩輪後的結果。



(a) 分別求出有幾個正三角形會在第一輪、第二輪和第三輪被移除？

(b) 試求第三輪時，剩餘的正三角形面積和。

(c) 進行 n 輪後，共有多少正三角形被移除？

（改編自翰林第二冊課本 1-2 例題 5）

Teacher: Check the figure. How many removed equilaterals in the first stage? In the second stage?

Student: There is one removed equilateral in the first stage. There are three more removed equilaterals in the second stage.

Teacher: Connect the midpoints of the remaining equilaterals in the 2nd stage. How many equilaterals would you remove in the third stage?

Student: I would remove 9 equilaterals in the third stage.

Teacher: Correct.
Now, let's find the area of the remaining triangles.
There are 3 triangles left in the first stage. There are 9 triangles left in the 2nd stage. How many triangles are left in the 3rd stage?

Student: Three to the power of three is twenty-seven.
There are 27 triangles left in the 3rd stage.

Teacher: The side length of the equilateral left in the first stage is one-half. The side length of the equilateral left in the 2nd stage is one-quarter. What is the side length of the equilateral left in the 3rd stage?

Student: It is one-eighth.

Teacher: The area of an equilateral is $\frac{\sqrt{3}}{4}s^2$, where s is the side length. What is the area of one equilateral in the third stage?

Student: $\frac{\sqrt{3}}{4} \times \left(\frac{1}{8}\right)^2 = \frac{\sqrt{3}}{256}$

Teacher: What is the area of all the remaining equilaterals in the third stage?

Student: $27 \times \frac{\sqrt{3}}{256} = \frac{27\sqrt{3}}{256}$

Teacher: So far, we have found the number of triangles and areas from one stage to another. Actually, we can develop a general formula for it by recognizing the pattern. For example, we notice that 1, 3, and 9 triangles are removed in the first three stages, respectively. We can apply the sum of the geometric series to find the number of total triangles removed after the nth stage. Who can explain the logic for solving this problem?

Student: The expression of finding the number of total triangles removed is $1 + 3 + 9 + \ldots + 3^{n-1}$

The sum would be $1 \times \frac{3^{n-1}-1}{3-1} = \frac{3^{n-1}-1}{2}$

Teacher: Not quite.

Note that there are n terms in the series, even though the last term is 3^{n-1}.

Student: The answer is $\frac{3^n-1}{2}$.

Teacher: Very good. You can always check whether your answer is correct by substituting the values of n.

老師： 觀察題目的圖形。第一輪移除了幾個正三角形？第二輪呢？

學生： 第一輪移除了一個正三角形，第二輪又移除了三個正三角形。

老師： 連接第二輪剩餘正三角形的中點，到第三輪會移除多少個正三角形？

學生： 九個。

老師： 沒錯。現在來找剩餘的正三角形面積和。

第一輪剩三個三角形，第二輪剩九個三角形，那麼第三輪會剩下多少個三角形？

學生： 3 的三次方是 27，第三輪剩下 27 個三角形。

老師： 第一輪剩下的正三角形的邊長，是 $\frac{1}{2}$；第二輪剩下的正三角形的邊長則是 $\frac{1}{4}$。

那第三輪剩下的正三角形的邊長是多少？

學生： 是 $\frac{1}{8}$。

老師：正三角形的面積公式是$\frac{\sqrt{3}}{4}s^2$，其中 s 是邊長。第三輪的一個正三角形的面積是多少？

學生：$\frac{\sqrt{3}}{4}\times\left(\frac{1}{8}\right)^2=\frac{\sqrt{3}}{256}$。

老師：第三輪所有剩餘的正三角形的面積是多少？

學生：$27\times\frac{\sqrt{3}}{256}=\frac{27\sqrt{3}}{256}$。

老師：到目前為止，已經找出經過一輪、兩輪、三輪後的三角形個數與面積。
實際上，觀察三輪的數字就可以寫出標準式。
例如，觀察到第一、二、三輪分別刪除了 1、3 和 9 個三角形，所以可以應用等比級數求和，來找到第 n 輪後刪除的總三角形數。
誰能解釋解決此問題的邏輯？

學生：求移除的總三角形數"$1+3+9+\ldots+3^{n-1}$。總和為 $1\times\frac{3^{n-1}-1}{3-1}=\frac{3^{n-1}-1}{2}$。

老師：不完全正確，即使最後一項是 3^{n-1}，但級數總共有 n 項。

學生：答案是 $\frac{3^n-1}{2}$。

老師：很好！同學可以把不同的數字代入 n 來驗算。

例題二

說明：本題是常見的複利生活應用題，也涉及對數的應用。老師猶有餘力時，可以向學生說明計算機的使用方式。

（英文）A deposit of ten thousand dollars is invested at the beginning of each month into an account that pays 0.1% interest compounded monthly.
(a) At the end of 5 years, what will the balance in the account be?
(b) How much time would it take for the balance to reach 1 million dollars?

（中文）小明每個月月初存入銀行 10,000 元，以月利率 0.1 %複利孳息。
(1) 五年後小明共可存到多少錢？
(2) 幾年後小明的存款可以達到百萬？

Teacher: Recall that the formula of compound interest is $P(1+r)^n$.
Applying this formula, what will the balance be after 5 years? Actually, you have to think of the period as 60 months.

Student: Substitute 10,000 for the variable P, 0.001 for the variable r and 60 for the variable t, we will get: $10{,}000 \times (1+0.001)^{60}$

Teacher: Based on the description, Min deposits 60 times since there are 60 months in five years. Who can change the answer?

Student: $10{,}000 \times (1.001)^{60} + 10{,}000 \times (1.001)^{59} + 10{,}000 \times (1.001)^{58} + \ldots + 10{,}000 \times (1.001)$.

Teacher: What kind of series is it? Do we have a formula for finding the sum?

Student: This is a geometric series. The first term is $10{,}000 \times 1.001$ and the common ratio is 1.001. The sum for the geometric series is $10{,}000 \times 1.001 \times \frac{1.001^{60}-1}{1.001-1}$

We can find the numerical answer by using a calculator. The balance after five years will be 618,665.

Teacher: Now, let's work on the second question. How much time will it take for the balance to reach 1 million?

Recall that the formula for finding the sum of a geometric series is $a_1 \cdot \frac{(r^n-1)}{r-1}$.

Substitute $a_1 = 10{,}000 \times 1.001 = 10{,}010$ and $r = 1.001$
We need to solve the equation $10{,}001{,}000 \times (1.001^n - 1) = 1{,}000{,}000$
Who knows what the next step is?

Student: Divide both sides by 10,001,000. We will have: $1.001^n - 1 = \frac{1000}{10001}$

Simplify this equation, it becomes: $1.001^n = \frac{11001}{10001}$

Take logarithms for both sides.

We will get: $n = (log\frac{11001}{10001}) \div log\,1.001$

The calculator shows $n \approx 95.3$ months

Teacher: If we deposit 10,000 dollars per month without interest, it takes 100 months to reach 1,000,000 dollars. In this question, compound interest is added to the savings account, taking less time to reach the balance of 1,000,000 dollars. The answer, 95 months, is less than 100 months. So it is a reasonable answer.

老師：回想一下之前教過的複利公式 $P(1+r)^n$，用這個公式來算出五年後小明總共可以存到多少錢。計算的時候要把五年看作是 60 個月。

學生：將 10,000 代入 P，0.001 代入 r，並將 60 代入 t，會得到：$10{,}000 \times (1+0.001)^{60}$。

老師：題目說，小明存款 60 次，因為五年有 60 個月。誰可以修正答案？

學生：$10{,}000 \times (1.001)^{60} + 10{,}000 \times (1.001)^{59} + 10{,}000 \times (1.001)^{58} + \ldots + 10{,}000 \times (1.001)$。

老師：這是什麼級數？要用什麼公式來求總和呢？

學生：這是一個等比級數。第一項是$10{,}000 \times (1.001)$，公比是1.001。

等比數列的總和是$10{,}000 \times 1.001 \times \frac{1.001^{60}-1}{1.001-1}$。用計算機算出來答案是 618,665。

老師：很好，現在來解第(2)小題。幾年後小明的存款可以達到百萬？

回想一下算等比數列和的公式：$a_1 \cdot \frac{(1-r^n)}{1-r}$。

代入$a_1 = 10{,}000 \times 1.001 = 10{,}010$ 和 $r = 1.001$。

解方程式 $10{,}001{,}000 \times (1.001^n - 1) = 1{,}000{,}000$，誰知道下一步要怎麼做？

學生：將兩邊都除以 10,001,000，變成 $1.001^n - 1 = \frac{1000}{10001}$。接著化簡得到$1.001^n = \frac{11001}{10001}$。

對兩邊取對數：$n = (log\frac{11001}{10001}) \div log\,1.001$。

用計算機算出$n \approx 95.3$ 個月。

老師：如果我們每個月存入 10,000 而不計利息，需要 100 個月才能存到一百萬。這題題目說，複利孳息，因此存到一百萬所需的時間較短。

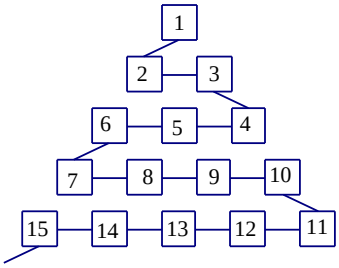
答案 95 個月小於 100 個月，所以是一個合理的答案。

例題三

說明：此題為數列與邏輯的應用，除了計算等差數列的和，尚需要邏輯判斷數字是由左而右或由右而左排列。

（英文）The snake model below is used to explain network operations. Number 1 is in the first row. Numbers 2 and 3 are in the second row. From left to right, numbers 6, 5, and 4 are in the third row. Numbers 7, 8, 9, and 10 are in the fourth row. What is the 67^{th} number that is in the 99^{th} row, counting from left to right?

（中文）下圖是從事網路工作者經常用來解釋網路運作的蛇形模型：



數字 1 出現在第 1 列；數字 2, 3 出現在第 2 列；數字 6, 5, 4(從左至右)出現在第 3 列；數字 7, 8, 9, 10 出現在第 4 列；依此類推。試問第 99 列，從左至右算，第 67 個數字為 ⑧⑨⑩⑪。

（94 年數乙指考）

Teacher: There is no shortcut to solving this problem.
Try to break this problem down into several steps.
If we want to count the numbers consecutively, sometimes we count from left to right, and sometimes we count from right to left.
For the 99^{th} row, which is the correct direction to count the number consecutively?

Student: In the odd rows, we count the numbers from right to left.
In the even rows, we count the numbers from left to right.
Therefore, we count the numbers in the 99^{th} row from right to left.

Teacher: Can you tell what the greatest number is in the 99^{th} row?

Student: It would be the sum of the first 99 positive integers.

$$(1 + 99) \times \frac{99}{2} = 4950.$$

Teacher: Let's consider the 99^{th} row as an arithmetic sequence. The first term is 4950, and the common difference is -1.
What is the 67^{th} term?

Student: 4950 minus 66 is 4884.

Teacher: Correct.
This question is all about the arithmetic sequence.
It is not very difficult if you can break it down.

老師：解題沒有捷徑，讓我們把問題一步一步分解。
如果要連續計數，從題目的圖可以看出，有的要從左到右，而有的則要從右到左。
若要算第 99 行，正確的方向是從哪邊？

學生：奇數行方向是從右到左；偶數行方向是從左到右。
因此第 99 行是從右到左算。

老師：能告訴我第 99 行的最大數是多少嗎？

學生：這將是前 99 個正整數的和：$(1+99)\times\frac{99}{2}=4950$。

老師：我們將第 99 行視為等差數列，首項是 4950，公差為−1。那第 67 項是多少呢？

學生：4950−66=4884。

老師：答對了，這個問題完全是在考等差數列的概念。將題目一步步分解，其實一點都不難。

例題四

說明：此題因為要求正方體的表面積和，所以會應用到特殊級數的求和公式。

參考網址：https://artofproblemsolving.com/wiki/index.php/2020_AMC_10A_Problems/Problem_10

（英文）Seven cubes, whose volumes are 1, 8, 27, 64, 125, 216, and 343 cubic units, are stacked vertically to form a tower in which the volumes of the cubes decrease from bottom to top. Except for the bottom cube, the bottom face of each cube lies completely on top of the cube below it. What is the total surface area of the tower (including the bottom) in square units?

（中文）七個正方體，體積分別為 1, 8, 27, 64, 125, 216 和 343，由下而上堆垛。體積最大的正方體置於最下層。因此，上層正方體的底面會覆蓋住下層正方體的頂面。試求此堆垛外露的表面積（含最大正方體的底面）？

Teacher: It is obvious that the side lengths of these cubes are 1, 2, 3, 4, 5, 6, and 7.
How do you find the surface area of these six cubes if they are not stacked?

Student: There are 6 sides to one cube.
The surface area of the seven cubes is: $6\times(1^2+2^2+\ldots+7^2)$.

Teacher: We can apply the formula of sum of perfect squares. The answer will be: $6\times\frac{7\times(7+1)\times(14+1)}{6}=840.$
Do you think this would be the answer to this problem?

Student: Not really. We have to take away the overlapped area because these cubes are stacked.

Teacher: Between each consecutive pair of cubes, the smaller cube's face on the upper layer is completely covered. An equal area of the larger cube's face on the lower layer is

covered as well. The covered area is thus $2s^2$, where s is the side length of the smaller cube on the upper layer. What is the total area of the overlapped parts of these cubes?

Student: The total area of the overlapped parts of the cubes is thus equal to:

$2(1^2 + 2^2 + \ldots + 6^2)$. It is: $6 \times \frac{6\times7\times13}{6} = 182$.

Teacher: Note that we only add up to six squared, rather than seven squared because the bottom of the greatest cube is not covered.

Therefore, the correct answer is: $840 - 182 = 658$.

老師：從題目可以明顯看出，這些立方體的邊長分別是 1、2、3、4、5、6 和 7。

如果這些立方體沒有疊在一起，怎麼算它們的表面積？

學生：一個立方體有 6 個面。七個立方體的表面積是：$6 \times (1^2 + 2^2 + \ldots + 7^2)$

老師：我們可以應用完全平方和公式，算出來是：$6 \times \frac{7\times(7+1)\times(14+1)}{6} = 840$。

這就是答案了嗎？

學生：不算，還要扣掉重疊的面積，因為這些立方體是疊在一起的。

老師：在每對相鄰的立方體之間，上層立方體的底面會蓋住下層正方體的頂面；同理，下層較大正方體也有同樣被蓋住的面積。

因此，覆蓋的面積為$2s^2$，s 是上層較小立方體的邊長。

接著算出這些立方體的重疊部分的總面積。

學生：$2(1^2 + 2^2 + \ldots + 6^2)$，重疊部分的總面積是 $6 \times \frac{6\times7\times13}{6} = 182$。

老師：這邊要這別注意，我們只加到 6 的平方，而不是 7 的平方，因為最大立方體的底部並沒有被蓋住。

最後正確答案是：$840 - 182 = 658$。

單元三 一維數據分析
One-dimensional Data Analysis

國立新竹科學園區實驗高級中等學校　周慧蓮老師

■ 前言 Introduction

一維數據分析就是只有一個變量的統計分析，例如身高、體重、成績與人數。分析者用一個簡單的指標來解讀大量的數據。本章節以學生於中小學習得的平均值與中位數等統計概念為基礎，延伸至數據的標準差與標準化等概念。教師在介紹英文數學詞彙時，可以有效地利用中英文詞彙的對應關係來幫助學生加深記憶。例如，介紹「標準差」這個概念時，可以指出其英文為 “standard deviation”，並強調英文縮寫 “SD” 與中文中「標準差」首字的對應關係。這樣的教學方法可以幫助學生理解並記憶專業術語，還能夠橫跨語言間的知識轉移。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
statistic	**統計學；統計數據**	**arithmetic mean**	**算術平均**
quartile	**四分位數**	**geometric mean**	**幾何平均**
percentile	**百分位數**	**weighted**	**加權**
25th- percentile	**第 25 百分位數**	**lower quartile**	**後標**
50th percentile	**第 50 百分位數；均標**	**upper quartile**	**前標**
75th percentile	**第 75 百分位數**	**first quartile**	**第一四分位數**

cumulative	累積的	**third quartile**	**第三四分位數**
relative	相對的	**round**	**將(數字)調為整數**
horizontal axis	**橫軸**	transformation	轉換
vertical axis	**縱軸**	translation	平移
vary	**變化**	variation	變化
variance	**變異數**	rescale	重新縮放
standard deviation	**標準差**	deviation	偏差
standardize	**標準化**	dispersion	離散

Method for Finding the Percentile

Say there are n data values. Follow these steps to find the m-th percentile:

1. Order the data set from least to greatest.

2. Solve $I = n \times \frac{m}{100}$

   (1) If I is not n integer, round I up to the nearest integer, say M. P_m is the M-th data value.

   (2) If I is an integer, P_m is the average of the I-th and $(I + 1)$-th data values.

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ The _____ mean of a numerical data set is the sum of _____ data divided by the number of data values.

例句(1)：**The** arithmetic **mean of a numerical data set is the sum of the data divided by the number of data values.**
資料集的算術平均是數據總和除以資料個數。

例句(2)：**The** weighted **mean of a numerical data set is the sum of** the weighted **data divided by the number of data values.**
資料集的加權平均是加權數據總和除以資料個數。

❷ The geometric mean is _____.

例句(1)：**The geometric mean** of two positive numbers, a and b, **is** the side length of a square whose area is the same as the area of a rectangle with side lengths *a* and *b*.
兩個正數的幾何平均數可以視為正方形的邊長值。其中，此正方形的面積與該二正數形成的長方形面積相等。

例句(2)：**The geometric mean is** the *n*th root of the product of n numbers.
幾何平均數是 *n* 個數據的乘積開 *n* 次方。

❸ The _____ falls above the lowest ____ of the data.

例句(1)：**The** first quartile **falls above the lowest** 25% **of the data**.
第一四分位數(Q_1)落在至少四分之一的數據之上。(至少有 25%的數據小於或等於 Q_1)。

例句(2)：**The** 75^{th} percentile is the number that **falls above the lowest** 75% **of the data**.
第 75 百分位數(P_{75})落在至少 75%的數據之上的數值。(至少有 75%的數據小於或等於 P_{75})。

❹ Order/Sort the values before finding the _____.

例句(1)：Don't forget to **sort the values before finding the** percentile.
求百分位數前別忘了將數值排序。

例句(2)：**Order the values before finding the** median.
求中位數前先將數值排序。

❺ ____ is a measure that examines how _____.

例句(1)：The deviation (from the mean) **is a measure that examines how** far each value is from the mean.
離均差是每個數據與平均值的差。

例句(2)：Precipitation **is a measure that examines how** much liquid falls from the sky onto the ground during a specific period.
降雨量是一定時間內從天空降落到地面的水量。

❻ What is the standard deviation of _____?

例句(1)：**What is the standard deviation of** the study hours listed in the table?
表格列示的研習時數數據之標準差為何？

例句(2)：**What is the standard deviation of** hourly wages?
時薪的標準差為何？

❼ _____ with a standard deviation of ____.

例句(1)：The batteries of this brand last an average of 12 hours, **with a standard deviation** of 0.7 hours.
這款電池的續航力維持 12 個小時，0.7 小時的標準差

例句(2)：On this math test, students averaged 83, **with a standard deviation of** 6 points.
這次的數學考試，學生的平均成績是 83 分，標準差是 6 分。

❽ A small/large standard deviation means that the data _____.

例句(1)：**A small standard deviation means that the data** are clustered around the mean.
小標準差代表資料集中在數據的均值附近。

例句(2)：**A large standard deviation means that the data** are more spread out.
大標準差代表資料比較分散。

❾ _____ when a real number *k* is _____ each value in the numerical data set.

例句(1)：The mean of the new data set can be found by adding *k* to the original mean **when a real number *k*** is added to **each value in the numerical data set**.
將原始資料同時加一個定數後，新資料的均值是舊資料均值加上該定數。

例句(2)：The standard deviation of the new data set can be found by multiplying *k* to the original standard deviation **when a positive number *k* is** multiplied to **each value in the numerical data set**.
將原始資料同時乘一個正數後，新資料的標準差是舊資料標準差乘以該正數。

❿ To standardize a value, we _____

例句(1)：**To standardize a value, we** subtract the mean from it and then divide this difference by the standard deviation.
將數值減去平均值再除以標準差，即是標準化數值。

例句(2)：**To standardize the data, we** shift the data by subtracting their mean and then rescale the data by dividing by their standard deviation.
將數據減去平均值再除以標準差，即是標準化數據。

■ 問題講解 Explanation of Problems

說明

In this section, we focus on one-dimensional data. One-dimensional data refer to the values recorded for one variable, such as heights, weights, and grades. Students will learn the differences among arithmetic, geometric, and weighted means. They will further learn how to apply the arithmetic mean and standard deviation to compare and standardize the data set. They will also learn the effect of these statistics when a data set is transformed.

∞ 運算問題的講解 ∞

例題一

說明：此題為均值的基本統計題，主要讓學生熟悉如何計算不同定義的平均值。

（英文）The following numbers represent your term grades for Chinese, English, Math, Science, and Social Studies.

88, 90, 85, 84, 80

(a) Find the arithmetic mean of your term grades.

(b) Your school adopts a weighted grade system depending on the teaching hours of each subject per week. Four, four, four, two, and two are the weekly teaching hours for Chinese, English, Math, Science, and Social Studies, respectively. What is your weighted mean grade?

（中文）以下的數字分別代表你的中文、英文、數學、自然和社會的學期成績。

88, 90, 85, 84, 80

(1) 試求你的成績的算術平均數。

(2) 你的學校依據各科的每週授課時數對學期成績進行加權，試求學期成績的加權平均數。

Teacher: The first one is a basic question in statistics. Most statistics questions have something to do with the arithmetic mean. So, who would like to take a shot at answering it?

Student: Let me do it. To find the arithmetic mean, we first add up all the numbers. The sum is 427. Divide four hundred and twenty-seven by five, which equals 85.4 (eighty-five point four)。

Teacher: Great.

We will apply similar logic to find the weighted mean. First, we must find the sum of the weighted grades. Each subject grade has to be multiplied by its weekly teaching hours. That is, multiply 88 by 4, multiply 90 by 4 and so on.

What is the sum?

Student: The sum of 352, 360, 340, 168 and 160 is 1380.

Teacher: What are the total teaching hours in a week?

Student: Sixteen hours.

Teacher: Your weighted grade is calculated by dividing 1,814 by 16, resulting in approximately eighty-six point two five.

老師： 第一個問題是統計學的基礎問題。大部分的統計問題都與算術平均有關。
那麼，誰想試著回答一下呢？

學生： 讓我來試試看。要找到算術平均，首先我們將所有的數字相加。總和是 427。
將 427 除以 5，得到 85.4。

老師： 太好了。
我們應用類似的邏輯來找到加權平均。首先，我們必須找到加權成績的總和。每個科目的成績必須乘以其每週的教學小時數。也就是說，將 88 乘以 4，90 乘以 4，以此類推。
總和是多少？

學生： 352、360、340、168 和 160 的總和是 1380。

老師： 一週的總教學小時數是多少？

學生： 16 小時。

老師： 那麼你的加權成績就會是 1814 除以 16，也就是大約 86.25。

例題二

說明：此題為幾何平均數的應用題，藉此讓學生了解幾何平均數與算術平均數的差異。

（英文）Consider an investment that grows by 10% in the first year, grows by 20% in the second year, but declines by 30% in the third year. What is the average growth rate of the investment?

（中文）某項投資第一年成長 10%，第二年成長 20%，卻在第三年減損 30%。此投資的平均成長率為何？

Teacher: If you add up 10%, 20%, and negative 30%, the sum is zero. It seems that there is no growth for this investment after three years.
Assume you have 100 in the beginning. One hundred times one hundred ten percent is 110. Your investment grows to 110 by the end of the first year. What will it be by the end of the second year?

Student: One hundred ten multiplied by one point two is 132. The investment is 132 by the end of the second year.

Teacher: Correct.

When we say the investment declines by 30%, it means that 70% of the initial investment is left. Can you tell me how much is left at the end of the third year?

Student: One hundred thirty-two multiplied by seventy percent is ninety-two point four.

Teacher: Very good.

The amount left at the end of the third year is less than one hundred. It means that your investment does not break even. The growth rate is absolutely not zero. As you can see from the previous calculation, multiplication is applied to find the balance. Here, we have to use the geometric mean concept to find the average growth rate. We see the product of multipliers in the consecutive three years first. We then find the cubic root of the product. The product of 1.1, 1.2, and 0.7 is 0.924. The cubic root of 0.924 is about 0.97, which is less than one. One minus 0.97 is three one-hundredths. We would say that the average rate of decline is 3%.

老師： 如果你把 10%、20%和負 30%相加，總和為零。看起來這個投資在三年後並沒有增長。

假設一開始你有 100 元。100 乘以 $\frac{110}{100}$ 等於 110。你的投資在第一年底增長到 110。到了第二年底會是多少呢？

學生： 110×1.2=132，投資在第二年底為 132。

老師： 正確。

當我們說投資下降 30%，表示剩下原始投資的 70%。你能告訴我在第三年底還剩下多少嗎？

學生： 132 乘以 $\frac{70}{100}$ 等於 92.4。

老師： 非常好。

第三年底剩下的金額少於 100。這表示你的投資並沒有收支平衡。增長率絕對不是零。如同之前的計算所示，我們使用乘法找到餘額。這裡，我們必須使用幾何平均的概念來找到平均增長率。首先，我們看連續三年的乘數的乘積。然後，找出該乘積的立方根。1.1、1.2 和 0.7 的乘積為 0.924。0.924 的立方根約為 0.97，小於 1。1 減去 0.97 得到 $\frac{3}{100}$。我們可以說平均下降率為 3%。

例題三

說明：此題是利用累計次數表來計算百分位數的基本題。建議教師先解說次數分配表、累積次數分配表與相對累積次數分配表的差異，再說明其與百分位數的關係。

（英文）To make a decision about the closing time, a fast food restaurant recorded the number of customers between 10:00 and 12:00 pm on weekdays. The results are summarized in the frequency table and cumulative frequency table below.

Number of customers	1	2	3	4	5	6	7	8	9	10
Frequency	37	8	5	10	5	18	5	6	4	2
Cumulative frequency	37	45	50	60	65	83	88	94	98	100

(1) What are the upper and lower quartiles?
(2) What is the 88th percentile?

