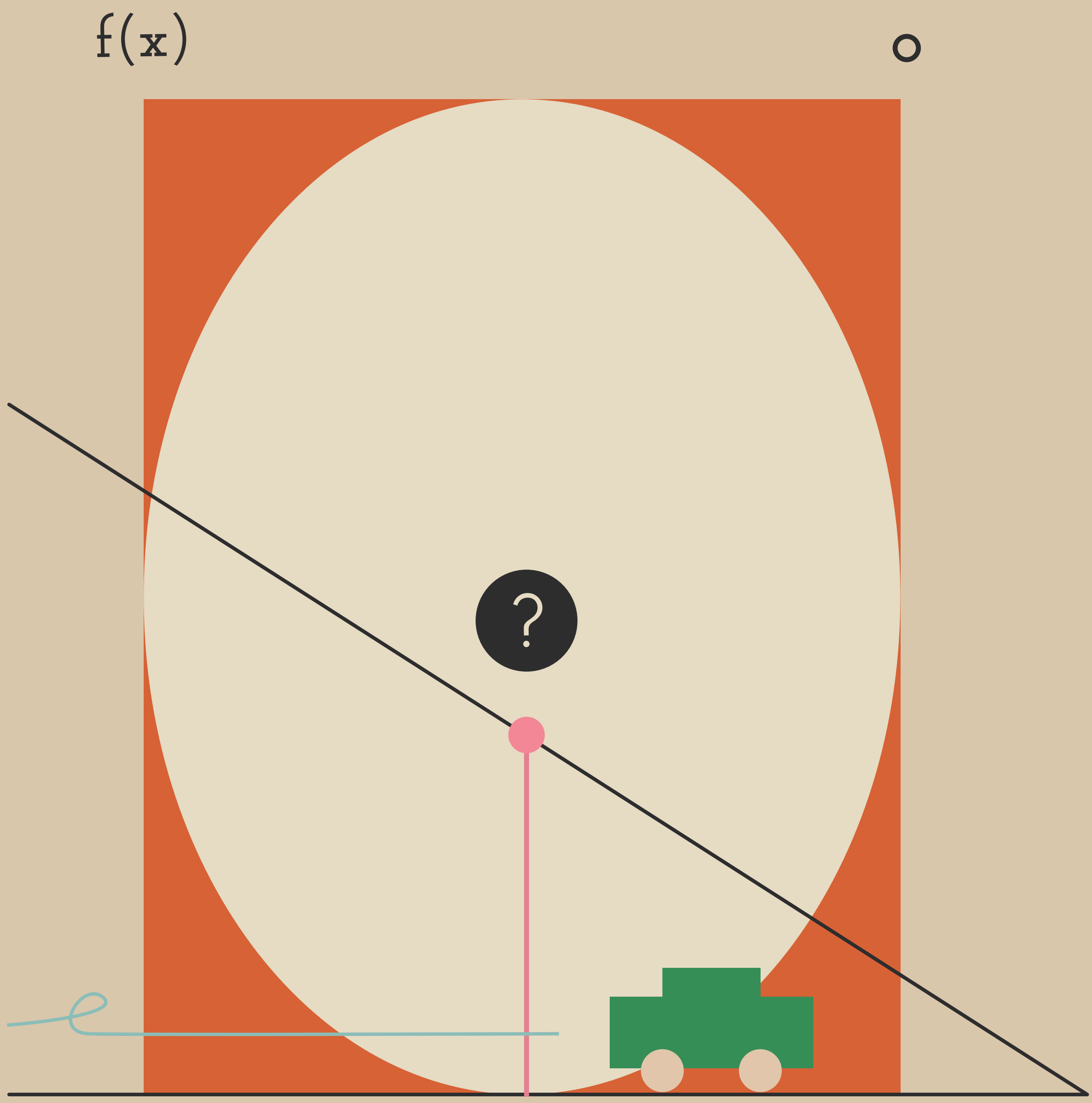
高中數學領域

雙語教學資源手冊 英語授課用語

A Reference Handbook for Senior High School Bilingual Teachers in the Domain of Mathematics: Instructional Language in English