（中文）某速食店為了決定閉門時間，記錄了週間晚上 10 點至 12 點的來客人數。
(1) 試分別求出來客數的前標和後標。
(2) 試求出來客數數據的第 88 個百分位數。

Teacher: The upper and lower quartiles are the 75^{th} and 25^{th} percentiles, respectively. The 25^{th} percentile means the value falls above 25% of the data. What is the lower quartile?

Student: It is 1 because 37% of the data are 1.

Teacher: How about the upper quartile?

Student: One to five only covers 65 percent of the data. The 75^{th} percentile means that the value falls above 75% of the data. It would be 6.

Teacher: Great!
How about the 88^{th} percentile?

Student: I knew it. It must be 7.

Teacher: Let's revisit the definition of percentile. The 88th percentile shows a value that is greater than 88% of the data. If we consider that "zero to seven people" make up 88% of the data, it shows that the number corresponding to the 88th percentile would be 8. This explanation also clarifies why the concept of the 100th percentile is invalid. The 100th percentile would suggest a value greater than 100% of the data. Since there cannot be a value surpassing all the data points, the notion of the 100th percentile is not applicable.)

老師： 前標和後標分別是第 75 個百分位數和第 25 個百分位數。第 25 個百分位數代表數值高於 25%的數據。後標是多少？

學生： 後標是 1，因為有 37%的數據是 1。

老師： 前標呢？

學生： 從 1 到 5 只包含了 65%的數據。第 75 百分位數表示數值高於 75%的數據。所以前標應該是 6。

老師： 太好了！

學生： 那麼第 88 百分位數呢？

老師： 我知道，是 7。

老師： 讓我們回顧一下百分位數的定義。第 88 個百分位數代表高於 88%的數據的數值。如果我們考慮到「零到七個人」組成了 88%的數據，這意味著對應於第 88 個百分位數的數字應該是 8。

這也解釋了為什麼不存在第 100 個百分位數。第 100 個百分位數代表一個高於 100%的數據的數值。因為沒有一個數值可以大於所有數據點，所以第 100 個百分位數的概念並不存在。

例題四

說明：此題用來計算變異數和標準差，教師可藉此引導學生解讀標準差的意義。

（英文）The hourly wages (in NTD) of different stores in the two cities are listed in the table.

City A	176	176	180	180	186	188	190	196
City B	180	180	182	182	186	186	188	188

(1) Find the arithmetic mean of the hourly wages in each city.
(2) Find the standard deviation of the hourly wages in City B. (Round your answer to the nearest hundredth.)
(3) Compare the greatness of the standard deviations between the two cities without calculating the actual value. Explain your reasoning.

（中文）下表列示兩個城市中不同商店提供的工讀時薪。

(1) 試分別求出兩個城市的時薪平均值。
(2) 試求 B 城市的時薪標準差。（四捨五入至小數點第二位）
(3) 在不計算 A 城市時薪標準差正確值的情況下，試比較兩個城市的時薪標準差大小並解釋原因。

Teacher: The arithmetic mean can be obtained by dividing the sum by 8. However, the great values of hourly wages would make our calculation complicated.
Let's begin by subtracting each value from 180 since all the values are centered around it. This adjustment will enable us to work with a more convenient range and avoid complicated calculations.
In the case of City A, we would have -4, -4, 0, 0, 6, 8, 10, and 16. The sum of these values is 32. The quotient of 32 divided by 8 is 4. We then add 4 back to 180. Here, we can find out that the arithmetic mean is 184.
Can you use this way to find the arithmetic mean of the hourly wages in City B?

Student: Let me try it. I would use 180 as the baseline as well. Subtracting 180 from each value listed in City B will lead to 0, 0, 2, 2, 6, 6, 8, and 8. Add up all these values, and then we get 32. Obviously, the arithmetic mean of the hourly wages in City B is the same as that in City A. It is 184.

Teacher: Thank you for your clear explanation.
Since we know the arithmetic means, how do we find the standard deviation?

Student: Find the deviations by taking the mean from each value. Then add them up.

Teacher: Not really.

Because some deviations would be negative, we have to square them before adding them up. The final step is to divide the sum of squared deviations by 8. Who would like to share your work?

Student: Subtracting 184 from each value listed in City B will lead to -4, -4, -2, -2, 2, 2, 4, and 4. The squared deviations are 16, 16, 4, 4, 4, 4, 16, and 16. The sum of 4 and 16 is 20. There are 4 pairs of 4 and 16. Multiplying 20 by 4 equals 80.

Teacher: Very good. You have applied the concept of weights to find the sum.
Next, divide 80 by 8 and take its square root. The square root of 10 is about 3.16. So, the standard deviation is about 3.16. It means a typical hourly wage in City B differs from the mean by 3.16.

Student: How can I compare the standard deviations in two cities without a detailed calculation?

Teacher: Check the dispersion. The more dispersed the data, the greater the standard deviation. Do you want to guess?

Student: I think the data in City B are closer to the mean because they have a smaller range. So, they should have a smaller standard deviation.

Teacher: Certainly. Well done!
This question is a good example of how the standard deviation matters when the two data sets have the same mean values.

老師： 算術平均可以通過將總和除以 8 來獲得。然而，時薪的大值會使我們的計算變得複雜。因為所有數值都集中在 180 附近，讓我們先將每個數值減去 180。這樣的調整使我們能夠處理一個更易著手的範圍，避免了複雜的計算。以 A 市為例，我們會得到-4、-4、0、0、6、8、10 和 16。這些值的總和為 32。32 除以 8 的商是 4。然後我們再將 4 加回到 180 上。這樣，我們可以得出算術平均為 184。你能用這種方法來找出 B 市的時薪算術平均嗎？

學生： 讓我試試看。我也使用 180 作為基準。將 B 市列出的每個數值減去 180 後，我們得到 0、0、2、2、6、6、8 和 8。將這些值相加，我們得到 32。顯然，B 市的時薪算術平均與 A 市相同，為 184。

老師： 感謝你清晰的解釋。
既然我們知道了算術平均，那麼如何找到標準差呢？

學生： 通過將每個值減去平均值來找到偏差，然後將它們相加。

老師： 實際上不是這樣的。
因為某些偏差可能為負值，所以我們必須在相加之前將它們平方。最後再將平

方偏差的總和除以 8。誰想分享一下你的計算結果？

學生： 將 B 市列出的每個值減去 184 後，我們得到-4、-4、-2、-2、2、2、4 和 4。這些偏差的平方分別是 16、16、4、4、4、4、16 和 16。4 和 16 的總和為 20。有 4 組 4 和 16。將 20 乘以 4 得到 80。

老師： 非常好。你應用了加權的概念來找到總和。

接下來，我們將 80 除以 8 並取其平方根。10 的平方根大約是 3.16。所以，標準差大約是 3.16。這意味著 B 市的基本時薪與平均值相差 3.16。

學生： 如何在不進行詳細計算的情況下比較兩個城市的標準差呢？

老師： 檢查散布程度。數據越分散，標準差越大。要猜猜看嗎？

學生： 我認為 B 市的數據更接近平均值，因為它們的範圍更小。所以，它們應該有較小的標準差。

老師： 沒錯。做得好！

這個問題很好地展示了當兩個數據集具有相同的平均值時，標準差的重要性。

例題五

說明：此題用來幫助學生理解數據的伸縮與平移對平均數和標準差的影響。

（英文）Followed by the previous question,

(a) Find the mean and standard deviation when each hourly wage is increased by 10.

(b) Find the mean and standard deviation when each hourly wage is raised by 10%.

(c) A new store in City B offers an hourly wage of NTD328 because this job needs a specific skill. How would it affect the mean and standard deviation of the hourly wages in City B?

（中文）根據前一題的敘述，

(1) 當每一家店的時薪增加 10 元時，試求出新數據的平均值和標準差。

(2) 當每一家店的時薪增加 10%時，試求出新數據的平均值和標準差。

(3) 乙城市開了一家新店，其技術活的工作時薪為 328 元。此數值將如何影響乙城市的時薪均值和標準差。

Teacher: The hourly wage in City B has a mean of 184 with a standard deviation of 3.16. Because each value is increased by 10, the new mean value will be increased by 10. The new mean is 194.

How about the standard deviation?

Student: Is it increased by 10?

Teacher: Good guess, but it is wrong.

The standard deviation considers the dispersion of the data. If every value is increased by 10, the difference between the data point and the mean stays the same. In this regard, the level of dispersion does not change. Therefore, what is the new standard deviation?

Student: The new standard deviation stays the same. It is still 3.16.

Teacher: Very good.

Now, let's consider the second situation.

What is the new mean when each hourly wage is raised by 10%?

Student: We multiply the original mean by 0.1.

It becomes 18.4.

Oh, no. It is less than 184.

We have to multiply it by 1.1.

Therefore, the new mean is 202.4.

Teacher: Correct.

How about the standard derivation?

Do you think the dispersions will change? Explain your reasoning.

Student: Take 180 and the mean 184 for example; the values become 198 and 202.4 after a 10% raise. The difference changes from 4 to 4.4. The dispersion changes because the difference is increased by 10%.

I think the new standard deviation is 3.16 multiplied by 1.1, which is about 3.48.

Teacher: Thank you for offering a clear explanation.

Now, let's work on the third question.

Do you guess that the new mean and standard deviation will increase or decrease?

Student: The sum of all the values increases a lot. Consequently, the squared difference between 328 and 184 also increases a lot. Therefore, I think both the new mean and standard deviation will increase.

Teacher: Your explanation makes sense.

Three hundred twenty-eight is an outlier. It makes the data more spread out and thus increases the new mean and standard deviation.

老師： 在乙城市，每小時工資的平均值為 184，標準差為 3.16。
因為每個值增加了 10，新的平均值將增加 10，也就變為 194。
標準差呢？

學生： 增加了 10 嗎？

老師： 猜得不錯，但不對。
標準差考慮了數據的分散程度。如果每個值增加了 10，數據點與平均值之間的差異保持不變。因此，在這情況下，數據的分散程度並未改變。
那麼，新的標準差是多少呢？

學生： 新的標準差不變。仍然是 3.16。

老師： 非常好。現在，我們考慮第二種情況。
當每小時工資增加 10%時，新的平均值是多少？

學生： 我們將原始平均值乘以 0.1，變成了 18.4。
哦，不。它比 184 還要小。我們應該是要乘以 1.1。因此，新的平均值是 202.4。

老師： 正確。標準差呢？你認為散布程度會改變嗎？說明一下你的推理。

學生： 以 180 和平均值 184 為例，經過 10%的增加後，值變為 198 和 202.4。
差異從 4 變為 4.4。散布程度因為差異增加了 10%而改變。
我認為新的標準差是 3.16 乘以 1.1，大約是 3.48。

老師： 謝謝你的詳細解釋。現在，讓我們來處理第(3)小題。
你覺得新的平均值和標準差會增加還是減少？

學生： 所有數值的總和大幅增加，於是 328 和 184 之間的平方差也大幅增加。
因此，我認為新的平均值和標準差都會增加。

老師： 你解釋得很有道理。
328 是一個離群值。它使數據更加分散，因此新的平均值和標準差會增加。

例題六

說明：此題是數學學科中心的推薦試題，數據計算不繁雜，教師可以利用此題教導學生數據的中位數和百分位數的關係，再進一步利用加權的概念計算算術平均數。(出處 https://ghresource.mt.ntnu.edu.tw/uploads/1615952695135lh9RBxTN.pdf)

（英文）The math grades of 40 students are listed below. A and B are missing.
Given the median is 65, find the 80th percentile and arithmetic mean.

Grade	30	40	50	60	70	80	90	100
frequency	1	A	1	18	6	6	B	2

（中文）全班 40 位同學某次數學小考成績如下表：

成績	30	40	50	60	70	80	90	100
次數	1	A	1	18	6	6	B	2

已知這 40 筆數據的中位數為 65，則(1)第 80 百分位數 P_{80}=_____分，(2)算術平均數μ =_____分。

（數學學科中心推薦試題）

Teacher: There are 40 data values. Who remembers how to find the median?

Student: The median is the average between the 20th and 21st values.

Teacher: According to the description, the median is 65. Sixty-five is the average between 60 and 70. So, we can conclude that the number of students who have grades no greater than 60 is 20.
How do we find the value of A?

Student: $1 + A + 1 + 18 = 20$. A is zero.

Teacher: The 80th-percentile is the number that falls above the lowest 80% of the data.
Forty multiplied by 0.8 is thirty-two.
The 80th-percentile should be the average of the 32nd and 33rd values.
What are the 32nd and 33rd values?

Student: The 32nd and 33rd values are 80 and 90, respectively.
Their average is 85.
The 80th-percentile is definitely 85.

Teacher: Very good. How about the arithmetic mean of all the grades?

Student: We have to find the value of B first. $6 + 6 + B + 2 = 20$. B is six.
We then multiply the grade with the corresponding frequency.
The products are 30, 0, 50, 1080, 420, 480, 540, and 200.
The sum is 2800. Divide 2800 by 40. The arithmetic mean is 70.

老師： 這裡有 40 個數據值。誰還記得如何找到中位數？
學生： 中位數是第 20 個和第 21 個數值的平均值。
老師： 根據題目敘述，中位數是 65。65 是 60 和 70 的平均值。所以，我們可以得出結論，成績不大於 60 的學生人數為 20。我們如何找到 A 的值？
學生： 1+A+1+18=20。A 為 0。
老師： 第 80 百分位數是超過 80%數據最小的數值。40 乘以 0.8 等於 32。第 80 百分位數應該是第 32 個和第 33 個數值的平均值。
請問數值是多少？
學生： 第 32 個和第 33 個數值分別是 80 和 90。它們的平均值是 85。第 80 百分位數為 85。
老師： 非常好。總成績的算術平均值是多少？
學生： 我們必須先找到 B 的值。6+6+B+2=20。B 是 6。然後，我們將成績與相應的次數相乘。乘起來分別為 30、0、50、1080、420、480、540 和 200。總和等於 2800。將 2800 除以 40，算術平均數是 70。

單元四 二維數據分析
Two-dimensional Data Analysis

國立新竹科學園區實驗高級中等學校　周慧蓮老師

■ 前言 Introduction

二維數據分析就是兩個變量的統計分析，常見的二維數據分析有讀書時數與成績或是氣溫與冷飲銷售量。在此章節，我們透過二維數據的散布圖討論兩個連續變數相關係數，並進一步計算最適直線，藉此解釋並預測兩個變量之間的相互關係。最適直線也被稱為「迴歸直線」。其中，「相關不代表因果關係 correlation does not imply causation.」的概念既是本章節的重要概念也是數據識讀的常見英文，值得教師於課堂多加說明。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
scatter plot	散布圖	**plot**	**繪圖**
trend	傾向	**relationship**	**關係**
tendency	趨勢	**causal**	**因果的**
correlation	**相關**	**causation**	**因果關係**
coefficient	**係數**	**fit**	適合
regress	迴歸	regression	迴歸
association	關聯	pattern	樣式

underlying	隱含的	direction	方向
strength	強度	reveal	揭露
		cluster	群集

Method for Finding the Correlation Coefficient
1. Standardize the paired variables (z_x, z_y) so that the units do not affect the measure. (Subtract the mean of the variable and divide by the standard deviation.) 2. Add up the product of the paired standardized values of the two variables. 3. Divide the previous result by n, the number of pairs. The result is the correlation coefficient.

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ In a scatterplot, _____.

例句(1)：**In a scatterplot,** the two data sets are graphed as ordered pairs.
在散布圖裡，這兩個變量的數據依序配對後圖示。

例句(2)：**In a scatterplot,** we use dots to represent the paired values for two variables.
在散布圖裡，我們用點來代表配對兩個變量的數值。

❷ There is (a) positive/negative/no association between _____ and _____.

例句(1)：**There is a positive association between** heights **and** weights.
身高和體重呈現正相關。

例句(2)：**There is a negative association between** the temperature **and** the amount of ice cream sold.
氣溫和冰淇淋銷售量呈現負相關。

❸ A pattern running from _____ to _____ represents _____.

例句(1)：**A pattern running from** the upper right **to** the lower left **represents** a positive correlation.
由右上到左下的圖形樣式代表正相關。

例句(2)：**A pattern running from the** upper left **to** the lower right **represents a** negative correlation.
從左上到右下的圖形樣式代表負相關。

❹ What conclusions can you make from ____?

例句(1)：**What conclusions can you make from** the line of best fit?
你可以從最適直線下什麼結論？

例句(2)：**What conclusions can you make from** the correlation coefficient?
你可以從相關係數下什麼結論？

❺ What tends to happen as _____?

例句(1)：**What tends to happen as** mobile phone use increases?
使用手機的時間增加時，可能會發生什麼事？

例句(2)：**What tends to happen as** the temperature decreases?
氣溫下降時，可能會發生什麼事？

❻ _____ tell whether _____.

例句(1)：Only time will **tell whether** the prediction was correct.
只有時間知道這個預測是否正確。

例句(2)：**Tell whether** the data shows a positive, negative, or no correlation.
判斷這數據呈現正相關、負相關還是沒有相關。

❼ Write an equation that models _____.

例句(1)：**Write an equation that models** the grades.
寫一個方程式來做成績的模型。

例句(2)：**Write an equation that models** the sales as a function of the temperature.
寫一個方程式做一個以氣溫對銷售量的模型。

❽ _____ show a (non)linear relationship.

例句(1)：When the data **show an approximately linear relationship**, you can model the data with a linear equation.
當資料大概呈現線性關係時，我們可以用線性方程式來做資料的模型。

例句(2)：When the data **show a nonlinear relationship,** using the correlation coefficient to describe the association between the two variables is not suitable.
若資料呈現非線性關係，並不適合用相關係數來描述兩個變數間的關係。

■ 問題講解 Explanation of Problems

ଓ 說明 ଓ

In this section, we focus on two-dimensional data. Two-dimensional data refer to the paired values recorded for two different variables, such as heights and weights. Students will learn how to use a scatterplot to check the strength of the association between the two variables. We can find the correlation coefficient between the two data sets by employing the standardization technique we learned in the previous section. We further use the correlation coefficient to find the line of best fit (sometimes called regression line) to model and predict the data.

運算問題的講解

例題一

說明：此題為散布圖的基本題，主要讓學生熟悉如何繪製散布圖並由圖形判斷變數間的關係。教師也可以進一步說明相關性不代表有因果關係存在。

（英文）A convenience store successively records the temperature and ice cream sales for 10 days. The data is shown in the table below.

Temperature (x)	26	30	32	36	31	28	29	28	34	33
Ice cream sales (y)	48	50	50	70	51	54	50	52	72	58

(1) Make a scatterplot of the data.

(2) Tell whether there is a positive, negative, or no association between the temperature and the ice cream sales.

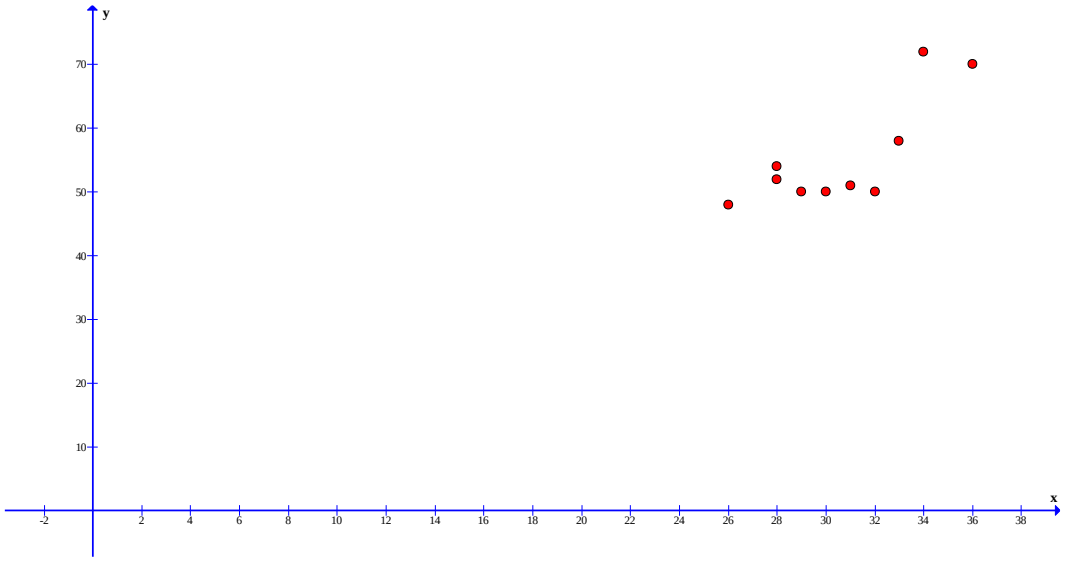
（中文）某超商記錄連續十天的氣溫(x)和冰淇淋銷售量(y)

(1) 試繪製散布圖。

(2) 根據散布圖判斷氣溫和冰淇淋銷售量的關係。

Teacher: A scatterplot can help us tell the data's tendency.
When you plot the scatter plot, make sure the scale is reasonable.
Can you demonstrate the scatterplot on the board?

Student:



Teacher: Great.
When you make the scatterplot, don't forget to label the x-axis and y-axis so that the readers can tell what the data represent.

A pattern running from the upper-right to the lower-left represents a positive correlation.

What tends to happen as the temperature increases?

Student: Ice cream sales increase as the temperature increases.

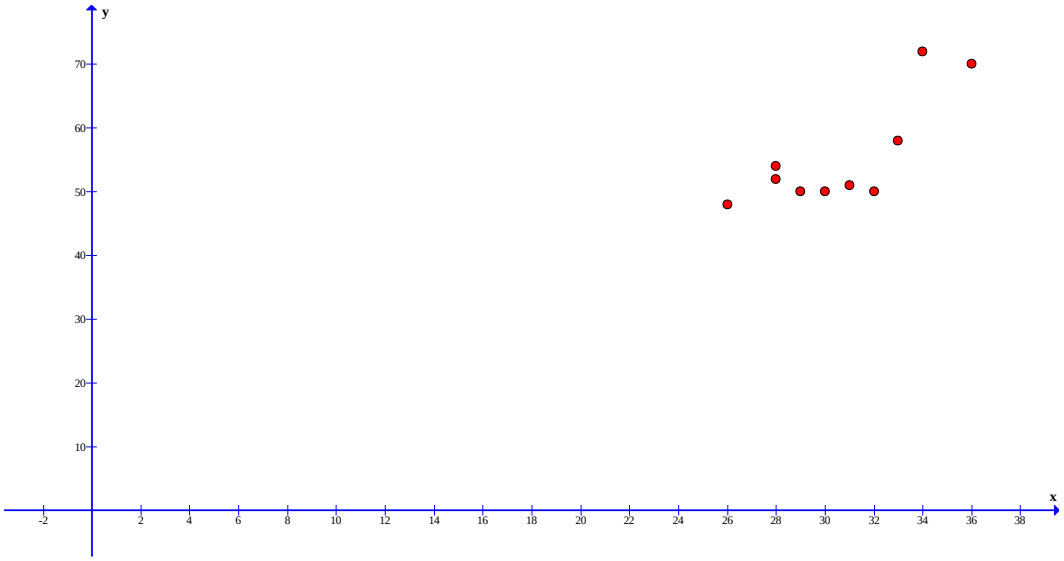
Teacher: Correct.

One thing I have to remind you about is that the relationship between the temperature and the ice cream sales is not necessarily causal.

Only a well-designed experiment can show causation.

老師： 散布圖可以幫助我們觀察資料的趨勢。畫散布圖時，要確保比例尺合理。你能在黑板上示範一下如何繪製散布圖嗎？

學生：



老師： 太棒了。在繪製散布圖時，別忘了標上 x 軸和 y 軸的，這樣讀者才可以知道資料代表什麼。

從右上到左下的一個圖形代表正相關。當溫度上升時，會發生什麼情況？

學生： 隨著溫度上升，冰淇淋的銷售量也會增加。

老師： 正確。

不過，要提醒大家的是，溫度和冰淇淋銷售之間的關係不一定是因果關係。

只有一個設計良好的實驗才會表現因果關係。

例題二

說明：此題是數學學科中心的推薦試題，數據計算不繁雜，教師可以利用此題教導學生數據的標準化、計算相關係數與迴歸直線。
(出處 https://ghresource.mt.ntnu.edu.tw/nss/p/MathematicsApproach05)

（英文）A convenience store records the temperature and ice cream sales, which are shown in the table.

Temperature (x)	26	28	30	32	34
Ice cream sales (y)	58	54	66	50	72

(1) Find the correlation coefficient between the temperature and the ice cream sales.
(2) Find the equation of the regression line.
(3) Predict the value of y when x equals 35 using the equation of the regression line.

（中文）某超商記錄 5 的氣溫(X)與冰淇淋銷售數量(Y)如下表：

氣溫(X)	26	28	30	32	34
冰淇淋銷售數量(Y)	58	54	66	50	72

(1) 求出氣溫與冰淇淋銷售數量的相關係數。
(2) 求對 X 的迴歸直線方程式。（直線方程式請以 $y=mx+k$ 的形式表示）
(3) 利用迴歸直線，預測當 X=35 時，Y 的值為多少？

（數學學科中心推薦試題）

Teacher: We have to standardize the values to find the correlation coefficient. To accomplish this, we need to calculate the mean and standard deviation first. What are the mean and standard deviations?

Student: I calculate the sum of 26, 28, 30, 32, and 34 first and divide the sum by 5. The mean for the temperature is 30. The squared deviation from the mean is 16, 4, 0, 4, and 16, respectively. Dividing the sum squared deviation by 5 leads to 8. The standard deviation is the square root of 8. It is two times the square root of 2 $(2\sqrt{2})$.

Teacher: Very good. Who wants to find the mean and standard deviation of the ice cream sales? Try to use 60 as the baseline to find the mean first. This can avoid complicated calculations.

Student: Let me try it. By using 60 as the baseline, the five differences are -2, -6, 6, -10, a 12, respectively. The sum of them is 0. The mean is 60!

Teacher: Great.

How about the standard deviation of the ice cream sales?

Student: Subtract 60 from each value; we have -2, -6, 6, -10, and 12, respectively. The squared difference is 4, 36, 36, 100, and 144, respectively. Divide the sum squared difference by 5 leads to 64. The square root of 64 is 8. The standard deviation is 8.

Teacher: Since we have the means and standard deviations of the two data sets, what are the standardized data sets?

Recall that to standardize a value, we subtract the mean from the given value and divide this difference by the standard deviation.

Use your notebook as scratch paper and write the answers in a table.

Student:

Temperature, x	$-\sqrt{2}$	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
Ice cream sales, y	$\frac{-1}{4}$	$\frac{-3}{4}$	$\frac{3}{4}$	$\frac{-5}{4}$	$\frac{3}{2}$

Teacher: The product of each pair is $\frac{\sqrt{2}}{4}$, $\frac{3\sqrt{2}}{8}$, 0, $\frac{-5\sqrt{2}}{8}$, $\frac{3\sqrt{2}}{2}$.

The sum is $\frac{3\sqrt{2}}{2}$. What is the correlation coefficient?

Student: Dividing the sum by five leads to the correlation coefficient. R is $\frac{3\sqrt{2}}{10}$.

Teacher: Well done.

Now, we can use $y' = \frac{3\sqrt{2}}{10}x'$ to model the standardized data sets.

How about the unstandardized data sets?

Student: Using the formula $y - \mu_y = r\frac{\sigma_y}{\sigma_x}(x - \mu_x)$, we have $y - 60 = \frac{3\sqrt{2}}{10} \cdot \frac{8}{2\sqrt{2}} \cdot (x - 30)$ (y minus mu y equals r times sigma y over sigma x and multiplied by x minus mu x).

It is $y = \frac{6}{5}x + 24$.

Teacher: Well done.

Now, let's use this equation to predict the y-value when x is 35.

Student: Divide 35 by five and then multiply by 6. It is 42.

Forty-two plus twenty-four is sixty-six.

Teacher: Thank you for the clear explanation.

老師：　我們必須將數值標準化以找出相關係數。這需要我們先計算平均值和標準差。平均值和標準差是多少？

學生：　首先，我計算 26、28、30、32 和 34 的總和，然後將其除以 5。溫度的平均值是 30。與平均值的平方差分別是 16、4、0、4 和 16。將平方差的總和除以 5 得到 8。標準差是 8 的平方根 $2\sqrt{2}$)。

老師：　非常好。誰來計算冰淇淋銷售額的平均值和標準差？
試著以 60 作為基準值，先找到平均值，這樣可以避免複雜的計算。

學生：　讓我來試試。以 60 為基準值，五個差異分別為 -2、-6、6、-10 和 12。總和為 0，平均值是 60。

老師：　太好了。冰淇淋銷售額的標準差如何？

學生：　從每個值中減去 60，我們分別得到 -2、-6、6、-10 和 12。
這些差的平方分別是 4、36、36、100 和 144。將平方差的總和除以 5 得到 64。64 的平方根是 8。標準差是 8。

老師：　由於我們已經有兩個資料集的平均值和標準差，標準化後的資料集是什麼？

學生：　回想一下，要將一個值標準化，我們將給定的值減去平均值，再將其除以標準差。
現在拿出筆記本畫表格，在表格中寫下答案。

學生：

氣溫(X)	$-\sqrt{2}$	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
冰淇淋銷售額(Y)	$\frac{-1}{4}$	$\frac{-3}{4}$	$\frac{3}{4}$	$\frac{-5}{4}$	$\frac{3}{2}$

老師：　每對數對的乘積分別是 $\frac{\sqrt{2}}{4}$、$\frac{3\sqrt{2}}{8}$、0、$\frac{-5\sqrt{2}}{8}$、$\frac{3\sqrt{2}}{2}$，總和是 $\frac{3\sqrt{2}}{2}$。
相關係數是多少？

學生：　將總和除以 5 得到相關係數。R 是 $\frac{3\sqrt{2}}{10}$。

老師：　很好。現在，我們可以用 $y' = \frac{3\sqrt{2}}{10}x'$ 來表示標準化資料集。那麼，未標準化的資料集呢？

學生：　使用公式 $y - \mu_y = r\frac{\sigma_y}{\sigma_x}(x - \mu_x)$，得到 $y - 60 = \frac{3\sqrt{2}}{10} \cdot \frac{8}{2\sqrt{2}} \cdot (x - 30)$。
也就是$y = \frac{6}{5}x + 24$。

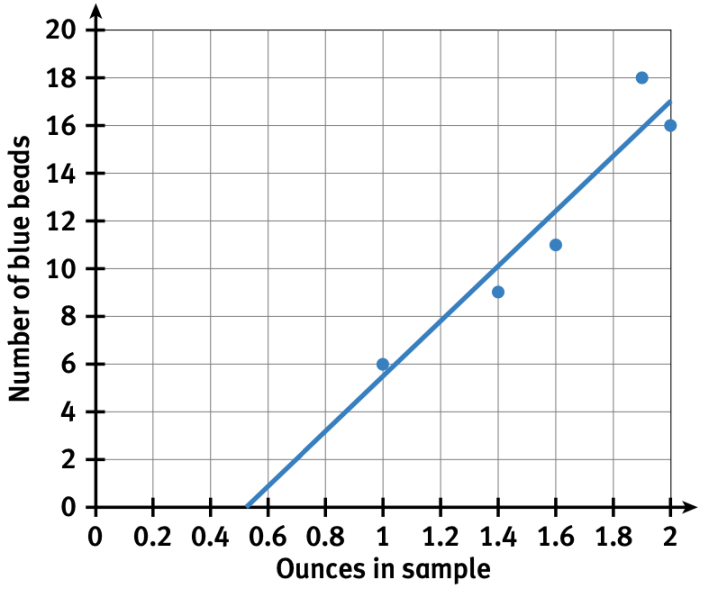
老師：　非常好。現在，讓我們使用這個方程式來預測當 x 為 35 時的 y 值。

學生： 將 35 除以 5，然後乘以 6，等於 42。42 加上 24 等於 66。
老師： 謝謝，解釋得很清楚。

例題三

說明：教案編撰者另行增加第一小題以討論相關的方向性。教師可以用第二題來提醒學生，避免複雜計算的前提下，如何選擇兩個點來概算迴歸直線。甚至進一步帶入迴歸直線與最小平方和的意義。
(出處 https://pre-ap.collegeboard.org/media/pdf/pre-algebra-1-cg-wr.pdf)

（英文）Eli is purchasing beads for an art project. The supply store sells the beads in bulk in multicolor assortments. Eli only wants to use blue beads for his project, but the store won't allow him to pick through the beads to select only blue ones. From the large bin of assorted beads, he takes five small samples of varying weight and counts the number of blue beads in each sample. The scatterplot represents his findings.



(1) Tell whether the data shows a positive, negative, or no correlation.

(2) To determine an equation for a trend line, Eli uses two points that he thinks to lie on the line: the number of blue beads for a sample that measures 0.6 ounces and the number of blue beads for a sample that measures 2 ounces. Which of the following is the correct equation of the trend line that Eli determined?

(A) $y = \frac{16}{1.4}x - \frac{41}{7}$　　(B) $y = \frac{15}{1.4}x - \frac{38}{7}$

(C) $y = \frac{1.4}{16}x + \frac{673}{400}$　　(D) $y = \frac{1.4}{15}x - \frac{1186}{75}$

（中文）艾麗準備為其美術專題購買珠子。商店成袋販售混色的珠子，不允許消費者選擇珠子顏色。但是艾麗的美術專題只需要藍色的珠子。艾麗挑選了五袋重量不一的混色珠子，紀錄各袋的重量與藍色珠子數量。重量與數量的散布圖如下。

(1) 判斷此數據是呈現正相關、負相關或沒有線性相關。

(2) 艾麗想利用假想的迴歸直線上的兩點來計算最適直線。她選定重量為 0.6 盎司與 2 盎司的對應點並求出其直線方程式。試問該直線方程式為何？

(A) $y = \frac{16}{1.4}x - \frac{41}{7}$　　(B) $y = \frac{15}{1.4}x - \frac{38}{7}$

(C) $y = \frac{1.4}{16}x + \frac{673}{400}$　　(D) $y = \frac{1.4}{15}x - \frac{1186}{75}$

（美國 Pre-AP Algebra I 示範試題）

Teacher: What tends to happen as the weight of the mixed beads increases?

Student: The number of blue beads increases.

Teacher: In the scatterplot, the pattern runs from lower-left to the upper-right. What would be the direction of the correlation?

Student: Positive correlation.

Teacher: Very good.

Now, Eli would like to find the regression equation.

To avoid the complicated calculation, he only selects two imaginary points that he thinks would be on the regression line.

The selection of these two points varies.

The key point is to make sure the given points are close to the imaginary line formed by the two imaginary points.

For example, I would say it is inappropriate for Eli to select $(1, 6)$ and $(1.9, 18)$

to find the fit line. Do you know why?

Student: When we draw the line formed by these two points, all the other given points are below this fit line.

Teacher: Correct.

If all the other given points are below this fit line, the linear equation will overestimate the number of blue beads.

Here, Eli imagined two points so that the fit line would be in the middle of the given points. According to the scatterplot, what are the coordinates of the beads with weights of 0.6 and 2.0 ounces?

Student: I think the coordinates are (0.6, 1) and (2, 17).

Teacher: Let's use the point-slope form to find the linear equation formed by these two points. Who wants to give it a try?

Student: $y - 17 = \frac{17-1}{2-0.6}(x - 2)$

(y minus seventeen equals seventeen minus one over two minus six-tenths multiplied by x minus two)

Teacher: By checking the slope, we can tell Choice A is the correct answer. If you are uncomfortable with the choice, you can simplify the equation to confirm the y-intercept.

老師： 當混色珠子的重量增加時，會發生什麼情況？

學生： 藍色珠子的數量會增加。

老師： 在散佈圖中，模式從左下到右上。相關性的方向會是什麼？

學生： 正相關。

老師： 非常好。

老師： 現在，艾莉想要找到迴歸方程式。為了避免複雜的計算，她只選擇了兩個她認為會在迴歸線上的假想點。

這兩個點的選擇可以有所不同。關鍵在於確保給定的點接近由這兩個假想點形成的最適直線。

例如，我會說艾莉選擇 (1, 6) 和 (1.9, 18) 來找到最適直線是不適當的。你知道為什麼嗎？

學生： 當我們畫出由這兩個點形成的線時，所有其他給定的點都在這條最適直線的下方。

老師： 正確。

如果所有其他給定的點都在這條最適直線的下方，線性方程式會高估藍色珠子的數量。

在這裡，艾莉假想了兩個點，使得最適直線位於給定點的中間。根據散佈圖，重量為 0.6 盎司和 2.0 盎司的珠子的座標是什麼？

學生： 座標是 (0.6, 1) 和 (2, 17)。

老師： 讓我們使用點斜式形式來找到由這兩個點形成的線性方程式。誰想試試看？

學生： $y - 17 = \frac{17-1}{2-0.6}(x - 2)$。

老師： 檢查斜率後，我們可以知道(A) 是正確答案。如果你對這個答案不太確定，你可以簡化方程式來確認 y 截距。

單元五 排列組合
Permutations and Combinations

國立新竹科學園區實驗高級中等學校　印娟娟老師

■ 前言 Introduction

本章排列組合涵蓋的內容在排列方面有重複排列及不重複排列，還包含窮舉法、樹狀圖、加法原理及乘法原理、直線全取排列…等；再由排列和組合的關係介紹兩者公式的關聯、巴斯卡性質及二項式定理。老師在介紹本章節時，可以依各主題之配合相關範例解說，讓學生透過解題實作，深化教材中的概念與知識。因為本章節的範例及概念內容較多，建議在示範例題時老師能反覆提問，讓學生能熟悉本章節的英文名詞及用語後，最後以應用問題或學測題來加深學生們對本章節的了解。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
permutation	**排列**	**combination**	**組合**
arrangement	排列	**binomial theorem**	**二項式定理**
tree diagram	樹狀圖	Venn diagram	文氏圖
factorial	**階乘**	Pascal's triangle	巴斯卡三角形
multiplication rule	乘法原理	coefficient	係數
addition rule	加法原理	inclusion	包容
exhaustive method	窮舉法	exclusion	排除

visualize	形象化	the inclusion-exclusion principle	取捨原理
repetition	重複	expression	算式

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ ____ is defined as _____.

例句：The value of 0! **is defined as** 1. (0! is read as zero factorial.)
0!（零階乘）被定義為 1。

❷ In how many ways _____?

例句：**In how many ways** can the letters of the word "permutations" be arranged if the word starts with p and ends with s?
將 permutations 這個字的字母重新排列，並以 P 開頭、S 結尾的排列共有多少種方式？

❸ _____ is read as _____.

例句：This expression can be written as 10! and it **is read as** "10 factorial."
這個算式可以寫成 10!且讀作 "十階乘(ten factorial)"。

❹ _____ consists of _____

例句：The question **consists of** three parts, and we need to finish them one by one.
這個題目是由三個部分組成，我們需要一個一個地完成。

❺ Refer to _____.

例句：**Refer to** the figure at the right, and find the measures of the indicated angles.
如右圖所示，找出標示的各角的度數。

❻ correspond to _____.

例句：Find the length of the arc which is **corresponding to** the chord.

找出與此弦相對應的弧的長度。

❼ We can conclude that _____.

例句：So, **we can conclude that** the two events are mutually exclusive.

因此，我們可以得出結論，這兩個事件是相互排斥的。

❽ There's a trick to _____.

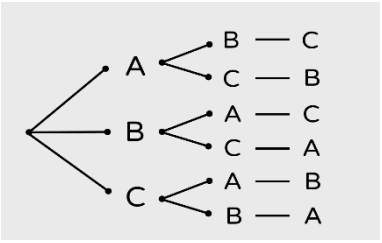
例句：**There's a trick to** get the answer quickly if you add the three equations and then divide the answer by two.

有一個小訣竅：如果你將三個式子相加，再除以二將能較快得到答案。

■ 問題講解 Explanation of Problems

ଓ 說明 ଓ

How many different ways can you arrange the three letters A, B and C in a row? If you list all the different ways, they are ABC, ACB, BAC, BCA, CAB, and CBA. So, it is clear that there are six different ways. This way is called an **exhaustive method**. In some cases, we can count the number of ways by listing them out or using a tree diagram, which helps us visualize the number of ways.



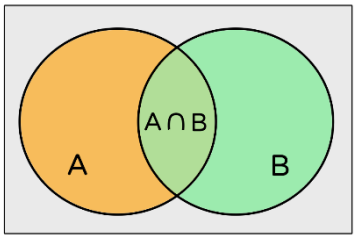
This tree diagram shows the 6 different ways for the permutations of the three letters A, B, and C.

When the number of the ways is too large to list them all, we could use the following rules to figure out the number of the ways.

Addition Rule: If one event can occur in *m* ways and a second event with no common outcomes can occur in *n* ways, then the first event or the second event can occur in *m*+*n* ways.

Multiplication Rule: If one event can occur in *m* ways and a second event can occur in *n* ways after the first event has occurred, then the two events can occur in $m \times n$ ways.

Inclusion-exclusion Rule



If *A* and *B* are both finite sets,

then: $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

$n(A \cup B)$ represents the number of elements in the union of *A* and *B*.

$A \cup B$ is read as *A* union *B* or the union of *A* and *B*.

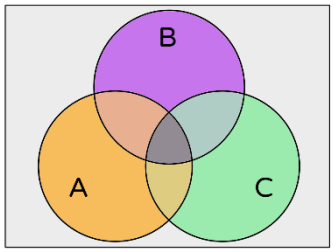
$n(A \cap B)$ represents the number of elements in the intersection of *A* and *B*.

$n(A \cap B)$ is read as *A* intersection *B* or the intersection of *A* and *B*.

Also, we can use a similar method to get

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C)$

$- n(C \cap A) + n(A \cap B \cap C)$.



Permutations

A permutation is an arrangement of objects in which the order is important.

How can we find the number of permutations of the ten letters A, B, C, D, F, G, H, I, J, and K in a row?

The Multiplication Rule can be extended to three or more elements in permutation.

You can find the number of permutations of the ten letters:

= (Choices for the 1st letter)(Choices for the 2nd letter)(Choices for the 3rd letter)(Choices for the 4th letter)(Choices for the 5th letter)...(Choices for the 10th letter)

$= 10 \cdot 9 \cdot 8 \cdot 7 \cdot \ldots \cdot 1$.

This expression can be written as 10! and read as "10 **factorial**."

For any positive integer n, the product of the integers from 1 to *n* is called ***n* factorial** and is written as $n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$.

How many ways can you arrange 4 of the 10 letters in a row?

You can find the number of permutations of the four letters:

= (Choices for the 1st letter)(Choices for the 2nd letter)(Choices for the 3rd letter)(Choices for the 4th letter)

$=10 \cdot 9 \cdot 8 \cdot 7$

$= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{10!}{6!}$

When permutations involve repeated symbols, they are called **repeated permutations (or permutations with repetition).**

If n objects are taken k times and the objects can be repeated, then there are

$n \cdot n \cdot \ldots \cdot n$ (k times) $= n^k$ ways of permutations

Core Concept

The number of permutations of n distinct objects in a row is given by:

$$P_n^n = n!$$

The number of permutations of n distinct objects taken r at a time, where $r \leq n$, is given by:

$$P_r^n = \frac{n!}{(n-r)!}$$

When $r = n$, we know: $P_r^n = \frac{n!}{(n-r)!} = \frac{n!}{(n-n)!} = \frac{n!}{0!}$ and also $P_n^n = n!$.

So, "**0!" is defined as 1**.

When $r = 0$, we know: $P_r^n = \frac{n!}{(n-r)!} = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1$.

So, the value of $P_0^n = 1$.

Core Concept

The number of permutations of n objects with m_1 of one type, m_2 of the second type, …, and m_k of the kth type is:

$$\frac{n!}{m_1!\, m_1!\, \ldots\, m_k!}$$

Combinations

A combination is a selection of objects in which order is **not** important.

Core Concept

The number of **combinations** of n distinct objects taken r at a time, where $r \leq n$, is given by

$$C_r^n = \frac{n!}{(n-r)!\, r!}$$

The permutation of n objects taken r at a time can be separated into two steps: take r objects at a time and then do the permutations of the r objects.

P_r^n (permutations of n objects taken r at a time)

$= C_r^n$ (combinations of n objects taken r at a time) × $r!$ (permutations of r objects)

Hence, $C_r^n = \frac{P_r^n}{r!} = \frac{n!}{(n-r)!\,r!}$ (n factorial over (n minus r) factorial times r factorial)

Binomial Theorem

We have learned Pascal's triangle to find binomial expansions. The table shows the coefficients in the expansions of $(x+y)^n$ correspond to combinations.

n	Pascal's Triangle as Numbers	Pascal's Triangle as Combinations	Binomial Expansion
0	1	C_0^0	$(x+y)^0 = 1$
1	1 1	C_0^1 C_1^1	$(x+y)^1 = 1x + 1y$
2	1 2 1	C_0^2 C_1^2 C_2^2	$(x+y)^2 = 1x^2 + 2xy + 1y^2$
3	1 3 3 1	C_0^3 C_1^3 C_2^3 C_3^3	$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$

The results in the table are generalized in the binomial theorem.

Core Concept

For any positive integer n,

$$(x+y)^n = C_0^n\,x^n + C_1^n\,x^{n-1}y^1 + C_2^n\,x^{n-2}y^2 + \ldots + C_{n-1}^n\,x^1y^{n-1} + C_n^n\,y^n$$

The formula for Pascal's triangle is given by:

$$C_k^n = C_{k-1}^{n-1} + C_k^{n-1}$$

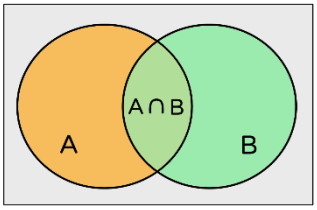
∞ 運算問題的講解 ∞

例題一

說明：取捨原理的應用。

（英文）How many integers in the range of 1 through 300 are multiples of 4 or 6?
（中文）試求 1 到 300 的整數中，是 4 的倍數或 6 的倍數的整數個數。

Teacher: Let A represent the multiples of 4, and B represent the multiples of 6.



To find the multiples of 4 or 6 is to find $n(A \cup B)$.
From the inclusion-exclusion rule,
$n(A \cup B)$ $(= n(A) + n(B) - n(A \cap B)$.
($A \cup B$ is read as A union B or the union of A and B.)
$n(A \cap B)$ represents the number of elements in the intersection of A and B.
$n(A \cap B)$ is read as A intersection B or the intersection of A and B.
Find out $n(A)$ and $n(B)$ first.

Student: $n(A) = 75$ and $n(B) = 50$.

Teacher: You are correct. What does $A \cap B$ represent? Also, find $n(A \cap B)$.

Student: $A \cap B$ represents the multiples of 12. $n(A \cap B) = 25$

Teacher: Very good. Then there are 100 integers are the multiples of 4 or 6 by using $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

老師：用 A 表示 4 的倍數，B 表示 6 的倍數，要找到 4 或 6 的倍數，就是要找到 $n(A \cup B)$。根據排容原理，$n(A \cup B) = n(A) + n(B) - n(A \cap B)$，先找出 $n(A)$和 $n(B)$。

學生：$n(A) = 75$、$n(B) = 50$.

老師：答對了，還要求出 $n(A \cap B)$，$A \cap B$ 是什麼意思？

學生：$A \cap B$ 表示 12 的倍數。$n(A \cap B) = 25$。

老師：很好。藉由 $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ 可得出總共有 100 個整數是 4 或 6 的倍數。

例題二

說明：了解運用加法原理或乘法原理的不同。

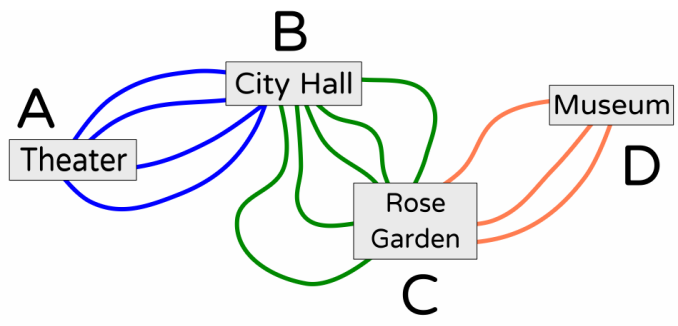
（英文）(a) For a city parade (See figure a), all floats need to start at the Theater, pass by the City Hall, cross the Rose Garden, and arrive at the Museum. How many ways can a float finish the parade?

(b) How many ways can a float finish the parade if all floats need to start at the Theater, pass by the City Hall **or** cross the Rose Garden, and arrive at the Museum (See figure b)?

（中文）(a) 在城市的花車遊行中，所有花車均需自戲院出發，經過市政府，玫瑰園，然後抵達博物館。如下圖(a)，該花車共有幾種不同的走法？

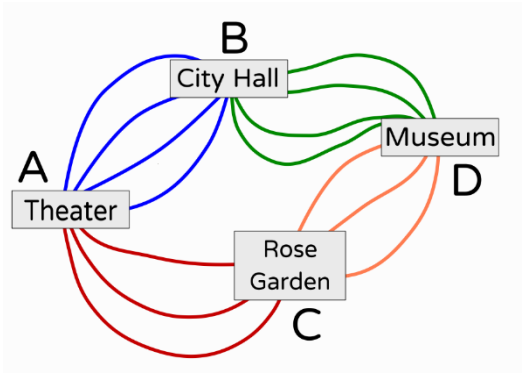
(b) 在城市的花車遊行中，所有花車均需自戲院出發，經過市政府或玫瑰園，然後抵達博物館。如下圖(b)，該花車共有幾種不同的走法？

(a)



A → B → C → D

(b)



A→ B or C →D

Teacher: In part (a), which rule should you apply?

Student: The multiplication rule.

Teacher: Correct. Using the multiplication rule would help because the floats need to pass through these spots. Find out the number of ways now.

Student: There are 60 different ways.

Teacher: Very good. In part (b), note that the float needs to "pass by the City Hall **or** cross the Rose Garden." Which rule could you use here?

Student: The addition rule.

Teacher: You are correct. What is the answer?

Student: There are 25 ways.

老師： 解(a)部分，應該要使用哪個規則呢？
學生： 乘法原理。
老師： 沒錯，可以用乘法原理，因為花車會經過這些地方。現在請找出共有多少種方法。
學生： 有 60 種不同的方法。
老師： 很好。在(b)小題的部分，要注意花車要「經過市政府或玫瑰園」。這邊要用哪個規則？
學生： 加法原理。
老師： 沒錯，答案是什麼呢？
學生： 有 25 種方法。

例題三

說明：了解重複排列的運用。

（英文）How many different car plates can be made if every car plate has three letters and four numerical digits?
（中文）若汽車車牌是由 3 個英文字母及 4 個數字 (0~9) 所組成。試問一共可以生產多少不同這樣規格的車牌？

Teacher: There are 26 letters in English, 10 different numerical digits, and all the letters and numbers can be repeated. So use the formula for a repeated permutation to find the answer.
Student: There are 175760000 car plates.
Teacher: You are correct. The answer is $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 175760000$.

老師： 英文字母全部有 26 個，數字有 10 個，且全部的字母和數字都可以重複。所以請使用重複排列的公式來找出答案。
學生： 共有 175,760,000 個。
老師： 沒錯，答案是 $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 175760000$。

例題四

說明：以乘法原理來計算排列方式，並介紹階層之運用。

（英文）Consider the number of permutations of the letters in the word April. How many ways can you arrange (a) all of the letters and (b) 3 of the letters?

（中文）將 APRIL 五個字母排列，共有多少種排列方式？若只取其中三個字母排列，則有幾種排列方式？

Teacher: As we just saw, the multiplication rule states that if there are n ways of doing something, and m ways of doing another thing after that, then there are $n \times m$ ways to perform both of these actions.
So, the number of permutations = (Choices for 1st letter)(Choices for 2nd letter)(Choices for 3rd letter)(Choices for 4th letter)(Choices for 5th letter)
Find out the number of ways that you can arrange all of the letters.

Student: The number of permutations is 120.

Teacher: You are correct. There are 120 ways you can arrange all letters in the word April. Also, you might notice that the number of permutations of 5 letters is 5!.
In part (b), the number of permutations = (Choices for 1st letter)(Choices for 2nd letter)(Choices for 3rd letter). Find the answer now.

Student: The number of permutations is 60.

Teacher: Good job!

老師： 正如所見，乘法原理表示，如果有 n 種方式做某件事，然後有 m 種方式再做另一件事，那麼進行這兩個動作的方法數就是 $n \times m$。
因此，排列方式=(第一個字母的選擇方式) × (第二個字母的選擇方式) × (第三個字母的選擇方式) × (第四個字母的選擇方式) × (第五個字母的選擇方式)。
請算出答案。

學生： 共有 120 種排列方式。

老師： 沒錯，“April”五個字母，有 120 種方法排列。同學應該有注意到 5 個字母的排列數量就是 5!。
第二個問題，若只取其中三個字母排列，排列方式則會= (第一個字母的選擇方式) × (第二個字母的選擇方式) × (第三個字母的選擇方式)，答案是多少？

學生： 共有 60 種排列方式。

老師： 很棒！

例題五

說明：運用階層來計算 n 個相異物排列的相關問題。

（英文）If we arrange the numbers 1, 3, 5, 7, and 9 into a 5-digit number, how many different numbers are there? If we list all the 5-digit numbers from least to greatest, what is the 76th number?

（中文）將 1, 3, 5, 7, 9 等五個數字排成 5 位數，共有多少種排法？若將這些 5 位數自小排到大，第 76 個數字為何？

Teacher: The number of permutations of 5 different numbers is given by $P_5^5 = 5!$. What is the product after you expand the factorial?

Student: 120.

Teacher: Good. To find the 76th number from least to greatest, we need to do it step by step. First, how many numbers are in all if the first digit is 1?

Student: $4! = 24$.

Teacher: Good. 24 numbers where the first digit is 1. It is the same situation when the first digit is 3, 5, or 7. The 76th number is one with a first digit of 7.

Step 2, on the condition of the first digit 7, the 2nd digit should start with 1.

7 1 ? ? ?

There are $3! = 6$ different numbers.

They are 71359 (73th), 71395(74th), 71539(75th), and 71593(76th).

So the 76th number is 71593.

老師： 5 個不同數字的排列數量為 $P_5^5 = 5!$，展開階乘後等於多少？

學生： 等於 120。

老師： 很好。現在一步一步來找到由小到大第 76 個數字。首先，如果最高位數是 1，總共有多少個數字？

學生： 有 $4! = 24$ 個數字。

老師： 很好，最高位數是 1 的時候有 24 個數字。最高位數是 3、5 或 7 時也是相同情況。第 76 個數字是以 7 為首位的數字。

第二步，以首位為 7 的條件下，第二位應該以 1 開頭。

7 1 ? ? ?

剩下有 3! = 6 種可能的數字。
依序是 71359(第 73 個)，71395(第 74 個)，71539(第 75 個)和 71593(第 76 個)。
因此第 76 個數字是 71593。

例題六

說明：有相同物的直線排列。

（英文）(a) How many different ways can you arrange all the letters of the word MISSISSIPPI?
(b) What is the number of permutations of the letter in the word MISSISSIPPI if the four S's are not coming together?
（中文）(a) 將密西西比 MISSISSIPPI 的英文字母重新排列，試問有多少種不同的排法？
(b) 如果四個 S 字母不能全部排列在一起，則有多少種不同的排法？

Teacher: The word MISSISSIPPI has one M, four I's, four S's, two P's and a total of 11 letters. Since some letters are repeated, we could use the formula to get $\frac{11!}{4!\,4!\,2!}$. Find the answer by expanding the factorial and dividing the common factor.
Student: 34650.
Teacher: Good. In part (b), it is more complicated to calculate the permutations if the 4 S's are not together. So, we can use the opposite way and combine the 4 S's together.

The number of permutations of the 4 S's not being together
= Total permutations − the number of permutations of the 4 S's together

We can take the 4 S's as one, then MISSISSIPPI → MIIIIPP SSSS.
So there is 1 M, 4 I's, 2 P's, and 1 SSSS.
We can think of the number of letters to be 8 when we tie the 4 S's together.
So the number of permutations is: $\frac{8!}{4!\,2!} = \frac{8\cdot7\cdot6\cdot5}{2} = 840$.

Find the number of permutations of 4 S's not coming together now.
Student: The total number of permutations of 4 S not coming together: = 34650 − 840 = 33810
Teacher: Excellent.

老師：MISSISSIPPI 這個詞有一個 M，四個 I，四個 S，兩個 P，共 11 個字母。

因為一些字母重複，我們可以列出階乘公式 $\frac{11!}{4!\,4!\,2!}$，展開後約分算出答案。

學生：答案是 34650。

老師：那(b)的話，4 個 S 不能全部排列在一起，計算排列組合更加複雜。所以我們可以使用相反的方式，將 4 個 S 組合在一起。

4 個 S 不能全部排列在一起 = 總排列數 − 4 個 S 在一起的排列數。我們可以將 4 個 S 視為一個字母，然後把 MISSISSIPPI 看作 MIIIIPP SSSS。

所以變成有 1 個 M，4 個 I，2 個 P 和 1 個 SSSS。當我們把 4 個 S 組合在一起時，可以視為有 8 個字母。因此排列數為：$\frac{8!}{4!\,2!} = \frac{8\cdot7\cdot6\cdot5}{2} = 840$。

現在算出 4 個 S 沒有全部排列在一起的排法。

學生：34650 − 840 = 33810。

老師：很好。

例題七

說明：運用組合公式。

（英文）May likes to order a hamburger at a restaurant. She can choose two different side dishes from a menu of nine. How many different combinations of side dishes could May choose?

（中文）小梅去餐廳點漢堡，還可選擇兩種不同的配菜。該餐廳提供 9 種配菜可供選擇。試問小梅有多少種不同的選擇方式？

Teacher: The order in which May chooses the side dishes is **not** important. So we should find the number of combinations.

$C_2^9 = \frac{9!}{7!\,2!}$

Expand the factorial and find the answer.

Student: There are 36 different combinations.

Teacher: Good job.

老師：小梅選擇配菜的順序不重要，因此這題要求的是組合的數目。

用 $C_2^9 = \frac{9!}{7!\,2!}$。展開來算出答案。

學生：有 36 種不同的選擇方式。

老師：很好。

例題八

說明：以乘法原理來計算排列方式，並介紹階層之運用。

（英文）Welcome Guest House has 8 rooms. One person lives in Room 01, 2 people in Room 02, 3 people in Room 03, 4 people in Room 04, …, and 8 people in Room 08. The guest house is holding a drawing activity to provide special gifts to two guests at night. In how many different ways could two guests who don't live in the same room be paired?

（中文）某民宿有 8 間房，第 1 間住 1 人，第 2 間住 2 人，第 3 間住 3 人，第 4 間住 4 人，...，第 8 間住 8 人。當晚進行抽獎活動，特獎有兩位。試問這 2 位中獎人不住在同一間房的情形有多少種？

（國家教育研究院「十二年國教數學素養導向課程」教師手冊）

Teacher: To solve this question, we need to know the total number of people living in the eight rooms. How many people live in the eight rooms?

Student: 36 people.

Teacher: Good. Second, should we use the permutation or combination formulas? Does the order matter?

Student: We should use the combination formula. The order doesn't matter.

Teacher: Correct. It is more difficult to directly find the number of different ways that two guests who do not live in the same room can be paired.

So, we can use the complement (set theory). Let A be the event where the 2 guests do not live in the same room. Then: $n(\mathrm{A}) = C_2^{36} - n(\mathrm{A'})$

It is easier to find the number of combinations when both of the guests live in the same room $n(\mathrm{A'})$.

Since only 1 person lives in Room 01, we could count the 2 guests from Room 02 which is C_2^2, $C_2^3$from Room 03, C_2^4 from Room 04, ..., and C_2^8 from Room 08. Now, find the sum of $C_2^2 + C_2^3 + C_2^4 + C_2^5 + C_2^6 + C_2^7 + C_2^8$.

Student: The sum is 84.

Teacher: Excellent. There is a trick to find the sum of $C_2^2 + C_2^3 + C_2^4 + C_2^5 + C_2^6 + C_2^7 + C_2^8$. By using the formula for Pascal's triangle: $C_k^n = C_{k-1}^{n-1} + C_k^{n-1}$, we can replace C_2^2 with C_3^3 (because $C_2^2 = 1 = C_3^3$), then we will get

$C_3^3 + C_2^3 + C_2^4 + C_2^5 + C_2^6 + C_2^7 + C_2^8$

$= C_3^4 + C_2^4 + C_2^5 + C_2^6 + C_2^7 + C_2^8 = C_3^5 + C_2^5 + C_2^6 + C_2^7 + C_2^8$

$= C_3^6 + C_2^6 + C_2^7 + C_2^8 = C_3^7 + C_2^7 + C_2^8 = C_3^8 + C_2^8 = C_3^9 = 84$

Now, find the answer of $C_2^{36} - (C_2^2 + C_2^3 + C_2^4 + C_2^5 + C_2^6 + C_2^7 + C_2^8)$

Student: It is 546.

Teacher: Excellent job! $C_2^{36} - (C_2^2 + C_2^3 + C_2^4 + C_2^5 + C_2^6 + C_2^7 + C_2^8)$

$= C_2^{36} - C_3^9 = \frac{36!}{34!\,2!} - \frac{9!}{6!\,3!} = 630 - 84 = 546$

So, there are 546 ways in which two paired guests do not live in the same room.

老師：要解這題，首先要先算出 8 間房間總共住多少人。

學生：36 人。

老師：很好。接下來，我們是要算排列還是組合？順序重要嗎？

學生：順序不重要，算組合。

老師：沒錯。

但要直接找到 2 位中獎人不住在同一間房的情形比較困難，我們可以補集合來求：

假設 2 位中獎人不住在同一間房是 A 事件，則：$n(A) = C_2^{36} - n(A')$。算出 n(A')，也就是 2 位中獎人住在同一間房的情況會比較簡單。

老師：第 1 間房間只住 1 個人，所以我們第 2 間房間選擇 2 位房客，C_2^2；

第 3 間房間選擇 2 位房客，C_2^3；第 4 間房間選擇 2 位房客，C_2^4，以此類推……，

第 8 間房間選擇 2 位房客，C_2^8。

接著，來求 $C_2^2 + C_2^3 + C_2^4 + C_2^5 + C_2^6 + C_2^7 + C_2^8$ 的和。

學生：總和為 84。

老師：答對了。算 $C_2^2 + C_2^3 + C_2^4 + C_2^5 + C_2^6 + C_2^7 + C_2^8$的和有個小訣竅，可以使用巴斯卡三角形公式：$C_k^n = C_{k-1}^{n-1} + C_k^{n-1}$

老師：我們把 C_2^2 換成 C_3^3（因為 $C_2^2 = 1 = C_3^3$），可以寫成：$C_3^3 + C_2^3 + C_2^4 + C_2^5 + C_2^6 + C_2^7 + C_2^8$

$= C_3^4 + C_2^4 + C_2^5 + C_2^6 + C_2^7 + C_2^8 = C_3^5 + C_2^5 + C_2^6 + C_2^7 + C_2^8$

$= C_3^6 + C_2^6 + C_2^7 + C_2^8 = C_3^7 + C_2^7 + C_2^8 = C_3^8 + C_2^8 = C_3^9 = 84$

老師：現在算出 $C_2^{36} - (C_2^2 + C_2^3 + C_2^4 + C_2^5 + C_2^6 + C_2^7 + C_2^8)$。

學生：546。

老師：做得很好！$C_2^{36} - (C_2^2 + C_2^3 + C_2^4 + C_2^5 + C_2^6 + C_2^7 + C_2^8) = C_2^{36} - C_3^9$

$= \frac{36!}{34!\,2!} - \frac{9!}{6!\,3!} = 630\text{-}84 = 546$

因此，2 位中獎人不住在同一間房的情形有 546 種。

∞ 應用問題 / 學測指考題 ∞

例題一

說明：應用窮舉法。

（英文）There are 8 buildings in a row, numbered 1, 2, 3, …, 8 from left to right. The telecom company plans to set up three base stations on the top of three buildings. To avoid signal interference, the base stations cannot be located in adjacent buildings. In how many different ways can the three base stations be set up if Building 3 is not an option for base station establishment?

（中文）有八棟大廈排成一列，由左至右分別編號 1, 2, 3, 4, 5, 6, 7, 8。今電信公司想選取其中三棟 大廈的屋頂分別設立一座電信基地台。若基地台不能設立於相鄰的兩棟大廈，以免訊號互相干擾，試問在 3 號大廈不設立基地台的情況下，有多少種設立基地台的選取方法？

(1) 12　(2) 13　(3) 20　(4)30　(5)35。

（111 年學測數學 B 試卷第 3 題）

Teacher: We know some restrictions in setting up the base stations from the given information. So we can use the exhaustive method to solve this question.

First, if one base station is located in Building 1, then we can list them as follows:

(i) the second one is located in Building 4. (1, 4, 6), (1, 4, 7), (1, 4, 8)

(ii) the second one is located in Building 5. (1, 5, 7), (1, 5, 8)

(iii) the second one is located in Building 6. (1, 6, 8)

Next, can you list all the arrangements if one base station is located in Building 2?

Student: (2, 4, 6), (2, 4, 7), (2, 4, 8), (2, 5, 7), (2, 5, 8), (2, 6, 8)

Teacher: Correct. The arrangements of setting up the base station in Building 1 and Building 2 are similar. How about the situation of setting up the base station in Building 4?

Student: (4, 6, 8).

Teacher: Very good. And you can find that we already listed all of the possible arrangements. So what is the answer?

Student: (2) 13

老師： 看完題目，已知設基地台的限制，因此可以用排除法來解這個題目。

首先，如果一座基地台設於 1 號大廈，我們可以列出：

(i) 第二座基地台位於 4 號大廈。 (1, 4, 6), (1, 4, 7), (1, 4, 8)

(ii) 第二座基地台位於 5 號大廈。 (1, 5, 7), (1, 5, 8)

(iii) 第二座基地台位於 6 號大廈。 (1, 6, 8)

接下來，如果一座基地台要設在 2 號大廈，能列出所有的排列方式嗎？

學生： (2, 4, 6)、(2, 4, 7)、(2, 4, 8)、(2, 5, 7)、(2, 5, 8)、(2, 6, 8)。

老師： 答對了。基地台設在 1 號和 2 號大廈的排列方式是相似的。那麼設在 4 號大廈的情況呢？

學生： (4, 6, 8)。

老師： 很好，然後你就會發現，我們已經列出所有可能的排列方式了，那麼答案是？

學生： (2) 13

例題六

（英文）An ice cream shop needs to prepare at least n buckets of ice cream with different flavors to satisfy its advertisement "there are more than 100 combinations of two ice creams with different flavors". How many ways can a customer choose two scoops (different or the same flavor) from n buckets of ice cream?

（中文）某冰淇淋店最少需準備 n 桶不同口味的冰淇淋，才能滿足廣告所稱「任選兩球不同口味冰淇淋的組合數超過 100 種」。試問來店顧客從 n 桶中任選兩球（可為同一口味）共有幾種方法？

(1) 101　(2) 105　(3) 115　(4) 120　(5) 225。

（111 年學測 A 試卷第 1 題）

Teacher: The given information shows more than 100 combinations of two ice creams with different flavors. We can get $C_2^n \geq 100$.

$$C_2^n = \frac{n!}{(n-2)!\,2!} = \frac{n\cdot(n-1)\cdot(n-2)!}{(n-2)!\,2!} = \frac{n\cdot(n-1)}{2}$$

So we get $\frac{n\cdot(n-1)}{2} \geq 100$.

Solve this inequality and find the least number n now. You have three minutes.

(After three minutes)

Ok. Time is up. What is the least number of n?

Student: 15.

Teacher: Good.

From the inequality $\frac{n\cdot(n-1)}{2} \geq 100$, we can simplify it to $n^2 - n - 200 \geq 0$.

Then, we get $n \geq \frac{1+\sqrt{801}}{2}$, where $28 \leq \sqrt{801} \leq 29$

So the least number of n is 15.

Now, can you find the number of combinations of choosing two flavors? Two scoops of the same flavor?

Student: The ways of two different flavors $= C_2^{15} = \frac{15\cdot14}{2} = 105$.

And the way of the same flavor is 15.

Teacher: Excellent. The ways of two flavors $= C_2^{15} = \frac{15 \cdot 14}{2} = 105$.

And their sum is 120. So the answer is (4) 120.

老師： 題目說:「任選兩球不同口味冰淇淋的組合數超過 100 種」,可以列出 $C_2^n \geq 100$。

$C_2^n = \frac{n!}{(n-2)!\,2!} = \frac{n \cdot (n-1) \cdot (n-2)!}{(n-2)!\,2!} = \frac{n \cdot (n-1)}{2}$，所以得到 $\frac{n \cdot (n-1)}{2} \geq 100$。

給你們三分鐘解出這個不等式，找到 n 的最小值。

（三分鐘後）好，時間到。n 的最小值是多少？

學生： 15。

老師： 答對了！從不等式 $\frac{n \cdot (n-1)}{2} \geq 100$，我們可以簡化為 $n^2 - n - 200 \geq 0$。

然後得到 $n \geq \frac{1+\sqrt{801}}{2}$，其中$28 \leq \sqrt{801} \leq 29$，所以 n 的最小值為 15。

老師： 現在，能找出任選兩球不同口味冰淇淋的組合數嗎？如果兩球口味相同呢？

學生： 任選兩球不同口味：$C_2^{15} = \frac{15 \cdot 14}{2} = 105$ 種；兩球口味相同有 15 種。

老師： 很棒！所以加起來是 120 種，答案是(4) 120。

單元六 機率
Probability

國立新竹科學園區實驗高級中等學校　印娟娟老師

■ 前言 Introduction

本單元涵蓋機率及期望值的內容。首先，介紹樣本空間與事件的定義，並運用窮舉法、樹狀圖、集合的性質等來計算樣本空間與事件的元素個數；然後介紹古典機率的定義及其相關性質；再由機率介紹期望值的定義及應用。老師在介紹本章節時，可以依各主題之配合相關範例解說，讓學生透過解題實作，深化教材中的概念與知識。因為學測等機率應用問題的單字較多，所以字彙部分仍以機率與期望值等主題相關內容為主。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
probability	**機率**	**expected value**	**期望值**
outcomes	結果	mutually exclusive	互斥的
event	事件	replacement	替換
sample space	**樣本空間**	rapid test kit	快篩檢驗試劑
trial	試驗	infectious	具傳染力的
random	隨機的	factoring	因式分解
intersection	交集	quadratic	二次的

union	聯集	consecutive	連續的
complement	**餘集**	token	代幣

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ For instance, _____.

例句：The situation is getting better now. **For instance,** the number of infected people last week was lower than the previous week.
現在情況正在好轉。例如，上週染疫人數就比前一週少。

❷ _____ be equally likely.

例句：The outcomes in a sample space S **are equally likely** if each outcome has the same probability of occurring.
樣本空間中的結果機會均等是指每個結果出現機會相等。

❸ In some cases, _____.

例句：In some cases, it is better to use the Venn diagram to find the answers.
在某些情況下，最好使用文氏圖來找答案。

❹ _____ have nothing in common.

例句：The events A and B **have nothing in common.**
事件 A 和事件 B 沒有交集。

❺ _____ at a time

例句：What is the probability of the event if we choose two red balls **at a time**?
我們一次取出兩個紅球的機率為何？

❻ It is your turn _____.

例句：Now, **it is your turn** to do the following question.

現在，輪到你來做下一題了。

■ 問題講解 Explanation of Problems

ଓ 說明 ଓ

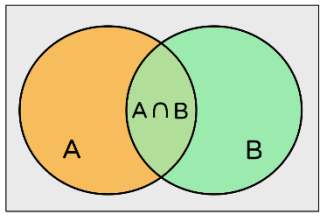
A probability experiment is a trial that has changing results. The possible results of a probability experiment are **outcomes**. A collection of one or more outcomes is an **event**. The set of all possible outcomes is called a **sample space. The probability of an event** is a measure of the chance that the event will occur. The probability of an event is a number between 0 and 1. When all outcomes are equally likely, the probability of event *A*(written as *P*(*A*)) can be found using the following:

$P(A) = \frac{n(A)}{n(S)}$ Where $n(A)$ is the number of outcomes in event *A*, and *n*(*S*) is the number of outcomes in the sample space.

For instance, when you roll a six-sided die, there are 6 possible outcomes: 1, 2, 3, 4, 5, or 6. The sample space $S = \{1, 2, 3, 4, 5, 6\}$. Then A represents the event that the number you rolled is even. In other words, $A = \{2, 4, 6\}$ (A is the set of 2, 4, and 6), and $P(A) = \frac{n(A)}{n(S)} = \frac{3}{6} = \frac{1}{2}$

The sum of the probabilities of all outcomes in a sample space is 1. So, when you know the probability of event *A*, you can find the probability of the complement of event *A*. The complement of event *A* consists of all outcomes that are not in *A* and is denoted by *A*'. The notation *A*' is read as "*A* prime."

If A and B are two events in the sample space S.



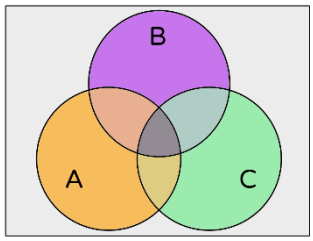
(1) $A \cap B$ (read as the intersection of events A and B) is the event that includes all outcomes that are both in A and B.

(2) $A \cup B$ (read as the union of the events of A and B) is the event that includes all outcomes that are in A or B.

(3) A'(read as A prime) is the complement of events that includes outcomes not in event A.

$A' = S - A$ and $A \cap A' = \phi$ (empty set)

Mutually exclusive. Two events are defined to be mutually exclusive if their intersection does not contain a sample point; that is, they have no outcomes in common.



For instance, A and A' are mutually exclusive.

The properties of probability:

1. $P(\phi) = 0, P(S) = 1$
2. $0 \leq P(A) \leq 1$
3. $P(A') = 1 - P(A); P(A \cap A') = 0$
4. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
5. $P(A \cup B \cup C)$

$= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$.

Expected Value

Let S be the sample space of a trial. $A_1, A_2, A_3, \ldots, A_n$ are mutually exclusive events, and $S = A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n$. For any $i = 1, 2, 3, \ldots,$ n, the probability of event A_i (denoted as $P(A_i)$) is p_i, and the value which corresponds to event A_i is m_i ($m_i \in R$), then the expected value:

$$E(x) = m_1 p_1 + m_2 p_2 + \ldots + m_n p_n$$

∞ 運算問題的講解 ∞

例題一

說明：了解樣本空間與機率。

（英文）When two six-sided dice are rolled, there are 36 possible outcomes, as shown. Find the probability of the following events.

(a) The sum is 8.　　(b) The sum is not a multiple of 5

（中文）同時擲兩顆公正骰子，則樣本空間共有 36 個元素。試求：

(1) 點數和是 8 的機率　　(2)點數和不是 5 的倍數的機率。

Teacher:

(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(6, 1)
(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)
(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)	(6, 3)
(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)	(6, 4)
(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)	(6, 5)
(1, 6)	(2, 6)	(3, 6)	(4, 6)	(5, 6)	(6, 6)

It is clear that the event (the sum is 8)= $\{(2,6),(3,5),(4,4),(5,3),(6,2)\}$.
So, what is the probability of part (a)?

Student: The probability = $\frac{5}{36}$.

Teacher: You are right. In part (b), the sum is from 2 to 12, and only 5 and 10 are the multiples of 5. So, it is easier to use $P(A') = 1 - P(A)$.
Find P (The sum is 5 or 10).

Student: $P(A)= \frac{7}{36}$.

Teacher: Very good. So $P(A') = 1 - P(A) = 1-\frac{7}{36} = \frac{29}{36}$.

老師：點數和是 8 的樣本數，包含$(2,6)$、$(3,5)$、$(4,4)$、$(5,3)$、$(6,2)$ 這五種情況。
第(1)小題的機率是多少呢？

學生：機率為 $\frac{5}{36}$。

老師：非常好。

接著第(2)小題，兩顆骰子的和介於 2 到 12 之間，其中只有 5 和 10 是 5 的倍數。因此，用 $P(A') = 1 - P(A)$比較快，先求出和為 5 或 10 的機率。

學生： $P(A)= \frac{7}{36}$

老師： 沒錯，所以 $P(A') = 1 - P(A) = 1-\frac{7}{36} = \frac{29}{36}$。

例題二

說明：運用組合公式來計算元素個數。

（英文）A bag contains 3 green, 3 blue, and 4 red balls. You randomly draw a ball out of the bag twice. The chance of any ball being drawn out is equally likely.
Find the probability of
(a) two red balls by drawing a ball with a replacement.
(b) two red balls by drawing a ball without replacement.
(c) two red balls if you select two balls at one time?

（中文）在袋子中有 3 顆綠球、3 顆藍球、4 顆紅球。若一次取一球共取兩次且每球被取出的機會均等，試求
(a) 取後放回，連續取兩球皆為紅球的機率？
(b) 取後不放回，連續取兩球皆為紅球的機率？
(c) 同時取兩球，兩球皆為紅球的機率？

Teacher: Look at the given information. We know there are 10 balls in all. What is the probability of drawing a red ball for the first time?

Student: $\frac{4}{10} = \frac{2}{5}$.

Teacher: Correct. In part (a), the 2nd ball is in the same condition as the first one because the first ball is replaced. So, the probability of 2 red balls with replacement is $\frac{2}{5} \cdot \frac{2}{5} = \frac{4}{25}$. In part (b), only 3 green, 3 blue, and 3 red balls are in the bag after the first ball is removed. So, the probability of selecting a red ball in the 2nd drawing is $\frac{1}{3}$. Find the probability of two red balls by drawing a ball without replacement.

Student: The probability is equal to: $\frac{2}{5} \cdot \frac{1}{3} = \frac{2}{15}$.

Teacher: Very good. Then what is the probability of drawing two red balls if you select two balls at a time? From the previous chapter, we know that: $n(S) = C_2^{10} = 45$ and $n(A) = C_2^4 = 6$. So apply the combination formula and find the probability now.

Student: $P(A) = \frac{C_2^4}{C_2^{10}} = \frac{6}{45} = \frac{2}{15}$.

Teacher: You are correct. Compare the answers of parts (b) and (c). You can find they are equal because drawing two red balls is similar to drawing one by one without replacement.

老師： 看完題目敘述，我們知道袋子裡總共有 10 顆球。
第一次取到紅球的機率是多少？

學生： $\frac{4}{10} = \frac{2}{5}$。

老師： 是的。(a)小題，紅球取後放回，所以取出第二顆球與第一顆球的情況相同。因此，取後放回所抽出 2 個紅球的機率是 $\frac{2}{5} \cdot \frac{2}{5} = \frac{4}{25}$。

老師： (b)小題，第一個球被取出後，袋子裡只剩下 3 顆綠球、3 顆藍球和 3 顆紅球。因此，第 2 次取到紅球的機率是 $\frac{1}{3}$。
現在請找出若球取後不放回，連續取兩球皆為紅球的機率。

學生： 機率是 $\frac{2}{5} \cdot \frac{1}{3} = \frac{2}{15}$。

老師： 非常好。那麼，如果你同時取兩球，抽出兩球皆為紅球的機率是多少？
運用前一章我們所學的，$n(S) = C_2^{10} = 45$，$n(A) = C_2^4 = 6$。現在用組合公式找出機率。

學生： $P(A) = \frac{C_2^4}{C_2^{10}} = \frac{6}{45} = \frac{2}{15}$。

老師： 答對了。比較一下(b)和(c)的答案，你可以發現是它們一樣的，因為同時取出兩顆紅球和取後不放回後再取出一顆紅球相似。

例題三

說明：運用組合公式來求解。

（英文）A bag contains 5 green and n red balls. You randomly choose two balls out of the bag, and the chance of any ball being selected is equally likely. If the probability of selecting two green balls is $\frac{2}{11}$, what is the value of n?

（中文）已知在袋子中有 5 顆綠球及 n 顆紅球。若每球被取出的機會均等，一次取兩球且兩球皆為綠球的機率為 $\frac{2}{11}$，則 n＝?

Teacher: The total possible outcomes of drawing two balls is $C_2^{5+n} = \frac{(n+5)(n+4)}{2}$ and of drawing two green balls = $C_2^5 = 10$. If the probability of selecting two green balls is $\frac{2}{11}$, then $\frac{C_2^5}{C_2^{5+n}} = \frac{2}{11}$. Use the combinations formula to solve for n.

Student: $n = 6$.

Teacher: You are right. From the equation $\frac{C_2^5}{C_2^{5+n}} = \frac{2}{11}$, we can expand it and get $\frac{\frac{5\cdot4}{2}}{\frac{(n+5)(n+4)}{2}} = \frac{2}{11}$. Then, we get $n^2 + 9n - 90 = 0$. By factoring this quadratic equation, we get $n = 6$.

老師： 取出兩顆球共有 $C_2^{5+n} = \frac{(n+5)(n+4)}{2}$ 種結果；取兩顆綠球的結果共有 $C_2^5 = 10$ 種。

如果一次取到兩顆綠球的機率是 $\frac{2}{11}$，那就代表 $\frac{C_2^5}{C_2^{5+n}} = \frac{2}{11}$。

用排列組合公式求 n。

學生： $n=6$。

老師： 答對了，我們可以展開 $\frac{C_2^5}{C_2^{5+n}} = \frac{2}{11}$ 得到 $\frac{\frac{5\cdot4}{2}}{\frac{(n+5)(n+4)}{2}} = \frac{2}{11}$。

然後進一步化簡成 $n^2 + 9n - 90 = 0$，最後因式分解二次方程式，答案是 $n=6$。

例題四

說明：以排列組合的方法來求機率。

（英文）According to a survey, 60% of people carry at least 1 reusable water bottle. If 3 people are selected randomly, what is the probability that at most 1 of them will be carrying a reusable water bottle?

（中文）根據一項調查，60%的人至少攜帶 1 個可重複使用的水壺。試求如果任意選擇 3 個人，最多只有 1 個人攜帶可重複使用的水壺的機率。

（改編自 Big ideas math-Algebra II 10.5 Practice B）

Teacher: The event "at most one of the three people carries a reusable water bottle" is equal to "none of them or exactly one of them carrying a reusable water bottle".
So, $P(n = 0) = C_3^3 \cdot (40\%)^3 = 0.064$
It is your turn to find out $P(n = 1)$.

Student: $P(n = 0) = C_1^3 \cdot (60\%) \cdot (40\%)^2 = 0.288$.

Teacher: Excellent. So the answer is: 0.064 + 0.288 = 0.352 or 35.2%

老師：「任選 3 個人，最多只有 1 個人帶水壺」事件等於「沒有人或只有 1 個人帶水壺」。因此，$P(n=0) = C_3^3 \cdot (40\%)^3 = 0.064$。換你們算 $P(n=1)$。

學生：$P(n = 0) = C_1^3 \cdot (60\%) \cdot (40\%)^2 = 0.288$。

老師：很好，所以答案是：0.064 + 0.288 = 0.352 或 35.2%。

例題五

說明：運用取捨原理及文氏圖求解。

（英文）A and B are two events in the sample space S, and $P(A)=\frac{2}{3}$, $P(B)=\frac{1}{2}$, and $P(A\cap B)=\frac{1}{4}$. Which of the following statement(s) is(are) true? (maybe more than one answer)

（中文）已知 A、B 為樣本空間 S 中的二個事件，其中 $P(A)=\frac{2}{3}$，$P(B)=\frac{1}{2}$ 及 $P(A\cap B)=\frac{1}{4}$。則下列選項哪些正確？（複選題）

(1) $P(A\cup B)=\frac{5}{6}$　(2) $P(B')=\frac{1}{2}$　(3) $P(A-B)=\frac{5}{12}$

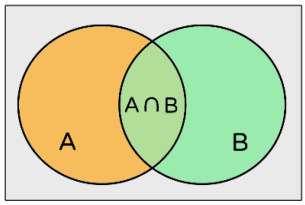
(4) $P(A'\cap B')=\frac{5}{6}$　(5) $P(A'\cap B)=\frac{1}{4}$

（改編自翰林第二冊課本第 3 章例題）

Teacher: We know that $P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$=\frac{2}{3}+\frac{1}{2}-\frac{1}{4}=\frac{11}{12}$

And $P(B')=1-P(B)$. Find $P(B')$ now.



Student: $P(B')=\frac{1}{2}$.

Teacher: Good. The next step is to find $P(A-B)$. It is clearer if we use a Venn diagram. By using the Venn diagram, we can find $P(A-B)=P(A)-P(A\cap B)$.
Is part (3) correct?

Student: Yes, it is correct. P $(A-B)=\frac{5}{12}$.

Teacher: Excellent.

In part (4), we know $(A\cup B)'=A'\cap B'$, and $P(A'\cap B')=1-P(A\cup B)=\frac{1}{12}$.

By using the Venn diagram, can you figure out what $P(A'\cap B)$ is?

Student: Yes, $P(A'\cap B)=\frac{1}{4}$.

Teacher: You are right. We can find that $P(A'\cap B)=P(B)-P(A\cap B)=P(B-A)=\frac{1}{4}$.

So the answers are (2), (3), and (5).

老師：我們知道 $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{2}{3} + \frac{1}{2} - \frac{1}{4} = \frac{11}{12}$。

且 $P(B') = 1 - P(B)$，請算出 $P(B')$。

學生：$P(B') = \frac{1}{2}$。

老師：很好，下一步是求 $P(A-B)$。畫出文氏圖會比較好理解，觀察後可以求 $P(A - B) = P(A) - P(A \cap B)$，選項(3)對嗎？

學生：對，$P(A - B) = \frac{5}{12}$。

老師：太棒了。接著看選項(4)，我們知道$(A \cup B)' = A' \cap B'$，

且 $P(A' \cap B') = 1 - P(A \cup B) = \frac{1}{12}$。可以用文氏圖求出 $P(A' \cap B)$ 是多少嗎？

學生：可以，$P(A' \cap B) = \frac{1}{4}$。

老師：答對了，$P(A' \cap B) = P(B) - P(A \cap B) = P(B-A) = \frac{1}{4}$。所以答案是(2)、(3)和(5)。

∞ 應用問題 / 學測指考題 ∞

例題一

（英文）A box contains 3 black balls and 2 white balls. You randomly choose one ball out of the box, and the chance of a ball being selected is equally likely. What is the game's expected value if the prize of selecting a black ball is $50 and the prize of selecting a white ball is $100?

（中文）有一箱子，內有 3 黑球與 2 白球。有一遊戲，從箱子中任取出一球。假設每一顆球被取出的機率都相同，若取出黑球可得獎金 50 元，而取出白球可得獎金 100 元，則下列哪一個選項是此遊戲的獎金期望值？

(1) 70 元　(2) 75 元　(3) 80 元　(4) 85 元　(5) 90 元

（100 年學測第 1 題）

Teacher: The chance of randomly selecting a black ball is $\frac{3}{5}$ and the chance of selecting a white ball is $\frac{2}{5}$. Find the answer by using the formula of the expected value now.

Student: 70 dollars.

Teacher: Good. The expected value $E(x) = m_b\, p_b + m_w\, p_w = 50 \cdot \frac{3}{5} + 100 \cdot \frac{2}{5} = 70$.

So, the answer is (1), 70 dollars.

老師：隨機取出黑球的機率是 $\frac{3}{5}$，取出白球的機率是 $\frac{2}{5}$。用期望值的公式計算答案。

學生：70。

老師：很好，期望值 $E(x) = m_b\, p_b + m_w\, p_w = 50 \cdot \frac{3}{5} + 100 \cdot \frac{2}{5} = 70$，答案是選項(1)70元。

例題二

（英文）An online game introduces a card mechanism of 10 consecutive draws. It takes 1500 tokens for each "10 consecutive draws." The probability of winning a gold card is 2% in the first nine draws and 10% in the tenth draw. One student has 23,000 tokens, and keeps playing the game until he doesn't have enough tokens to continue. What is the expected value of the number of gold cards drawn by the student?

（中文）有一款線上遊戲推出「十連抽」的抽卡機制，「十連抽」意思為系統自動做十次的抽卡動作。若每次「十連抽」需用 1500 枚代幣，抽中金卡的機率在前九次皆為 2%，在第十次為 10%。今某生有代幣 23000 枚，且不斷使用「十連抽」，抽到不能再抽為止。則某生抽到金卡張數的期望值為_____張。

（111 年學測 A 試卷第 13 題）

Teacher: What is the expected value for one round of 10 consecutive draws if the chance of winning a gold card is 2% on the first nine draws and 10% on the tenth draw?

Student: It is 0.28.

Teacher: Yes, you are correct. It is $9 \cdot 2\% + 10\% = 0.28$. Then, how many rounds can a student play with 23,000 tokens?

What is the quotient if 23,000 is divided by 1500?

Student: The quotient is 15.

Teacher: Very good! We know that this student could play 15 rounds of 10 consecutive draws. The expected value of the 15 rounds is equal to 0.28 times 15, and the product is 4.2. So, the answer is 4.2 gold cards.

老師： 一次十連抽，在前九次抽中金卡的機率為 2%，第十次為 10%，求一次十連抽的期望值是多少？

學生： 是 0.28。

老師： 沒錯，答對了。期望值為 $9 \cdot 2\% + 10\% = 0.28$。那麼，某生有 23,000 枚代幣，他可以抽幾次？

23,000 除以 1500 是多少？

學生： 15。

老師： 非常好！我們知道他可以玩 15 次十連抽。

15 次的期望值等於$0.28 \times 15 = 4.20$。因此答案，抽到金卡張數的期望值為 4.2 張。

例題三

（英文）A bag contains 10 balls in blue, green, and yellow. You randomly choose two balls out of the bag, and the chance of a ball being selected is equally likely. The probability of selecting two blue balls is $\frac{1}{15}$, and the probability of selecting two green balls is $\frac{2}{9}$. What is the probability of selecting two balls of different colors?

（中文）袋中有藍、綠、黃三種顏色的球共 10 顆。今從袋中隨機抽取兩顆球(每顆球被抽中的機率相等)，若抽出的兩顆球皆為藍色的機率為 $\frac{1}{15}$，皆為綠色的機率為 $\frac{2}{9}$，則從袋中隨機抽出兩球，此兩球為相異顏色的機率為何？(化為最簡分數)。

（111 學測數學 B 試卷第 16 題）

Teacher: According to the given information, we know that the probability of choosing two blue balls is $\frac{1}{15}$, and $\frac{2}{9}$ for green balls. Assume there are *n* blue balls, then the probability of choosing two blue balls: $\frac{1}{15} = \frac{C_2^n}{C_2^{10}}$.

We can expand it and get: $\frac{\frac{n\cdot(n-1)}{2}}{45} = \frac{1}{15}$.

Then we get: $n^2 - n - 6 = 0$, and *n* = 3. There are 3 blue balls.

Now, it is your turn to find the number of green balls in the same way.

You have three minutes.

(After three minutes.) How many green balls are there?

Student: 5 green balls.

Teacher: Great. So the number of yellow balls is 2. By using the property of complement in probability, P(two balls of different colors) $= 1 - P$(two balls of the same color) $= 1 - [P(\text{two blue}) + P(\text{two green}) + P(\text{two yellow})]$

where $P(\text{two yellow}) = \frac{C_2^2}{C_2^{10}} = \frac{1}{45}$.

Find the probability of two balls of different colors now.

Student: The probability is $\frac{31}{45}$.

Teacher: Yes, you are correct.

老師：看完題目已知，抽出的兩顆球皆為藍色的機率為 $\frac{1}{15}$，皆為綠色的機率為 $\frac{2}{9}$。假設有 n 顆藍球，那麼兩顆球皆為藍色的機率：$\frac{1}{15} = \frac{C_2^n}{C_2^{10}}$。

展開得到 $\frac{\frac{n\cdot(n-1)}{2}}{45} = \frac{1}{15}$，化簡 $n^2 - n - 6 = 0$，$n = 3$，因此袋中有 3 顆藍球。

現在換你們用同樣的方法求綠球的數量，給大家三分鐘。

（三分鐘後）有多少綠球呢？

學生：有 5 顆綠球。

老師：太好了，所以算下來袋中會有 2 顆黃球。

接著用機率補集的性質，P(兩顆相異顏色的球) $=1-$ P(兩顆相同顏色的球)$=1-$ $[P(\text{兩顆藍球}) + P(\text{兩顆綠球}) + P(\text{兩顆黃球})]$

其中 $P(\text{兩顆黃球}) = \frac{C_2^2}{C_2^{10}} = \frac{1}{45}$。

現在找出兩球為相異顏色的機率。

學生：機率是 $\frac{31}{45}$。

老師：答對了！

例題四

（英文）We randomly select three different numbers from the integers 1 through 9, and the chance of each number being selected is equally likely. What is the probability that the product of three numbers is a perfect square?

（中文）從 1, 2, 3, 4, 5, 6, 7, 8, 9 這九個數中任意取出三個相異的數，每數被取出的機率皆相等，則三數乘積是一完全平方數的機率為何？（化成最簡分數）

（110 年學測填充題 C）

Teacher: According to the given information, we can list out all the outcomes of this event: (1, 2, 8), (1, 4, 9), (2, 3, 6), (2, 4, 8), (2, 8, 9), (3, 6, 8).
So, the total number of the outcomes as a perfect square is 6.
Now, it's your turn to find the total number of outcomes in the sample space.
What is it?

Student: $C_3^9 = 84$.

Teacher: Good. So, the probability of the event is equal to $\frac{6}{84} = \frac{1}{14}$.

老師：看完題目，可以列出乘積是完全平方數的所有結果：
(1, 2, 8)，(1, 4, 9)，(2, 3, 6)，(2, 4, 8)，(2, 8, 9)，(3, 6, 8)，總共有 6 個。
現在換大家計算樣本空間的總數是多少。

學生：$C_3^9 = 84$。

老師：很好，所以機率為 $\frac{6}{84} = \frac{1}{14}$。

例題五

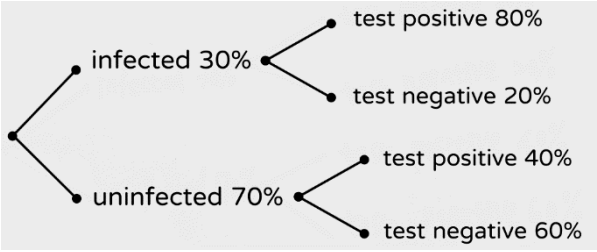
（英文）It is known that 30% of the population in an area is infected with an infectious disease. The rapid test kits for the infectious disease have positive and negative results. The probability of a rapid test kit is known to have an 80% chance of identifying an infected person as positive and a 60% chance of identifying an uninfected person as negative. To reduce the error evaluation that the infected person was misjudged as negative, experts suggest three consecutive tests. Among those who test negative in a single test, the probability of being infected is P. However, among those who test negative in three consecutive tests, the probability of being infected is P'. Which of the following values is the closest to the value of $\frac{P}{P'}$?

（中文）已知某地區有 30% 的人口感染某傳染病。針對該傳染病的快篩試劑檢驗，有陽性或陰兩結果。已知該試劑將染病者判為陽性的機率為 80%，將未染病者判為陰性的機率則為 60%。為降低該試劑將染病者誤判為陰性的情況，專家建議連續採檢三次。若單次採檢判為陰性者中，染病者的機率為 P；而連續採檢三次皆判為陰性者中，染病者的機率為 P'。試問' $\frac{P}{P'}$ 最接近哪一選項？

(1) 7　(2) 8　(3) 9　(4) 10　(5) 11

（111 年學測數學 A 試卷第 5 題）

Teacher: We can use a tree diagram to show the result of rapid test kits for infected people and uninfected people:



We can find that:

$$P = \frac{\text{test negative and is infected}}{\text{test negative and is infected} + \text{test negative and is uninfected}}$$

$$= \frac{30\%\cdot 20\%}{30\%\cdot 20\% + 70\%\cdot 60\%} = \frac{1}{8}$$

And

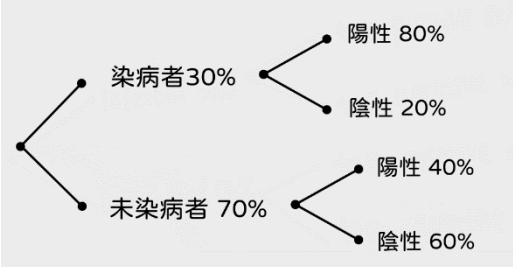
$$P' = \frac{\text{3 consecutive tests negative and is infected}}{\text{test negative and is infected} + \text{test negative and is uninfected}}$$

Try to find the value of P' now.

Student: $P' = \frac{30\%\cdot(20\%)^3}{30\%\cdot(20\%)^3 + 70\%\cdot(60\%)^3} = \frac{1}{64}$.

Teacher: Good. So, we get: $\frac{P}{P'} = 8$. The answer is (B).

老師：可以畫樹狀圖來表示染病者和非染病者的快篩結果：



得到：

機率 $P = \frac{\text{陰性染病者}}{\text{陰性染病者+陰性未染病者}} = \frac{30\%\cdot 20\%}{30\%\cdot 20\%+70\%\cdot 60\%} = \frac{1}{8}$

老師：機率 $P' = \frac{\text{3 次皆為陰性染病者}}{\text{陰性染病者+陰性未染病者}}$。

現在試著算出 P'。

學生：$P' = \frac{30\%\cdot(20\%)^3}{30\%\cdot(20\%)^3+70\%\cdot(60\%)^3} = \frac{1}{64}$。

老師：非常好。所以得出：$\frac{P}{P'} = 8$。答案是(B)。

單元七 三角比、廣義角與極坐標
Trigonometric Ratios, Generalized Angles and Polar Coordinates

國立新竹科學園區實驗高級中等學校　印娟娟老師

■ 前言 Introduction

本單元三角比首先介紹銳角三角比的正弦、餘弦、正切及三角比的基本性質；接著將角度的概念推廣至廣義角及其性質；最後介紹極坐標與直角坐標的轉換。因為本單元英文字彙非常多，建議老師在介紹內容及範例時配合繪圖解說，可以讓學生強化對三角比之各英文字彙的記憶並了解問題內容。同時，本單元包含許多基本性質及關係式，建議在示範例題時老師能反覆提問，讓學生能熟悉學習的內容及性質，最後以應用問題或學測題來加深學生們對本單元學習內容的運用。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
ratio	比	vertex	頂點
trigonometric	三角的	counterclockwise	逆時針的
sine (sin)	正弦	clockwise	順時針的
cosine (cos)	餘弦	generalized angle	廣義角
tangent (tan)	正切	standard position angle	標準位置角
abbreviation	縮寫	**quadrant**	象限
opposite side	對邊	coincide	重合

adjacent side	鄰邊	rotation	旋轉
hypotenuse	斜邊	**coterminal angle**	同界角
angle of elevation	仰角	complementary angle	餘角
angle of depression	俯角	rectangular coordinate system	直角坐標系統
directed angle	**有向角**	polar axis	極軸
initial side	始邊	**polar coordinates**	極坐標
terminal side	終邊	**reference angle**	參考角
inclination	傾斜	offset distance	偏移距離
ratio	比	vertex	頂點

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ ____ with reference to _____.

例句：Please find the ratios of sine, cosine and tangent **with reference to** triangle ABC.

請參考三角形 ABC 來求出正弦、餘弦及正切的值。

❷ ____ is adjacent to _____.

例句：$\overline{BC}$ **is adjacent to** $\angle C$.

$\overline{BC}$ 是 $\angle C$ 的鄰邊。

❸ _____ coincide with _____.

例句：The terminal side of $\angle A$ **coincides with** line k.

$\angle A$ 的終邊和直線 k 重合。

❹ **convert _____ to _____.**

例句：How do you convert 35 degrees Celsius to Fahrenheit?

如何將攝氏 35 度轉換成華氏？

❺ **differ by _____.**

例句：The length of the two tunnels **differs by** 2 kilometers.

兩座隧道的長度相差 2 公里。

❻ **move on to _____.**

例句：Now, let us **move on to** the next item on the question.

接下來，讓我們繼續下一個題目。

❼ **_____ so that _____.**

例句：The like terms can be combined **so that** they can be solved efficiently.

可以將相似的項目組合起來，以便更有效地處理。

❽ **apply to _____**

例句：The generalized rule **applies to** all different angles.

這廣義的規則適用於所有不同的角度。

■ 問題講解 Explanation of Problems

☙ 說明 ❧

The ratio of the sides of a right triangle is called the trigonometric ratio. We will define them with reference to the right triangle ABC, as shown below.

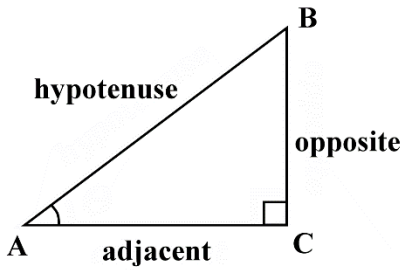
In the right triangle ABC with $\angle C = 90°$, $\angle A$ is acute, and the three sides opposite the angles $\angle A, \angle B, and\ \angle C$ are a, b, and c.

Then the three common trigonometric ratios are:

$$\sin A = \frac{opp}{hyp}\ \left(\frac{opposite\ side}{hypotenuse}\right) = \frac{a}{c}$$

$$\cos A = \frac{adj}{hyp}\ \left(\frac{adjacent\ side}{hypotenuse}\right) = \frac{b}{c}$$

$$\tan A = \frac{opp}{adj}\ \left(\frac{opposite\ side}{adjacent\ side}\right) = \frac{a}{b}$$



The abbreviations opp, adj, and hyp refer to the lengths of the side opposite $\angle A$, the side adjacent to $\angle A$, and the hypotenuse, respectively.

In a right triangle, the hypotenuse is the longest side, greater than the other two sides.

Thus, we can find: $0 < \sin A < 1$, $0 < \cos A < 1$, and $\tan A > 0$

The table gives the trigonometric ratios for the special angles 30°, 45°, and 60°.

Angles	sin	cos	tan
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

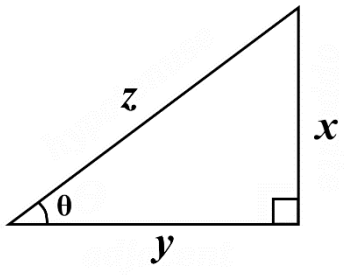
Trigonometric Identities

If θ is an acute angle, we can find the following trigonometric identities to be true by using the definitions of the trigonometric ratios.

1. $\tan\theta = \frac{\sin\theta}{\cos\theta}$
2. $\sin^2\theta + \cos^2\theta = 1$
3. $\sin(90° - \theta) = \cos\theta\ ;\ \cos(90° - \theta) = \sin\theta$

We can sketch a right triangle with an acute angle θ as shown.

By definition, $\sin\theta = \frac{opp}{hyp} = \frac{x}{z}$ and $\cos\theta = \frac{adj}{hyp} = \frac{y}{z}$.



$\frac{\sin\theta}{\cos\theta} = \frac{\frac{x}{z}}{\frac{y}{z}} = \frac{x}{z} \cdot \frac{z}{y} = \frac{x}{y} = \tan\theta$

So, it is true that $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

Similarly, we can show $\sin^2\theta + \cos^2\theta = 1$ as follows:

$\sin^2\theta + \cos^2\theta = (\frac{x}{z})^2 + (\frac{y}{z})^2 = \frac{x^2}{z^2} + \frac{y^2}{z^2} = \frac{x^2+y^2}{z^2} = \frac{z^2}{z^2} = 1.$

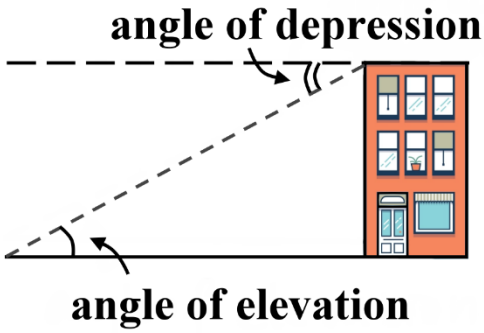
(using the Pythagorean Theorem, $x^2 + y^2 = z^2$)

The other acute angle in the right triangle is $(90° - \theta)$.

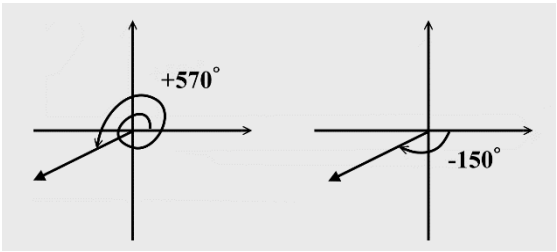
$\sin(90° - \theta) = \frac{y}{z} = \cos\theta$ and $\cos(90° - \theta) = \frac{x}{z} = \sin\theta$.

The angle of elevation and angle of depression

The angle of elevation and the angle of depression are often used in trigonometry word problems. For example, if a person stands and looks up at an object, the **angle of elevation** is the angle between the horizontal line of sight and the object. Similarly, if a person stands and looks down at an object, **the angle of depression** is the angle between the horizontal line of sight and the object.



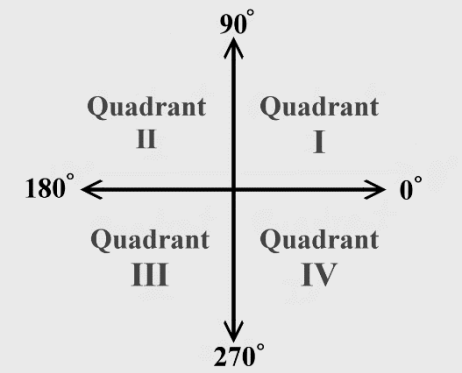
An angle is considered standard if its vertex is located at the origin and the initial side lies on the positive axis.



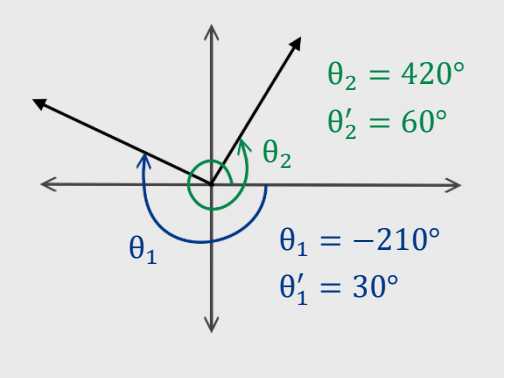
If the rotation is counterclockwise, the angle has a positive measure. In contrast, if the rotation is clockwise, the angle is negative. **Besides,** the terminal side of an angle can rotate more than 360°.

In the coordinate plane, the x-axis and y-axis divide the plane into four **quadrants**. When referring to the coordinate plane, the measure of an angle θ in the 1st quadrant is from 0° to 90°(0° < θ < 90°).

The measure of an angle θ in the 2nd quadrant is from 90° to 180° (90° < θ < 180°), and so on. Angles in the standard position with terminal sides lying on the x-axis or y-axis are called **quadrantal angles**. The measures of quadrantal angles are all multiples of 90° ($\theta = n \cdot 90°$, n is an integer).



The **reference angle** θ' of an angle θ is a positive acute angle formed by the terminal side of θ and the x-axis, as shown on the right. If an angle lies in Quadrant I or Quadrant IV, its reference angle is equal to the positive acute angles between its terminal side and the positive x-axis.

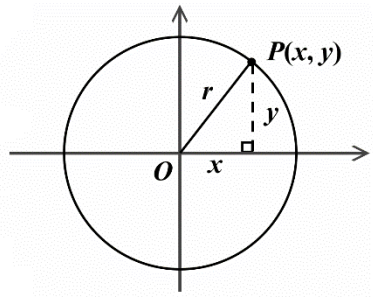


For an angle that lies in Quadrant II or III, its reference angle is equal to the positive acute angles between its terminal side and the negative x-axis.

For example, the reference angles of $\theta_1(-210°)$and $\theta_2(420°)$ are 30° and 60°, respectively.

Trigonometric Ratios of Generalized Angles

We can generalize the right-triangle definitions of trigonometric ratios so that they can apply to any angle in a standard position. Let P(x, y) be a point on the terminal side of the angle θ and r is the hypotenuse of the triangle formed by the terminal side $\overline{OP}$ and the x-axis as shown, then we can find r = $\sqrt{x^2 + y^2}$.



The definitions of the trigonometric ratios are as follows:

sin $\theta = \frac{y}{r}$, cos $\theta = \frac{x}{r}$, and tan $\theta = \frac{y}{x}$

The following table shows the values of the trigonometric ratios in different quadrants.

Quadrant	I	II	III	IV
sin θ	+	+	−	−
cos θ	+	−	−	+
tan θ	+	−	+	−

If angle θ is a quadrantal angle, then the *x*-coordinate or *y*-coordinate is 0 because the terminal side of angle θ lies on the *x*-axis or *y*-axis. When θ = 90°, *x* = 0 and *y* = *r*. Then, sin 90° = 1 , cos 90° = 0 , and tan 90° = undefined.

The following table shows the values of the trigonometric ratios if θ is a quadrantal angle.

Quadrantal angle	0°	90°	180°	270°
sin θ	0	1	0	-1
cos θ	1	0	-1	0
tan θ	0	undefined	0	undefined

Trigonometric Ratios Formula

There are more formulas besides $\sin(90° - \theta) = \cos\theta$ &$\cos(90° - \theta) = \sin\theta$, which are listed in the Trigonometric Identities.

The formulas are listed as follows:

1. $\sin(180° - \theta) = \sin\theta$; $\cos(180° - \theta) = -\cos\theta$
2. $\sin(180° + \theta) = -\sin\theta$; $\cos(180° + \theta) = -\cos\theta$
3. $\sin(-\theta) = -\sin\theta$; $\cos(-\theta) = \cos\theta$
4. $\tan(180° - \theta) = -\tan\theta$; $\tan(180° + \theta) = tan\theta$;
   $\tan(-\theta) = -\tan\theta$ ($\theta \neq \pm 90°$ or their coterminal angles.)

Polar Coordinate System

The polar coordinate system is a two-dimensional coordinate system in which points are given by an angle and a distance from a central point known as the pole. The pole coincides with the origin, and the polar axis coincides with the x-axis in the Rectangular Coordinate System.

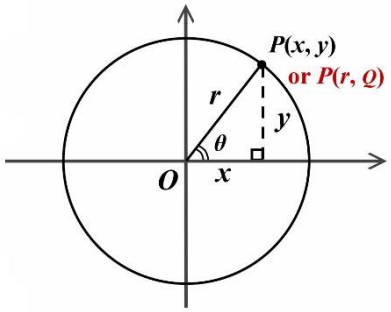
So, a relationship can be established between the polar coordinates and the rectangular coordinates for a point.

Converting between Polar and Rectangular Coordinates

The polar coordinates $P(r, \theta)$ can be converted to the rectangular coordinates (x, y) by:

$x = r \cos \theta$

$y = r \sin \theta$



From the above formulas, r and θ can be defined in terms of x and y as follows:

r = $\sqrt{x^2 + y^2}$

sin $\theta = \frac{y}{r}$, cos $\theta = \frac{x}{r}$

∞ 運算問題的講解 ∞

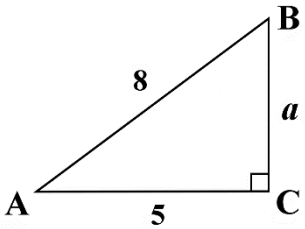
例題一

說明：運用三角比的定義求解。

（英文）Let $\angle A$ be an acute angle such that cos $A = \frac{5}{8}$. Find the values of sin A and tan A.

（中文）已知$\angle A$ 為銳角且 cos $A = \frac{5}{8}$，求 sin A 和 tan A 的值。

Teacher: Sketch a right triangle showing an acute angle A.
Label the adjacent side 5 and the hypotenuse 8.
Now, find the opposite side.



Student: The opposite side is $\sqrt{39}$. ($\sqrt{39}$ is read as the square root of thirty-nine.)

Teacher: You are correct. From the Pythagorean Theorem, we get $a^2 + 5^2 = 8^2$.
(a^2 is read as a squared.)
So, $a = \sqrt{64 - 25} = \sqrt{39}$. Find the values of sin A and tan A.

Student: sin $A = \frac{\sqrt{39}}{8}$ and tan $A = \frac{\sqrt{39}}{5}$.

Teacher: Both of the answers are correct.

老師： 畫一個∠A 為銳角的直角三角形。標上鄰邊為 5，斜邊為 8。現在，找出對邊。

學生： 對邊是$\sqrt{39}$。

老師： 你答對了。根據畢氏定理，我們得到 $a^2 + 5^2 = 8^2$。

所以，$a = \sqrt{64 - 25} = \sqrt{39}$。求 $\sin A$ 和 $\tan A$ 的值。

學生： $\sin A = \frac{\sqrt{39}}{8}$，$\tan A = \frac{\sqrt{39}}{5}$。

老師： 都答對了！

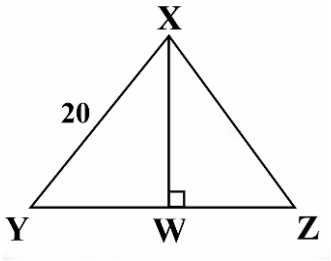
例題二

說明：運用三角比求邊長。

（英文）In $\triangle XYZ$, $\overline{XW} \perp \overline{YZ}$ ($\overline{XW}$ is perpendicular to $\overline{YZ}$), $\sin Y = \frac{3}{5}$, $\tan Z = \frac{12}{5}$ and $\overline{XY} = 20$. Find the perimeter of $\triangle XYZ$.

（中文）在$\triangle XYZ$ 中，已知 $\overline{XW} \perp \overline{YZ}$， $\sin Y = \frac{3}{5}$ ，$\tan Z = \frac{12}{5}$ ，且 $\overline{XY} = 20$。求$\triangle XYZ$ 的周長。

Teacher: In $\triangle XYW$, $\sin Y = \frac{3}{5}$ and $\overline{XY} = 20$.



Because $\sin Y = \frac{opp}{hyp} = \frac{XW}{XY}$.

So, $\frac{3}{5} = \frac{XW}{20}$. $\overline{XW} = 12$.

It's your turn to find the length of $\overline{YW}$.

Student: $\overline{YW} = 16$.

Teacher: Correct. You can find $\overline{YW}$ by applying the Pythagorean Theorem.

Next, in $\triangle XWZ$, $\tan Z = \frac{opp}{adj} = \frac{\overline{XW}}{\overline{WZ}} = \frac{12}{5}$.

What is the length of $\overline{WZ}$?

Student: $\overline{WZ} = 5$.

Teacher: Excellent. Now, find the length of $\overline{XZ}$ and the perimeter of $\triangle XYZ$.

Student: $\overline{XZ} = 13$ and the perimeter of $\triangle XYZ = 54$.

Teacher: Excellent.

老師：　在△XYW 中，$\sin Y = \frac{3}{5}$，且 $\overline{XY} = 20$。

因為 $\sin Y = \frac{\text{對邊}}{\text{斜邊}} = \frac{XW}{XY}$，所以 $\frac{3}{5} = \frac{XW}{20}$，$\overline{XW} = 12$。現在換你們找出 $\overline{YW}$ 的長度

學生：　$\overline{YW} = 16$.

老師：　正確。可以應用畢氏定理找到 $\overline{YW}$ 的長度。

接下來，在△XWZ 中，$\tan Z = \frac{\text{對邊}}{\text{鄰邊}} = \frac{\overline{XW}}{\overline{WZ}} = \frac{12}{5}$。$\overline{WZ}$ 的長度是多少？

學生：　$\overline{WZ} = 5$。

老師：　太好了。現在算出 $\overline{XZ}$ 的長度和 △XYZ 的周長。

學生：　$\overline{XZ} = 13$，△XYZ 的周長為 54。

老師：　非常棒！

例題三

說明：運用三角比的基本關係。

（英文）If α is an acute angle and $\sin\alpha + \cos\alpha = \frac{4}{3}$, evaluate:

(1) $\sin\alpha \times \cos\alpha$.　(2) $\sin^3\alpha + \cos^3\alpha$.

（中文）已知α為銳角且 $\sin\alpha + \cos\alpha = \frac{4}{3}$，求下列各式的值。

(1) $\sin\alpha \times \cos\alpha$　(2) $\sin^3\alpha + \cos^3\alpha$

Teacher:　We can square each side of the original equation $\sin\alpha + \cos\alpha = \frac{4}{3}$ and get:

$(\sin\alpha + \cos\alpha)^2 = \frac{16}{9}$.

Expand the left side and get: $\sin^2\alpha + 2\sin\alpha \cdot \cos\alpha + \cos^2\alpha = \frac{16}{9}$.

Now, find the value of $\sin\alpha \cdot \cos\alpha$ by applying the identity $\sin^2\alpha + \cos^2\alpha = 1$.

Student:　The value of $\sin\alpha \cdot \cos\alpha$ is $\frac{7}{18}$.

Teacher:　Yes, you are correct. Next, evaluate Part (2) $\sin^3\alpha + \cos^3\alpha$.

Student:　We can factor $(\sin^3\alpha + \cos^3\alpha)$ as:

$(\sin\alpha + \cos\alpha)(\sin^2\alpha - \sin\alpha \cdot \cos\alpha + \cos^2\alpha)$.

Try to find the answer by using the result in Part (1) $\sin\alpha \cdot \cos\alpha = \frac{7}{18}$.

Teacher: The value is $\frac{22}{27}$.

Student: Excellent. The value of $(\sin^3\alpha + \cos^3\alpha)$

$= (\sin\alpha + \cos\alpha)(\sin^2\alpha - \sin\alpha \cdot \cos\alpha + \cos^2\alpha)$

$= \frac{4}{3} \times \left(1 - \frac{7}{18}\right) = \frac{22}{27}$.

老師： 我們先將原方程式 $\alpha + \cos\alpha = \frac{4}{3}$ 兩邊平方，得到：$(\sin\alpha + \cos\alpha)^2 = \frac{16}{9}$。

左邊展開，得到：$\sin^2\alpha + 2\sin\alpha\cos\alpha + \cos^2\alpha = \frac{16}{9}$。

現在，利用恆等式 $\sin^2\alpha + \cos^2\alpha = 1$ 找出 $\sin\alpha \cdot \cos\alpha$ 的值。

學生： $\sin\alpha \cdot \cos\alpha$ 的值為 $\frac{7}{18}$。

老師： 是的，答對了。接下來，算第(2)小題 $\sin^3\alpha + \cos^3\alpha$。

老師： 我們可以將$(\sin^3\alpha + \cos^3\alpha)$ 因式分解為：

$(\sin\alpha + \cos\alpha)(\sin^2\alpha - \sin\alpha \cdot \cos\alpha + \cos^2\alpha)$

試著代入第(1)小題算出來的 $\sin\alpha \cdot \cos\alpha = \frac{7}{18}$。

學生： 答案是 $\frac{22}{27}$。

老師： 很好。 $(\sin^3\alpha + \cos^3\alpha)$ 的值

$= (\sin\alpha + \cos\alpha)(\sin^2\alpha - \sin\alpha \cdot \cos\alpha + \cos^2\alpha)$

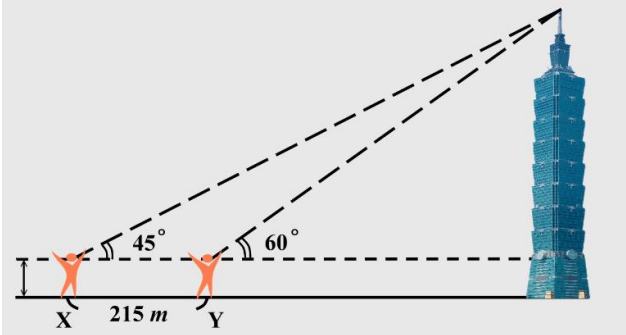
$= \frac{4}{3} \times \left(1 - \frac{7}{18}\right) = \frac{22}{27}$。

例題四

說明：運用仰角或俯角於三角測量應用問題。

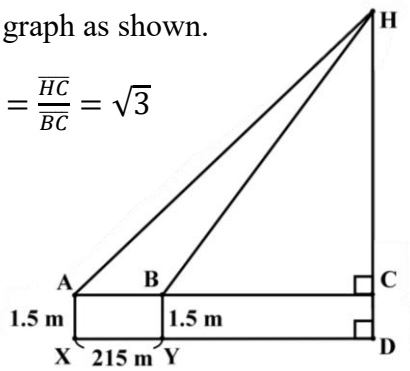
（英文）Min wants to measure the height of Taipei 101. He stands at point X viewing Taipei 101, and the angle of the elevation is 45°. Then, he walks 215 m toward Taipei 101, and the angle of the elevation is 60°at point Y. If his eye level is 1.5 m above the ground, what is the height of Taipei 101?

（中文）小明要測量台北 101 大樓的高度。他先在地上 X 點測得樓頂仰角為 45 度，然後朝 101 大樓方向走 215 公尺後在 Y 點測得樓頂仰角為 60 度。若小明的眼睛高度為 1.5 公尺，試求台北 101 大樓的高度為何？



（示意圖）

Teacher: According to the given information, we can draw a graph as shown.



Then we get: $\tan A = \frac{\overline{HC}}{\overline{AC}} = 1$ (tan 45°) and $\tan B = \frac{\overline{HC}}{\overline{BC}} = \sqrt{3}$ ≈1.732(tan 60°).

So, $\overline{HC} = \overline{AC}$ and $\overline{HC} = 1.732\ \overline{BC}$.

Now, find the length of $\overline{BC}$.

Student: $\overline{BC}$ = 293.72 m.

Teacher: You are correct.

Because $\overline{AC} = \overline{HC} = 1.732\ \overline{BC}$ and also $\overline{AC} = \overline{AB} + \overline{BC} = 215 + \overline{BC}$.

So, $0.732\ \overline{BC} = 215$ and $\overline{BC} = 293.72$ m.

Next, find the height of Taipei 101.

Student: The height of Taipei 101 is about 510 m.

Teacher: Excellent.

The height $\overline{HD} = \overline{HC} + \overline{CD}$

$= 1.732\ \overline{BC} + 1.5$

≈ 510.

The answer 510 m is close to the real height of 509.2 m of Taipei 101.

老師：　根據已知條件，我們可以繪製以下示意圖。然後我們得到：

$\tan A = \frac{\overline{HC}}{\overline{AC}} = 1$ (tan 45°)，以及 $\tan B = \frac{\overline{HC}}{\overline{BC}} = \sqrt{3} \approx 1.732$ (tan 60°)。

因此，$\overline{HC} = \overline{AC}$ 且 $\overline{HC} = 1.732\ \overline{BC}$。

現在，找出 $\overline{BC}$ 的長度。

學生：　$\overline{BC} = 293.72$ 公尺。

老師：　答對了。因為 $\overline{AC} = \overline{HC} = 1.732$，且 $\overline{AC} = \overline{AB} + \overline{BC} = 215 + \overline{BC}$.。所以，

$0.732\ \overline{BC} = 215$，$\overline{BC} = 293.72$ 公尺。

接下來，算出台北 101 的高度。

學生：　台北 101 的高度約為 510 公尺。

老師：　很棒。高度是 $\overline{HD} = \overline{HC} + \overline{CD} = 1.732\ \overline{BC} + 1.5 \approx 510$

答案 510 公尺很接近台北 101 的實際高度 509.2 公尺。

例題五

說明：求廣義角的三角比。

（英文）Let $A(5, -12)$ be a point on the terminal side of an angle θ in the standard position.

Evaluate $\sin\theta$, $\cos\theta$, and $\tan\theta$.

（中文）若 $A(5, -12)$為標準位置角 θ 終邊上的一點，試求 $\sin\theta$、$\cos\theta$ 和 $\tan\theta$。

Teacher:　The trigonometric ratios of the generalized angles are:

$\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, and $\tan\theta = \frac{y}{x}$ $(r = \sqrt{x^2 + y^2})$.

Find r when $x = 5$ and $y = -12$.

Student:　$r = \sqrt{x^2 + y^2} = 13$.

Teacher:　Correct. Then, can you explain the three trigonometric ratios?

Student:　$\sin\theta = \frac{-12}{13}$, $\cos\theta = \frac{5}{13}$, and $\tan\theta = \frac{-12}{5}$.

Teacher:　Good job.

老師：　一般角的三角比為：

$\sin\theta = \frac{y}{r}$，$\cos\theta = \frac{x}{r}$，$\tan\theta = \frac{y}{x}$ (其中 $r = \sqrt{x^2 + y^2}$)

當 $x = 5$、$y = -12$ 時，r 是多少。

學生：$r=\sqrt{x^2+y^2}=13$。

老師：正確。那麼，你能說明一下這三個三角比嗎？

學生：$\sin\theta=\frac{-12}{13}$，$\cos\theta=\frac{5}{13}$，$\tan\theta=\frac{-12}{5}$。

老師：做得很好。

例題六

說明：求廣義角的同義角、象限及參考角。

（英文）(a) Find the smallest positive angle and the greatest negative angle that are coterminal with the angle 2023°.

(b) Find the Quadrant where the angle 2023° is located and its reference angle

（中文）(a) 試求 2023°最小的正同界角與最大的負同界角。

(b) 試求 2023°所在之象限及其參考角。

Teacher: There are many coterminal angles of 2023°, depending on what multiple of 360° is added or subtracted.
Because $2023°=5\times 360°+223°$,
223° is the smallest positive coterminal angle.
What is the greatest negative coterminal angle of 2023°?

Student: $-137°$.

Teacher: Good. Next, we know the coterminal angle 223° is located in the 3rd Quadrant.
Find the reference angle.

Student: The reference angle is $223°-180°=43°$

Teacher: Excellent.

老師：題目要求的 2023°，根據增減 360°的倍數會得到許多同界角。
因為 $2023°=5\times 360°+223°$，所以 223° 是最小的正同界角。那最大負同界角是多少呢？

學生：$-137°$。

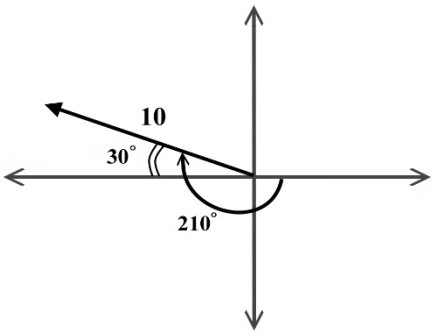
老師： 很好。接下來，我們知道同界角 223° 位於第三象限。找出其參考角。
學生： 參考角是 $223°-180°=43°$。
老師： 太棒了。

例題七

說明：極坐標與直角坐標的轉換。

（英文）(a) Evaluate tan(−210°).
(b) Convert the polar coordinates (10, −210°) to (x, y) in the rectangular coordinate system.
（中文）(a) 試求 tan(−210°) 之值。
(b) 將極坐標 (10, −210°) 轉換成直角坐標 (x, y)。

Teacher: The angle −210º is coterminal with 150º.
The reference angle is 30º. Find tan(−210º).



Student: $\tan(-210º) = -\tan 30º = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$.

Teacher: Good job. The tangent ratio is negative in Quadrant II.
So, $\tan(-210º) = -\tan 30º$.
Next, convert the polar coordinates (10, −210°) to (x, y).
In Quadrant II, the sine ratio is positive, and the cosine ratio is negative.
Find out the values of the coordinates (x, y).

Student: $x = r \cdot \cos\theta = 10 \cdot \cos(-210º) = 10 \cdot (-\cos 30º) = -5\sqrt{3}$.
$y = r \cdot \sin\theta = 10 \cdot \sin(-210º) = 10 \cdot \sin 30º = 5$.

Teacher: Yes, you are correct. The rectangular coordinates are: $(-5\sqrt{3}, 5)$.

老師： 角度-210° 與 150° 互為同界角，參考角是 30°。
求出 $\tan(-210^\circ)$是多少？

學生： $\tan(-210^\circ) = -\tan 30^\circ = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$。

老師： 很好。在第二象限，正切是負的。所以，$\tan(-210^\circ) = -\tan 30^\circ$。
接下來，將極坐標 $(10, -210°)$ 轉換為直角坐標 (x, y)。在第二象限，正弦為正，餘弦為負。找出坐標 (x, y) 的值。

學生： $x = r \cdot \cos\theta = 10 \cdot \cos(-210^\circ) = 10 \cdot (-\cos 30^\circ) = -5\sqrt{3}$.
$y = r \cdot \sin\theta = 10 \cdot \sin(-210^\circ) = 10 \cdot \sin 30^\circ = 5$.

老師： 是的，答對了。直角坐標是：$(-5\sqrt{3}, 5)$。

例題八

說明：斜率 m 與正切 $\tan\theta$的關係。

（英文）The terminal side of an angle θ coincides with the line $3x + 4y = 0$. Find:

(1) $\tan\theta$

(2) $\frac{3\sin\theta - 2\cos\theta}{\sin\theta + 2\cos\theta}$

（中文）若角θ的終邊與直線 $3x + 4y = 0$ 重和，試求：

(1) $\tan\theta$

(2) $\frac{3\sin\theta - 2\cos\theta}{\sin\theta + 2\cos\theta}$

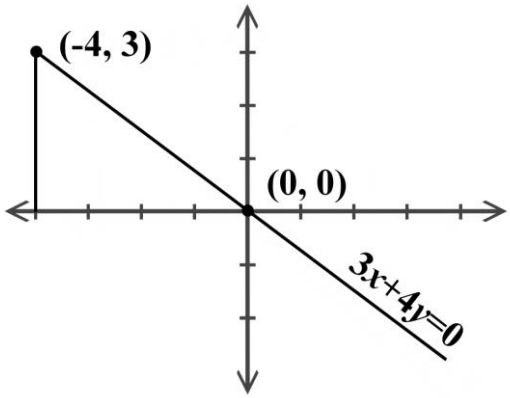
Teacher: Draw a sketch of the line. The slope of a line is the change in the y coordinate with respect to the change in x coordinate.

Hence, the slope of the line $m = \frac{\text{change in } y}{\text{change in } x} = \tan\ \theta$.

No matter whether the terminal side of angle θ lies in the Quadrant II or Quadrant IV of line $3x + 4y = 0$, the tangent ratio stays the same.

$\tan(180^\circ + \theta) = \tan\theta$.

Find the tangent ratio now.



Student: $\tan\theta = -\frac{3}{4}$.

Teacher: Yes, you are correct. The line $3x + 4y = 0$ has the slope-intercept form $y = -\frac{3}{4}x$.

So, the slope is $-\frac{3}{4}$ which is also $\tan\ \theta$.

Next, when the terminal side of angle θ is in Quadrant II, we can find $\sin\ \theta = \frac{3}{5}$ and $\cos\ \theta = -\frac{4}{5}$.

What are the sine and cosine ratios if θ is in Quadrant IV?

Student: $\sin\theta = -\frac{3}{5}$ and $\cos\theta = \frac{4}{5}$

Teacher: Very good. Now, find Part (b) when θ is in Quadrants II or IV.
Compare the values in both Quadrants. Are they the same?

Student: Yes, they are the same. The value is $-\frac{17}{5}$.

Teacher: Excellent.

老師：先畫出這條直線的示意圖。一條直線的斜率是 y 坐標的變化量除以 x 坐標的變化量。因此，直線的斜率就是 $\frac{y\text{的變化量}}{x\text{的變化量}} = \frac{\Delta y}{\Delta x} = \tan\ \theta$。

無論角θ的終邊位於線條 $3x + 4y = 0$ 的第二象限還是第四象限，正切都是相同的。因為 $\tan(180^\circ + \theta) = \tan\theta$。現在求$\tan\theta$是多少。

學生：$\tan\theta = -\frac{3}{4}$。

老師：是的，你說得對。直線 $3x + 4y = 0$ 的斜截式是 $y = -\frac{3}{4}x$。所以，斜率是$-\frac{3}{4}$，也

就是 tan θ。接下來，當角 θ 的終邊位於第二象限時，我們可以找到

$\sin\theta = \frac{3}{5}$ 和 $\cos\theta = -\frac{4}{5}$。

如果 θ 在第四象限，正弦和餘弦會是多少？

學生： $\sin\theta = -\frac{3}{5}$，$\cos\theta = \frac{4}{5}$。

老師： 非常好。接著，當 θ 在第二象限或第四象限時，算出第(2)小題。比較 θ 在兩個象限的值，是相同的嗎？

學生： 是的，算出來都是 $-\frac{17}{5}$。

老師： 太棒了。

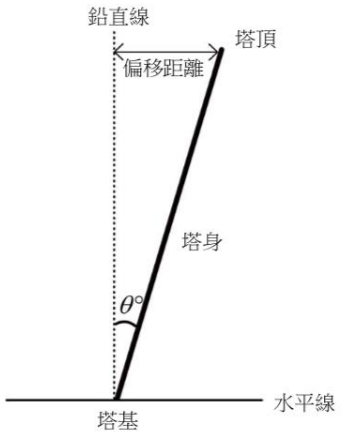
∞ 應用問題 / 學測指考題 ∞

例題一

說明：運用三角比測量距離。

（英文）There are two towers of the same height. The inclinations of the two towers are α^o and β^o, respectively. If $\sin \alpha^o = \frac{1}{5}$, $\sin \beta^o = \frac{7}{25}$ and the offset distance of the two towers differs by 20 meters, find the difference in the distance from the tops of their towers to the ground.

（中文）假設有塔高相等的兩座鐵塔，它們的傾斜度 $\alpha°$, $\beta°$分別滿足 $\sin \alpha^o = \frac{1}{5}$, $\sin \beta^o = \frac{7}{25}$。已知兩座鐵塔的偏移距離相差 20 公尺，試求它們的塔頂到地面之距離相差多少公尺。

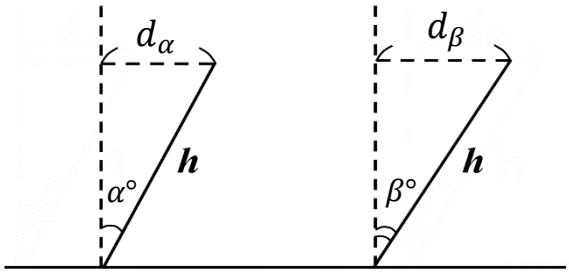


（111 年學測數學 B 第 20 題）

Teacher: Assume that the height of the two towers is h.
Because the inclinations of the two towers are α^o and β^o, respectively,

$\sin \alpha^o = \frac{1}{5} = \frac{d_\alpha}{h}$ and

$\sin \beta^o = \frac{7}{25} = \frac{d_\beta}{h}$.



Then $d_\alpha = \frac{1}{5}h$ and $d_\beta = \frac{7}{25}h$.

Solve h by knowing that $d_\beta - d_\alpha = 20$.

Student: $h = 250$.

Teacher: Yes, you are correct. We get $h = 250$ because $d_\beta - d_\alpha = 20 = \frac{2}{25}h$.

Let's move on. Find $\cos \alpha$ and $\cos \beta$ before getting the distance of the two towers from the top to the ground.

Student: $\cos \alpha = \frac{\sqrt{24}}{5} = \frac{2\sqrt{6}}{5}$ and $\cos \beta = \frac{24}{25}$.

Teacher: Excellent.
So, the difference in the distance from the tops of the two towers
$= h \cdot \cos\alpha - h \cdot \cos\beta$
$= (100\sqrt{6} - 240)$ m.

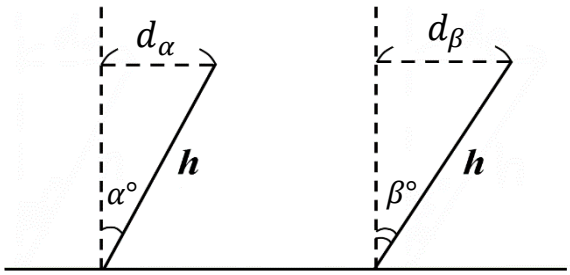
老師：假設兩座鐵塔的塔高均為 h。因為它們的傾斜度分別為α^o 及 β^o。所以

$\sin\alpha^o = \frac{1}{5} = \frac{d_\alpha}{h}$ 且

$\sin\beta^o = \frac{7}{25} = \frac{d_\beta}{h}$。

因此 $d_\alpha = \frac{1}{5}h$ 且$d_\beta = \frac{7}{25}h$。

已知兩座鐵塔的偏移距離($d_\beta - d_\alpha$)相差 20 公尺，求出塔高 h。



學生：$h = 250$.

老師：正確。因為$d_\beta - d_\alpha = 20 = \frac{2}{25}h$，我們可以求得塔高為 250 公尺。

接下來，在求塔頂到地面之距離前先找出 $\cos\alpha$和 $\cos\beta$之值。

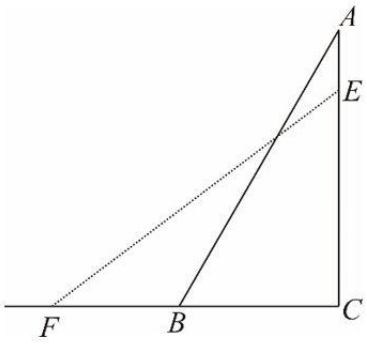
學生：$\cos\alpha = \frac{\sqrt{24}}{5} = \frac{2\sqrt{6}}{5}$，$\cos\beta = \frac{24}{25}$。

老師：太棒了。所以它們的塔頂到地面之距離差為
$h \cdot \cos\alpha - h \cdot \cos\beta$
$= (100\sqrt{6} - 240)$ 公尺。

例題二

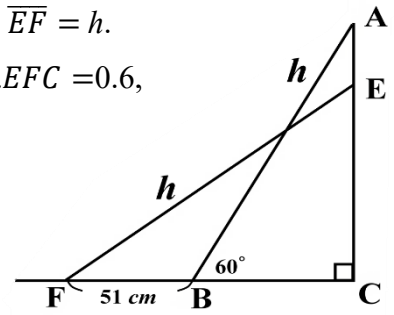
（英文）The ladder $\overline{AB}$ is leaning against the wall $\overline{AC}$ which is perpendicular to the ground $\overline{BC}$. The angle formed by the ladder and the ground is 60° ($\angle ABC = 60°$). If the ladder is pulled from point B to point F ($\overline{BF} = 51$ cm) and $\sin\angle EFC = 0.6$, find the length of the ladder $\overline{AB}$.

（中文）如右圖所示（只是示意圖），將梯子$\overline{AB}$ 靠在與地面垂直的牆 $\overline{AC}$ 上，測得與水平地面的夾角∠ABC 為 60°。將在地面上的底 B 沿著地面向外拉 51 公分到點 F （$\overline{BF}$ = 51 公分），此時梯子$\overline{EF}$ 與地面的夾角 $\angle EFC$ 之正弦值為 $\sin\angle EFC = 0.6$，則梯子長 $\overline{AB}$ =_____公分。



（107 年學測數學選非選題 2）

Teacher: Assume the length of the ladder is h, then $\overline{AB} = \overline{EF} = h$.
Given that $\angle ABC = 60°$, $\overline{BF} = 51$ cm, and $\sin\angle EFC = 0.6$,
find $\overline{BC}$ in terms of h.



Student: $\overline{BC} = 0.5\,h$.

Teacher: You are correct. Because $\cos 60° = \frac{1}{2} = \frac{\overline{BC}}{h}$. So, $\overline{BC} = 0.5\,h$.

Similarly, find cos ∠EFC and $\overline{FC}$ in terms of h.

Student: $\cos\angle EFC = 0.8$ and $\overline{FC} = 0.8h$.

Teacher: Correct. We can find $\cos\angle EFC = \sqrt{1 - 0.6^2} = 0.8$, and therefore $\overline{FC} = 0.8h$.
Now, solve the equation for h by using the equation $\overline{FC} = 51 + \overline{BC}$.

Student: $h = 170$.

Teacher: Excellent. So, the length of the ladder $\overline{AB}$ is 170 cm.

老師：假設梯子的長度為 h，則 $\overline{AB} = \overline{EF} = h$。已知 $\angle ABC = 60°$，$\overline{BF} = 51$ 公分，且 $\sin\angle EFC = 0.6$。$\overline{BC}$ 要如何以 h 表示？

學生：$\overline{BC} = 0.5\,h$。

老師： 沒錯。因為 $60° = \frac{1}{2} = \frac{\overline{BC}}{h}$，所以 $\overline{BC} = 0.5\ h$。同樣地，算出 cos∠EFC，並以 h 表示 $\overline{FC}$。

學生： cos∠EFC =0.8，$\overline{FC}$ =0.8h.

老師： 正確。我們可以求得 cos∠EFC = $\sqrt{1-0.6^2} = 0.8$，$\overline{FC}$ =0.8h。
現在，用 $\overline{FC} = 51 + \overline{BC}$ 這個方程式解 h。

學生： h=170。

老師： 太棒了。所以梯子的長度 $\overline{AB}$ 是 170 公分。

單元八 三角比的性質
Properties of Trigonometric Ratios

國立新竹科學園區實驗高級中等學校　印娟娟老師

■ 前言 Introduction

本單元三角比的性質主要介紹三角形面積公式、正弦定理、餘弦定理。首先，由夾角為銳角、直角或鈍角等不同角度來證明三角形面積公式；再由三角形面積公式推導出正弦定理；最後證明餘弦定理及由餘弦定理推導出海龍公式。在問題詳解中附上各定理的英文證明及相關用語供老師參考。三角形面積公式、正弦定理、餘弦定理的應用廣泛，老師在介紹本章節時可以先以圖形標示已知條件，再依已知條件來說明哪一項定理或公式較適用，同時深化學生對本單元的內容與公式。歷屆學測題目中運用正弦或餘弦定理相當多，最後以學測題來加深學生們對本章節的了解。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
altitude	**高度**	necessarily	必然地
obtuse angle	鈍角	**proportional**	成比例的
included angle	夾角	**circumscribe**	外接
tetrahedron	**四面體**	observation	觀察
thus	因此	simultaneously	同時地

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ ____ not necessarily _____.

例句：The included angle is **not necessarily** acute when we apply the triangle area formula.

當使用三角型面積公式時，夾角未必必須是銳角。

❷ _____ be proportional to _____.

例句：The profits are **proportional to** the revenue.

獲利和收入成正比。

❸ _____ circumscribes _____.

例句：Circle O **circumscribes** the square $ABCD$.

圓 O 是正方形 $ABCD$ 的外接圓。

❹ Thus, we can _____.

例句：**Thus, we can** get the length of $\overline{AB}$ by knowing that $\triangle ABC$ is isosceles.

因此，由 $\triangle ABC$ 是等腰三角形可以得到 $\overline{AB}$ 的長度。

❺ _____ derive from _____.

例句：The law of sines can be **derived from** the formulas of the area of $\triangle ABC$.

正弦定理可以由 $\triangle ABC$ 面積公式推導出來。

❻ No matter whether _____, _____.

例句：**No matter whether** $\angle B$ is acute, right, or obtuse, the formula of length $\overline{CD}$ is still equal to $b \sin A$.

無論 $\angle B$ 是銳角、直角或鈍角，$\overline{CD}$ 的長度公式都是 $b \sin A$。

■ 問題講解 Explanation of Problems

☙ 說明 ❧

Areas of Triangles

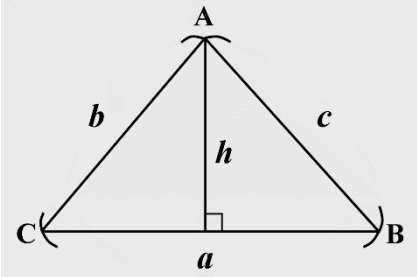
In $\triangle ABC$, a, b, and c are the three sides opposite $\angle A, \angle B, and \angle C$.

The area of $\triangle ABC = \frac{1}{2} ab\ sinC = \frac{1}{2} bc\ sinA = \frac{1}{2} ca\ sinB$

The area of a triangle is area = $\frac{1}{2}$ × base × height. If we know two sides and the included angle of a triangle, then we can find the height by using the trigonometric ratios, and from this, we can find the area.

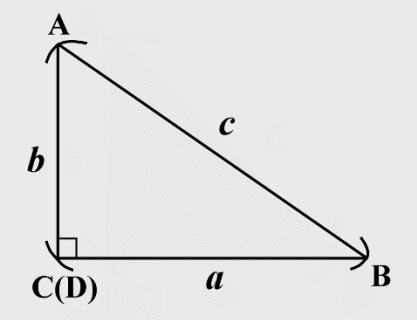
We can discuss three different cases as follows.

(1) If $\angle C$ is an acute angle, we can draw an altitude $\overline{AD}$ from point A to the opposite side $\overline{BC}$. Then the height of the triangle is given by $h = b \cdot \sin C$. Thus, the area of $\triangle ABC$ $= \frac{1}{2}$ × base × height $= \frac{1}{2} ab\ sinC$.

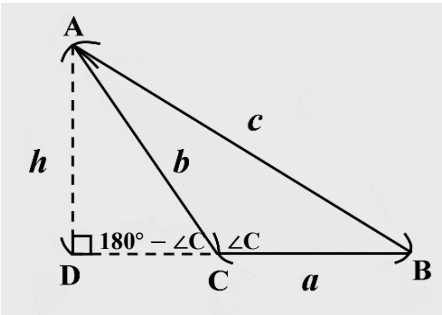


(2) If $\angle C$ is a right angle, then the triangle's height is given by $h = b = b \cdot \sin C$ ($\sin 90° = 1$).

So, the area of $\triangle ABC$ $= \frac{1}{2}$ × base × height $= \frac{1}{2} ab \sin C$.



(3) If $\angle C$ is an obtuse angle, we can draw an altitude $\overline{AD}$ from point A to the baseline $\overleftrightarrow{BC}$. Then the height of the triangle is given by $h = b \cdot \sin(180° - \angle C) = b \cdot \sin C$ because $\sin(180° - \angle C) = \sin C$. The area of $\triangle ABC$ $= \frac{1}{2}$ × base × height $= \frac{1}{2} ab\ sinC$.



We can conclude that the area of $\triangle ABC$ is $\frac{1}{2} ab\ sinC$ and the included angle is not necessarily acute. Similarly, we can get the area of $\triangle ABC$ $= \frac{1}{2} bc \sin A = \frac{1}{2} ca\ sinB$.

The Law of Sines

In $\triangle ABC$, a, b, and c are the three sides opposite $\angle A, \angle B, and \angle C$. R is the radius of the circumscribed circle of $\triangle ABC$.

We have $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$.

The law of sines says that in any triangle, the lengths of the sides are proportional to the sines of the corresponding opposite angles.

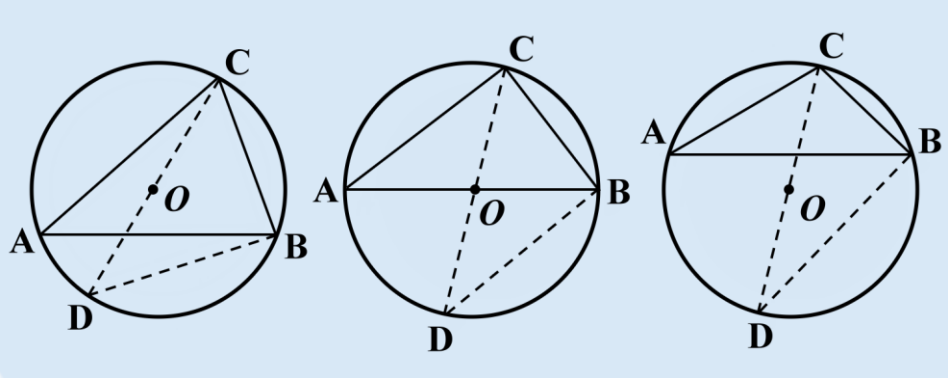
To see why the law of sines is true, we can derive from the formulas we get from the formula:

The area of $\triangle ABC = \frac{1}{2} ab\, sinC = \frac{1}{2} bc\, sinA = \frac{1}{2} ca\, sinB$

Multiplying all by $\frac{2}{abc}$ gives $\frac{sinC}{c} = \frac{\sin A}{a} = \frac{\sin B}{b}$

Rewriting the proportion gives the law of sines.

$\frac{a}{\sin A} = \frac{b}{sinB} = \frac{c}{\sin C}$.



We can derive $\frac{a}{\sin A} = 2R$ in three cases: $\triangle ABC$ is acute, right, or obtuse.

Draw $\overrightarrow{CO}$ and intersect the circumscribed circle O at point D.

We know $\angle A \cong \angle D$ because the two angles intercept the same arc.

So, $\frac{a}{\sin A} = \frac{a}{sinD}$.

In $\triangle CDB$, $\frac{a}{\sin D} = \frac{\overline{CD}}{sin\angle CBD} = 2R$ since $\overline{CD} = 2R$ and $sin\angle CBD = 90° = 1$.

Hence, we have $\frac{a}{\sin A} = \frac{b}{sinB} = \frac{c}{\sin C} = 2R$.

The Law of Cosines

In $\triangle ABC$, a, b, and c are the three sides opposite $\angle A, \angle B, and \angle C$.

We have

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

We derive only the first of the three equations because the other two can be derived in the same way.

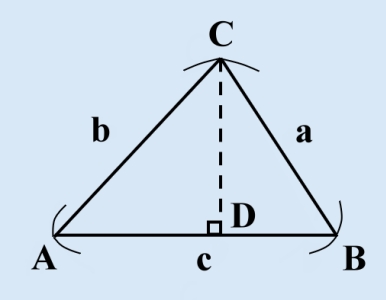
We can draw an altitude $\overline{CD}$ from C to the opposite side $\overline{AB}$ ($\overleftrightarrow{AB}$).

When $\triangle ABC$ is:

(1) an acute triangle

$\overline{BD} = \overline{AB} - \overline{AD}$

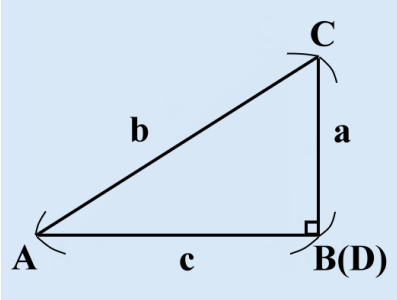
$= c - b\cos A$



(2) a right triangle

$\overline{BD} = \overline{AB} - \overline{AD}$

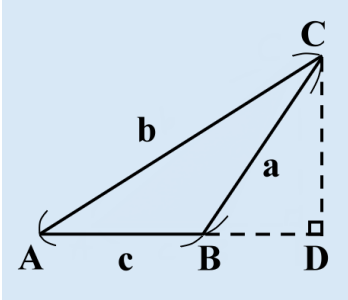
$= c - b\cos A$



(3) an obtuse triangle

$\overline{BD} = \overline{AD} - \overline{AB}$

$= b\cos A - c$



No matter whether $\angle B$ is acute, right, or obtuse, we get:

$\overline{CD} = b\sin A$ and $\overline{BD} = |c - b\cos A|$.

By applying the Pythagorean Theorem, we get:

$$\overline{BC}^2 = \overline{CD}^2 + \overline{BD}^2$$

$$a^2 = (b\sin A)^2 + (c - b\cos A)^2$$

$$= b^2\sin^2 A + c^2 - 2bc\cos A + b^2\cos^2 A$$

$$= b^2(\sin^2 A + \cos^2 A) + c^2 - 2bc\cos A$$

$$= b^2 + c^2 - 2bc\cos A$$

Also, we have $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

Heron's Formula

By the law of cosines, we know: $\cos C = \frac{a^2+b^2-c^2}{2ab}$.

The area of $\triangle ABC = \frac{1}{2}\, ab \sin C$

$$= \frac{1}{2}\, ab \sqrt{1-\cos^2 C}$$

$$= \frac{1}{2}\, ab \sqrt{(1+\cos C)(1-\cos C)}$$

$$= \frac{1}{2}\, ab \sqrt{(1+\frac{a^2+b^2-c^2}{2ab})(1-\frac{a^2+b^2-c^2}{2ab})}$$

$$= \frac{1}{2}\, ab \sqrt{(\frac{(a+b)^2-c^2}{2ab})(\frac{c^2-(a-b)^2}{2ab})}$$

$$= \sqrt{\frac{(a+b+c)(a+b-c)(c-a+b)(c+a-b)}{16}}$$

$$= \sqrt{(\frac{a+b+c}{2})(\frac{a+b-c}{2})(\frac{b+c-a}{2})(\frac{a+c-b}{2})}$$

$$= \sqrt{s\,(s-c)(s-a)(s-b)} \text{ where s} = \frac{a+b+c}{2}$$

$$= \sqrt{s\,(s-a)(s-b)(s-c)} \text{ where s} = \frac{a+b+c}{2}$$

∞ 運算問題的講解 ∞

例題一

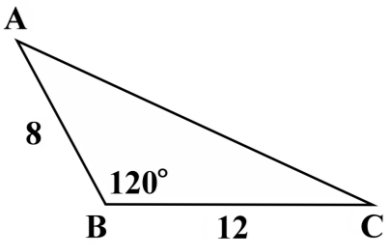
說明：運用三角形的面積公式。

（英文）In $\triangle ABC$, $\overline{AB}=8, \overline{BC}=12, and\ cosB=-\frac{1}{2}$. Find the area of $\triangle ABC$.

（中文）已知$\triangle ABC$中，$\overline{AB}=8$，$\overline{BC}=12$且$cosB=-\frac{1}{2}$，求$\triangle ABC$的面積。

Teacher: Sketch a triangle showing $\overline{AB}=8, \overline{BC}=12$, and $\cos B=-\frac{1}{2}$.

Since the area of $\triangle ABC=\frac{1}{2}ca\sin B$, so we need to find the sin B first. In fact, we know $\angle B=120°$ by given that $\cos B=-\frac{1}{2}$.



What is the value of sin B?

Student: $\sin B=\frac{\sqrt{3}}{2}$.

Teacher: Very good. Then find the area now.

Student: The area is $24\sqrt{3}$.

Teacher: Excellent. Let us move on to example 2.

老師：畫一個三角形，標出 $\overline{AB}=8$，$\overline{BC}=12$，且$\cos B=-\frac{1}{2}$。

學生：因為$\triangle ABC$ 的面積 $=\frac{1}{2}ca\sin B$，所以我們需要先找到 $\sin B$。而事實上，根據 $\cos B=-\frac{1}{2}$，我們知道 $\angle B=120°$。那麼 $\sin B$ 的值是多少？

學生：$\sin B=\frac{\sqrt{3}}{2}$。

老師：很好，現在算出面積。

學生：面積為 $24\sqrt{3}$。

老師：太棒了。我們繼續看例題二。

例題二

說明：運用正弦定理求解。

（英文）In△ ABC, $\angle A = 45°, \angle C = 75°$ and $\overline{AC} = 12$. If circle O circumscribes △ ABC, what is the area of circle O?

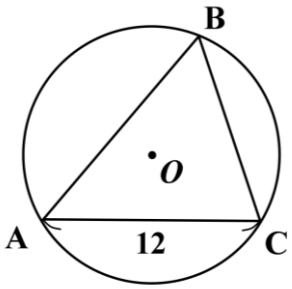
（中文）在△ ABC中，已知$\angle A = 45°$，$\angle C = 75°$且$\overline{AC} = 12$。圓 O 為△ ABC的外接圓，試求圓 O 的面積。

（改編自三民第二冊課本第 1 章例題）

Teacher: If $\angle A = 45°, \angle C = 75°$ and $\overline{AC} = 12$, then $\angle B = 60°$.

Apply the law of sines, $\frac{12}{\sin 60°} = 2R$.

Find the radius R and the area of the circle O now.



Student: $R = 4\sqrt{3}$ and the area of circle O is 48π.

Teacher: Yes. Both of the answers are correct.

老師： 如果$\angle A = 45°$，$\angle C = 75°$，且 $\overline{AC} = 12$，則$\angle B = 60°$。

應用正弦定理，$\frac{12}{\sin 60°} = 2R$。現在找出半徑 R 和圓 O 的面積。

學生： $R = 4\sqrt{3}$，圓 O 的面積為 48π。

老師： 是的，這兩個答案都正確。

例題三

說明：運用正弦定理解三角形。(SSA 情形的解可能為 0、1 或 2 組解)。

（英文）Please determine the remaining sides and angles of $\triangle ABC$ based on the given side and angle conditions for each problem.

(1) $\overline{AB} = 7\sqrt{2}, \overline{AC} = 7,$ and $\angle B = 45°$.

(2) $\overline{AB} = 10, \overline{AC} = 6,$ and $\angle B = 30°$.

(3) $\overline{AB} = 10, \overline{AC} = 5,$ and $\angle B = 60°$.

（中文）請根據各小題給定的邊角條件，求出$\triangle ABC$其餘的邊與角之值。

(1) $\overline{AB} = 7\sqrt{2}$，$\overline{AC} = 7$且$\angle B = 45°$ 。

(2) $\overline{AB} = 10$，$\overline{AC} = 6$ 且$\angle B = 30°$ 。

(3) $\overline{AB} = 10$，$\overline{AC} = 5$且$\angle B = 60°$ 。

Teacher: This question includes the three different situations in SSR, which means only two sides and one angle opposite to one of the sides are given.

In Part (1), by applying the law of sine's with $\overline{AB} = 7\sqrt{2}, \overline{AC} = 7,$ and $\angle B = 45°$

$$\frac{\overline{AB}}{\sin \angle C} = \frac{\overline{AC}}{\sin \angle B}$$

$$\frac{7\sqrt{2}}{\sin \angle C} = \frac{7}{\sin 45°}$$

Find the value of $\sin \angle C$ and the measure of $\angle C$ now.

Student: $\sin \angle C = 1$ and $\angle C = 90°$.

Teacher: Excellent. So, $\angle C = 90°$, $\angle B = 45°$, and $\angle A = 45°$.

$\triangle ABC$ is an isosceles right triangle. The three sides are $\overline{AB} = 7\sqrt{2}$，$\overline{AC} = 7$ and $\overline{BC} = 7$.

In Part (2), similarly

$$\frac{10}{sin \angle C} = \frac{6}{sin 30°}$$

So, $sin \angle C = \frac{5}{6}$.

By using the calculator, we know that there are two solutions: $\angle C = 56.4°$or $\angle C = 123.6°$.

Then find the measure of $\angle A$.

Student: $\angle A = 93.6°$ or $26.4°$.

Teacher: Very good.

So, the three angles of $\triangle ABC$ are: $\angle A = 93.6°$, $\angle B = 30°$, and $\angle C = 56.4°$ or $\angle A = 26.4°$, $\angle B = 30°$, and $\angle C = 123.6°$.

Also, we can find the third side $\overline{BC}$ by using the calculator.

$$\frac{\overline{BC}}{\sin 93.6°} = \frac{6}{\sin 30°} \text{ or } \frac{\overline{BC}}{\sin 26.4°} = \frac{6}{\sin 30°}$$

$\overline{BC}$ =11.98 or 5.34.

So, the three missing parts of $\triangle ABC$ are: $\angle A = 93.6°$, $\angle C = 56.4°$, and $\overline{BC}$ =11.98 or $\angle A = 26.4°, \angle C = 123.6°$, and $\overline{BC}$ =5.34.

Teacher: In Part (3), by applying the law of sine's with $\overline{AB} = 10, \overline{AC} = 5,$ and $\angle B = 60°$

$$\frac{\overline{AB}}{\sin \angle C} = \frac{\overline{AC}}{\sin \angle B}$$

$$\frac{10}{\sin \angle C} = \frac{5}{\sin 60°}$$

So, $\sin \angle C = 2 \sin 60° = \sqrt{3}$

Find the measure of $\angle C$.

Student: There is no solution.

Teacher: Excellent. Because $0 \leq sin\theta \leq 1$ when θ is between 0°and 180°

So, there is no solution.

老師：這個問題包含了 SSR 中的三種不同情況，也就是只給出兩邊和其中一邊的對應角。第(1)小題，$\overline{AB} = 7\sqrt{2}$，$\overline{AC} = 7$，$\angle B = 45°$，應用正弦定理

$$\frac{\overline{AB}}{\sin \angle C} = \frac{\overline{AC}}{\sin \angle B}$$

$$\frac{7\sqrt{2}}{\sin \angle C} = \frac{7}{\sin 45°}$$

所以，找出sin C 及$\angle C$。

學生：$\sin \angle C = 1$ and $\angle C = 90°$.

老師：很好。所以，$\angle C = 90°$，$\angle B = 45°$，$\angle A = 45°$。$\triangle ABC$ 是一個等腰直角三角形。三邊長分別為 $\overline{AB} = 7\sqrt{2}$，$\overline{AC} = 7$ 和 $\overline{BC} = 7$。

第(2)小題，同樣地，$\frac{10}{sin \angle C} = \frac{6}{sin 30°}$，所以，$sin \angle C = \frac{5}{6}$。

用計算器計算，得到答案會有兩個：$\angle C = 56.4°$ 或 $\angle C = 123.6°$。接著算出$\angle A$的度數。

學生：$\angle A = 93.6°$ 或 $26.4°$。

老師： 非常好。因此，△ABC 的三個角度為：$\angle A = 93.6°$，$\angle B = 30°$，$\angle C = 56.4°$ 或是 $\angle A = 26.4°$，$\angle B = 30°$，$\angle C = 123.6°$。接著一樣，我們可以用計算器算出第三邊 $\overline{BC}$。

$$\frac{\overline{BC}}{\sin 93.6°} = \frac{6}{\sin 30°} \text{ 或 } \frac{\overline{BC}}{\sin 26.4°} = \frac{6}{\sin 30°}$$

$\overline{BC} = 11.98$ 或 5.34。

老師： 第(3)小題，$\overline{AB} = 10$，$\overline{AC} = 5$，$\angle B = 60$，應用正弦定理

$$\frac{\overline{AB}}{\sin \angle C} = \frac{\overline{AC}}{\sin \angle B}$$

$$\frac{10}{\sin \angle C} = \frac{5}{\sin 60°}$$

所以，$\sin \angle C = 2\sin 60° = \sqrt{3}$。求$\angle C$

學生： 這題無解。

老師： 非常好。因為當 θ在 0°到 180°之間時，$0 \le sin\theta \le 1$。所以無解。

例題四

說明：運用正弦定理求解。

（英文）In △ABC, the three sides opposite $\angle A, \angle B, and\ \angle C$ are $a = 5$, $b = 7$, and $c = 9$. If the radius of the circumscribing circle is $3\sqrt{3}$, find

(1) sin A : sin B : sin C (the ratio of sin A to sin B to sin C)

(2) the measure of the greatest angle.

（中文）已知△ABC的三邊長 $a = 5$，$b = 7$ 且 $c = 9$，若外接圓半徑 R 為 $3\sqrt{3}$，試求：

(1) sin A : sin B : sin C。

(2) 最大角的度數。

（改編自龍騰第二冊課本單元 10 例題）

Teacher: By applying the law of sines, in Part (1) we have $\frac{a}{sin\ A} = \frac{b}{sinB} = \frac{c}{sin\ C}$.

Find the answer for Part (1).

Student: sin A: sin B: sin C = a: b: c = 5: 7: 9.

Teacher: You are right. In Part (2), we know that the greatest angle is opposite the greatest side. So, the greatest angle in △ABC is $\angle C$.

Since $\frac{c}{\sin C} = 2R$. Find sin C now.

Student: $\sin C = \frac{c}{2R} = \frac{9}{2(3\sqrt{3})} = \frac{\sqrt{3}}{2}$.

Teacher: Excellent. By getting sin C = $\frac{\sqrt{3}}{2}$, we know $\angle C = 60°$ or 120°.

Are they both satisfied?

Student: No, only 120° is the solution.

Teacher: Yes, you are right. The three sides of $\triangle ABC$ are 5, 7, 9.

We know that $\triangle ABC$ is an obtuse triangle because $5^2 + 7^2 < 9^2$.

So, the greatest angle $\angle C = 120°$.

老師： 第(1)小題，可以應用正弦定理 $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$。

學生： $\sin A : \sin B : \sin C = a : b : c = 5 : 7 : 9$。

老師： 答對。第(2)小題，我們知道最大角對應於最長的邊。所以，在$\triangle ABC$ 中，最大角是 $\angle C$。而 $\frac{c}{\sin C} = 2R$。現在找出 sin C。

學生： $\sin C = \frac{c}{2R} = \frac{9}{2(\frac{9\sqrt{3}}{3})} = \frac{\sqrt{3}}{2}$

老師： 很好。因為 $\sin C = \frac{\sqrt{3}}{2}$，我們得到 $\angle C = 60°$ 或 120°。這兩者都符合題目要的嗎？

學生： 不，只有 120° 是答案。

老師： 沒錯。$\triangle ABC$ 的三個邊長為 5、7、9。我們知道$\triangle ABC$ 是一個鈍角三角形，因為 $5^2 + 7^2 < 9^2$。所以，最大角為$\angle C = 120°$。

例題五

說明：運用餘弦定理求解。

（英文）a, b, and c are the three sides of $\triangle ABC$ and $(a+b+c)(a+c-b)=ac$.

Find the measure of $\angle B$.

（中文）已知 a、b、c 為$\triangle ABC$的三邊長且$(a+b+c)(a+c-b)=ac$，試求$\angle B$的度數。

（改編自翰林第三冊課本第一章例題）

Teacher: The only given information is $(a+b+c)(a+c-b)=ac$.

By simplifying the equation, we get:

$a^2+c^2+2ac-b^2=ac$

$a^2+c^2-b^2=-ac$

Which one should we use? The law of sines or the law of cosines?

Student: The law of cosines.

Teacher: Yes, you are correct, because $\cos B = \frac{a^2+c^2-b^2}{2ac}$ by the law of cosines.

Find $\cos B$ now.

Student: $\text{Cos } B = \frac{a^2+c^2-b^2}{2ac} = \frac{-ac}{2ac} = -\frac{1}{2}$

Teacher: Very good. So, we know $\angle B = 120°$.

老師：題目只有一個已知條件：$(a+b+c)(\text{a}+\text{c}-\text{b})=ac$。簡化方程後，我們得到：

$a^2+c^2+2ac-b^2=ac$

$a^2+c^2-b^2=-ac$

接著我們應該使用正弦定理還是餘弦定理？

學生：餘弦定理。

老師：是的沒錯，因為根據餘弦定理，$\cos B = \frac{\text{a}^2+c^2-\text{b}^2}{2\text{ac}}$。現在算 $\cos B$。

學生：$\text{Cos } B = \frac{\text{a}^2+c^2-\text{b}^2}{2\text{ac}} = \frac{-ac}{2ac} = -\frac{1}{2}$

老師：非常好。所以得出$\angle B = 120°$。

例題六

說明：運用餘弦定理及海龍公式求解。

（英文）In $\triangle ABC$, $\overline{AB}=12$, $\overline{BC}=8$, $\overline{AC}=10$, and $\overline{\mathrm{CM}}$ is a median. Find

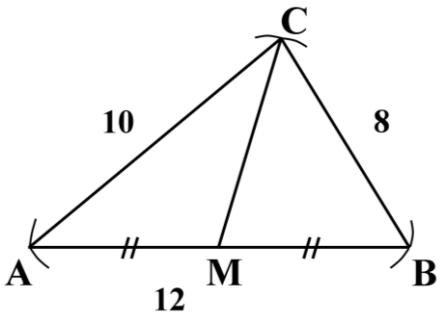
(1) The length of $\overline{CM}$.

(2) The area of $\triangle ABC$.

（中文）已知△ ABC 中，$\overline{AB}=12$，$\overline{BC}=8$，$\overline{AC}=10$，且 $\overline{CM}$ 為中線。試求：

(1) $\overline{CM}$ 的長度。

(2) △ ABC的面積。



（改編自龍騰第二冊課本單元 10 例題）

Teacher: By knowing the lengths of three sides in $\triangle ABC$, we have

$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{10^2+12^2-8^2}{2\cdot10\cdot12} = \frac{180}{240} = \frac{3}{4}$

In $\triangle AMC$, we can get $\overline{CM}$ by the law of cosine again with $\overline{AC}=10$ and $\overline{AM}=\frac{1}{2}\overline{AB}=6$($\overline{CM}$ is a median). What is the length of $\overline{CM}$?

Student: $\overline{CM}=\sqrt{46}$.

Teacher: Very good.

Student: In Part (2), we can use Heron's formula.

The area of $\triangle ABC=\sqrt{s\,(s-a)(s-b)(s-c)}$ where s = $\frac{a+b+c}{2}$ = 15.

Find the area of $\triangle ABC$ now.

The area of $\triangle ABC=\sqrt{s\,(s-a)(s-b)(s-c)}=\sqrt{15\cdot7\cdot5\cdot3}=15\sqrt{7}$

Teacher: Yes, you are correct.

老師：由已知 $\triangle ABC$ 的三邊長，我們可以求出

$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{10^2+12^2-8^2}{2\cdot10\cdot12} = \frac{180}{240} = \frac{3}{4}$。

老師：在$\triangle AMC$ 中，我們可以再次使用餘弦定理來求得 $\overline{CM}$，其中 $\overline{AC}=10$ 且 $\overline{AM}=\frac{1}{2}\overline{AB}=6$ （$\overline{CM}$ 是中線）。請問 $\overline{CM}$ 的長度是多少？

學生：$\overline{\mathrm{CM}}=\sqrt{46}$。

老師：非常好。第(2)小題，我們可以使用海龍公式。

$\triangle ABC$ 的面積 $=\sqrt{s\,(s-a)(s-b)(s-c)}$，其中 $s=\frac{a+b+c}{2}=15$。現在求出 $\triangle ABC$ 的面積。

學生：$\triangle ABC$ 的面積 $=\sqrt{s\,(s-a)(s-b)(s-c)}=\sqrt{15\cdot 7\cdot 5\cdot 3}=15\sqrt{7}$。

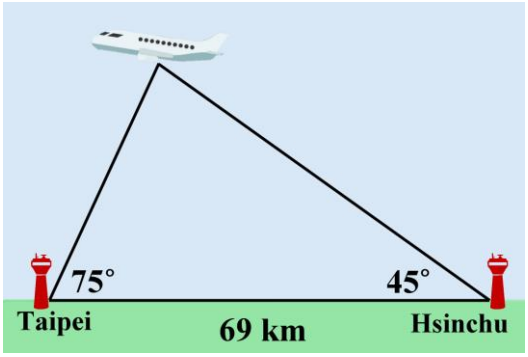
老師：答對了。

例題六

說明：運用正弦定理於測量距離的應用問題。

（英文）An airplane passes directly overhead the observation stations in Taipei and Hsinchu, which are 69 km apart.

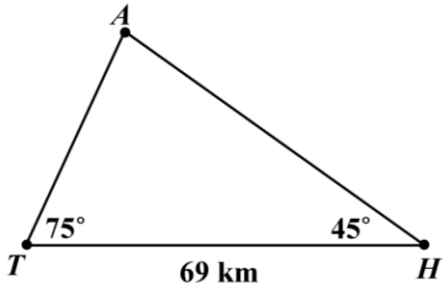
When the airplane is between these two stations, its angle of elevation is simultaneously observed to be 75° at Taipei and 45° at Hsinchu. How far is the airplane from Taipei?



（示意圖）

（中文）一架飛機飛過台北和新竹的觀測站，同一時間台北和新竹觀測站測得該飛機的仰角分別為 75°及 45°。已知台北與新竹兩觀測站相距 69 公里，求該飛機與台北觀測站的距離。

Teacher: When we mark all the given information on the graph, we can find the measure of $\angle A$. What is the measure of $\angle A$?



Student: $\angle A$ is 60°.

Teacher: Good. It is better to apply the law of sines in this kind of ASA questions. By applying the law of sines, we have

$$\frac{69}{\sin 60^\circ}=\frac{\overline{AT}}{\sin 45^\circ}$$

What is the value of $\overline{AT}$?

Student: $\overline{AT}=23\sqrt{6}$.

Teacher: Excellent. Since $\sin 60° = \frac{\sqrt{3}}{2}$ and $\sin 45° = \frac{\sqrt{2}}{2}$, we have $\overline{AT} = \frac{69 \times \frac{\sqrt{2}}{2}}{\frac{\sqrt{3}}{2}} = 23\sqrt{6}$

The distance between the airplane and Taipei station is $23\sqrt{6}$ km.

老師：當我們在圖上標記所有給定的資訊後，我們可以確定這是一個 ASA 的情況，可以應用正弦定理。同時，我們需要求邊長為 69 公里的對應角。

現在算出角度。

學生：答案是 60°。

老師：很好。接下來，$\frac{69}{\sin 60°} = \frac{\overline{AT}}{\sin 45°}$。$\overline{AT}$是多少？

學生：$\overline{AT} = 23\sqrt{6}$

老師：非常棒。由於 $\sin 60° = \frac{\sqrt{3}}{2}$，$\sin 45° = \frac{\sqrt{2}}{2}$，$\overline{AT} = \frac{69 \times \frac{\sqrt{2}}{2}}{\frac{\sqrt{3}}{2}} = 23\sqrt{6}$

飛機與台北觀測站之間的距離是 $23\sqrt{6}$ 公里。

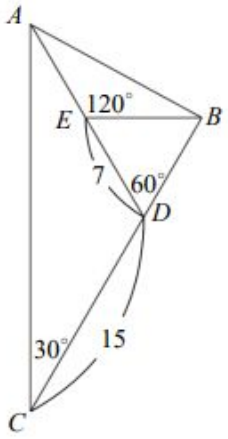
∞ 應用問題 / 學測指考題 ∞

例題一

說明：運用三角比測量距離。

（英文） In $\triangle ABC$, $\overline{AD}$ and $\overline{BC}$ intersect at D, and $\overline{BE}$ and $\overline{AD}$ intersect at E. Besides, $\angle ACB = 30°$ $\angle EDB = 60°$, and $\angle AEB = 120°$. If $\overline{CD} = 15$ and $\overline{ED} = 7$, find the length of $\overline{AB}$.

（中文）如圖（此為示意圖），在 $\triangle ABC$ 中，$\overline{AD}$ 交 $\overline{BC}$ 於 D 點，$\overline{BE}$ 交 $\overline{AD}$ 於 E 點，且$\angle ACB = 30°$，$\angle EDB = 60°$，$\angle AEB = 120°$。若 $\overline{CD} = 15$，$\overline{ED} = 7$，則$\overline{AB}$ =_____。



（108 年學測數學選填題 E）

Teacher: Refer to the graph above, $\angle ADC = 180^o - \angle ADB = 120^o$.
And by the given information, $\angle DAC = 30^o$.
So, $\triangle ADC$ is an isosceles triangle with $\overline{AD} = \overline{CD} = 15$.
In $\triangle BED$, $\angle BED = 180^o - \angle AEB = 60^o$
Also, $\angle BDE = 60^o$ by the given information.
What kind of triangle is $\triangle BED$?

Student: An equilateral triangle.

Teacher: Correct. So, we know $\overline{BD} = \overline{DE} = 7$.
So far, we know the two sides $\overline{BD}$ & $\overline{AD}$, and the included angle $\angle ADB$.
Which formula should we apply to get $\overline{AB}$?

Student: The law of cosines.

Teacher: Yes, you are right. We have:

$$\overline{AB}^2 = \overline{BD}^2 + \overline{AD}^2 - 2 \cdot \overline{BD} \cdot \overline{AD} \cdot \cos \angle ADB$$
$$= 7^2 + 15^2 - 2 \cdot 7 \cdot 15 \cdot \cos 60^o = 169$$

So, $\overline{AB} = 13$.

老師：根據上方的圖形，$\angle ADC = 180^o - \angle ADB = 120^o$。根據題目的已知條件，得出 $\angle DAC = 30$。所以，△ADC 是一個等腰三角形，且 $\overline{AD} = \overline{CD} = 15$。
在△BED 中，$\angle BED = 180^o - \angle AEB = 60^o$。同樣地，根據題目給的資訊得到 $\angle BDE = 60^o$。
△BED 是什麼類型的三角形？

學生：是一個正三角形。

老師：正確。因此我們知道 $\overline{BD} = \overline{DE} = 7$。
到目前為止，我們已經知道了兩個邊長 $\overline{BD}$ 和 $\overline{AD}$，以及$\angle ADB$。應該使用哪個公式來求$\overline{AB}$？

學生：使用餘弦定理。

老師：沒錯。得到：

$$\overline{AB}^2 = \overline{BD}^2 + \overline{AD}^2 - 2 \cdot \overline{BD} \cdot \overline{AD} \cdot \cos \angle ADB$$
$$= 7^2 + 15^2 - 2 \cdot 7 \cdot 15 \cdot \cos 60^o = 169$$

$\overline{AB} = 13$。

例題二

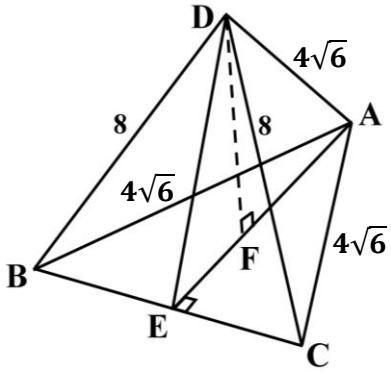
說明：運用餘弦定理求解。

（英文）In the tetrahedron $ABCD$, $\overline{AB} = \overline{AC} = \overline{AD} = 4\sqrt{6}$, $\overline{BD} = \overline{CD} = 8$ and $\cos\angle BAC = \frac{1}{3}$. Find the distance from point D to plane ABC. (Note: Simplify the radical expression to its simplest form.)

（中文）在四面體 $ABCD$ 中，$\overline{AB} = \overline{AC} = \overline{AD} = 4\sqrt{6}$，$\overline{BD} = \overline{CD} = 8$，且 $\cos\angle BAC = \frac{1}{3}$，則點 D 到平面 ABC 的距離為 _____。（化成最簡根式）

（110 學測數學選填題 G）

Teacher: We can draw a diagram and mark all the given information as shown.



It is obvious that both $\triangle ABC$ & $\triangle DBC$ are isosceles triangles.

If we draw the altitude (also median) $\overline{AE} \perp \overline{BC}$ and $\overline{DF} \perp$ plane ABC, then F is on $\overline{AE}$.

What is $\overline{BC}$ if $\cos\angle BAC = \frac{1}{3}$?

Student: $\overline{BC} = 8\sqrt{2}$.

Teacher: Good. Since E is the midpoint of $\overline{BC}$, $\overline{BE} = \frac{1}{2}\overline{BC} = 4\sqrt{2}$.

Thus, we can get $\overline{AE}$ by knowing $\triangle ABE$ is a right triangle.

$\overline{AE}^2 = \overline{AB}^2 - \overline{BE}^2$

$= 96 - 32 = 64$

So, $\overline{AE} = 8$.

Similarly, what is $\overline{DE}$?

Student: $\overline{DE} = 4\sqrt{2}$.

Teacher: Excellent.

Now, we can find $\overline{DF}$ and we know that $\overline{DF} \perp \overline{AE}$.

But the three sides of $\triangle ADE$ are 8, $4\sqrt{2}$, and $4\sqrt{6}$, and $(4\sqrt{6})^2 = 8^2 + (4\sqrt{2})^2$.

So, $\triangle ADE$ is a right triangle, and point E coincides with point F.

Therefore, $\overline{DF}$ $(\overline{DE}) = 4\sqrt{2}$.

老師： 我們可以根據題目所給的已知條件畫出示意圖。很明顯，△ABC 和 △DBC 都是等腰三角形。

如果我們畫出高（同時也是中線） $\overline{AE} \perp \overline{BC}$，且$\overline{DF}$ ⊥ 平面ABC，那麼 F 點就會在 $\overline{AE}$ 上。

如果 $\cos\angle BAC = \frac{1}{3}$，那 $\overline{BC}$ 是多少？

學生： $\overline{BC} = 8\sqrt{2}$.

老師： 很好。由於 E 是$\overline{BC}$的中點，$\overline{BE} = \frac{1}{2}\overline{BC} = 4\sqrt{2}$。因此，由於△$ABE$ 是一個直角三角形，我們可以來求$\overline{AE}$。

$\overline{AE}^2 = \overline{AB}^2 - \overline{BE}^2$

$= 96 - 32 = 64$。因此，$\overline{AE} = 8$。

同樣地，$\overline{DE}$ 是多少？

學生： $\overline{DE} = 4\sqrt{2}$.

老師： 非常好。現在，我們可以試著求出 $\overline{DF}$，而我們已經知道 $\overline{DF} \perp \overline{AE}$。

但是 △ADE 的三邊長分別為 8、$4\sqrt{2}$ 和 $4\sqrt{6}$，且$(4\sqrt{6})^2 = 8^2 + (4\sqrt{2})^2$。

因此，△ADE 是一個直角三角形，E 點與 F 點重合。

所以，$\overline{DF}$ $(\overline{DE}) = 4\sqrt{2}$.

國內外參考資源 More to Explore

國家教育研究院樂詞網 查詢學科詞彙 https://terms.naer.edu.tw/search/	
教育雲：教育媒體影音 為教育部委辦計畫雙語教學影片 https://video.cloud.edu.tw/video/co_search.php?s=%E9%9B%99%E8%AA%9E	
Oak Teacher Hub 國外教學及影音資源，除了數學領域還有其他科目 https://teachers.thenational.academy/	
CK-12 國外教學及影音資源，除了數學領域還有自然領域 https://www.ck12.org/student/	
Twinkl 國外教學及影音資源，除了數學領域還有其他科目，多為小學及學齡前內容 https://www.twinkl.com.tw/	

Khan Academy 可汗學院，有分年級數學教學影片及問題的討論 https://www.khanacademy.org/	
Open Textbook (Math) 國外數學開放式教學資源 http://content.nroc.org/DevelopmentalMath.HTML5/Common/toc/toc_en.html	
MATH is FUN 國外教學資源，還有數學相關的小遊戲 https://www.mathsisfun.com/index.htm	
PhET: Interactive Simulations 國外教學資源，互動式電腦模擬。除了數學領域，還有自然科 https://phet.colorado.edu/	
Eddie Woo YouTube Channel 國外數學教學影音 https://www.youtube.com/c/misterwootube	

國立臺灣師範大學數學系陳界山教授網站	
國高中數學雙語教學相關教材 https://math.ntnu.edu.tw/~jschen/index.php?menu=Teaching_Worksheets	
2023 年第四屆科學與科普專業英文(ESP)能力大賽	
科學專業英文相關教材，除了數學領域，還有其他領域 https://sites.google.com/view/ntseccompetition/%E5%B0%88%E6%A5%AD%E8%8B%B1%E6%96%87%E5%AD%B8%E7%BF%92%E8%B3%87%E6%BA%90/%E7%9B%B8%E9%97%9C%E6%95%99%E6%9D%90?authuser=0	

高中數學領域雙語教學資源手冊：英語授課用語

[十年級下學期]

A Reference Handbook for Senior High School Bilingual Teachers in the Domain of Mathematics: Instructional Language in English

[10th grade 2nd semester]

- 研編單位：國立臺灣師範大學雙語教學研究中心
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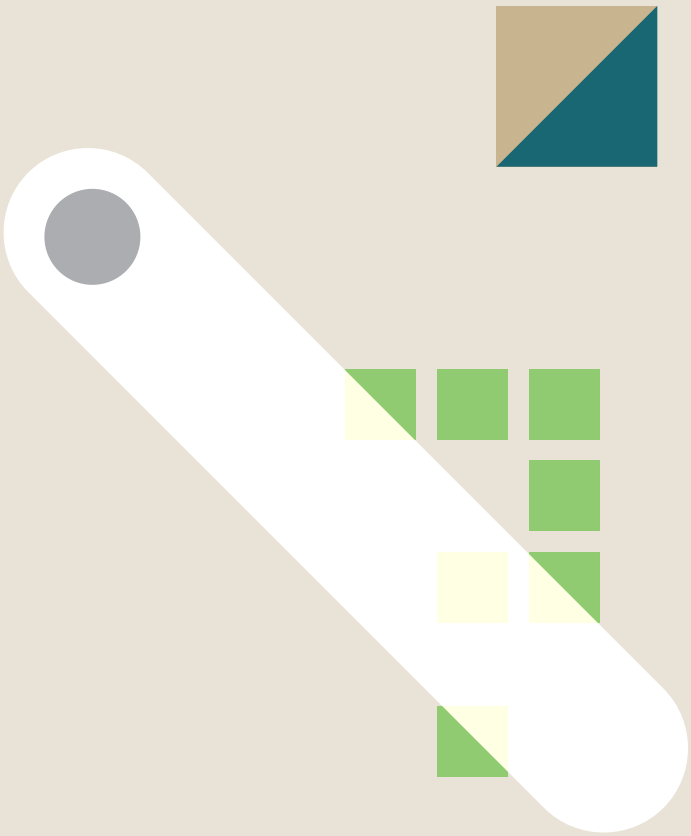
- 編輯排版：吳依靜
- 封面封底：JUPE Design



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