〔高三上學期〕



臺師大雙語教學研究中心 發行

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單元一　數列及其極限
Sequence and Limit

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■ 前言 Introduction

本單元將介紹數列極限的概念，採用直觀的方法讓學生從例子中歸納結論，介紹數列極限的求法及探討無窮數列的收斂與發散，進而介紹數列極限的運算性質。並由兩數列的比較，引導出夾擠定理，最後，從連續複利認識常數 e。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
finite sequence	**有限數列**	**infinite sequence**	**無窮數列**
convergent sequence	**收斂數列**	limit	極限
divergent sequence	**發散數列**	infinity	無窮大
constant sequence	常數數列	convergence	收斂
divergence	發散	infinite geometric sequence	無窮等比數列
sandwich/squeeze theorem	**夾擠定理**	**continuous compounding**	**連續複利**

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ ________ is denoted as ________.

例句：The limit of a sequence a_n **is denoted as** $\lim_{n\to\infty} a_n$. (The limit of a_n as n approaches infinity.)

數列< a_n >的極限，記作 $\lim_{n\to\infty} a_n$。

❷ Determine whether ________.

例句：**Determine whether** the following infinite sequences converge or not.

判斷下列無窮數列是否收斂。

❸ As ________ gets bigger, ________.

例句：**As** n **gets bigger,** $(1+\frac{1}{n})^n$ will approach a fixed value denoted as e.

當 n 越來越大，$(1+\frac{1}{n})^n$ 會趨近一個定值，記為 e。

❹ If ____________ satisfies_________, evaluate _________.

例句：**If** $< a_n >$ **satisfies** $3n^2-1 \le n^2 a_n \le 3n^2+1$, **evaluate** $\lim_{n\to\infty} a_n$.

已知數列< a_n >滿足 $3n^2-1 \le n^2 a_n \le 3n^2+1$，試求 $\lim_{n\to\infty} a_n$ 的值。

■ 問題講解 Explanation of Problems

說明

[The limit of a sequence]

Observe an infinite sequence $<\frac{1}{n}>$. As n gets larger, $\frac{1}{n}$ tends towards 0.

The sequence $<\frac{1}{n}>$ is called **a convergent sequence**, and the fixed value of 0 is called **the limit of the sequence**, denoted as $\lim_{n\to\infty}\frac{1}{n}=0$. (The limit of $\frac{1}{n}$ is 0 as n approaches infinity.)

What is the limit of n^2 as n approaches infinity? Obviously, as n gets larger without bound, so does n^2. The sequence does not approach a fixed value, so it is called **a divergent sequence.**

Definition: The limit of a sequence Given an infinite sequence: 1. As n gets larger, $<a_n>$ tends towards a fixed value of L. The sequence $<a_n>$ is called a convergent sequence, and the fixed value of L is called the limit of the sequence, denoted as $\lim_{n\to\infty}a_n=L$ 2. As n gets larger, $<a_n>$ does not move towards some value. The sequence $<a_n>$ is called a divergent sequence.

[Convergence and divergence of a sequence $<r^n>$] Now consider the infinite geometric sequence whose first term and common ratio are both $r(r\neq 0)$, denoted as $<r^n>$: $r, r^2, r^3, r^4, \cdots$ We can conclude that: 1. If $r=1$, each item of the sequence is equal to 1, the sequence converges, and its limit is 1. 2. If $0<r<1$ or $-1<r<0$, r^n tends towards 0 as n gets larger. Therefore, the sequence converges and its limit is 0. 3. If $r=-1$, the sequence is $-1, 1, -1, 1, \ldots$. The sequence will not approach a fixed value, so the sequence diverges. 4. If $r>1$ or $r<-1$, r^n will not approach a fixed value as n gets larger. Therefore, the sequence diverges

Properties of Limits

If L and M are real numbers and $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = M$, then

(1) Sum Rule： $\lim\limits_{n\to\infty}(a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n = L + M$.

(2) Difference Rule： $\lim\limits_{n\to\infty}(a_n - b_n) = \lim\limits_{n\to\infty} a_n - \lim\limits_{n\to\infty} b_n = L - M$.

(3) Product Rule： $\lim\limits_{n\to\infty}(a_n \cdot b_n) = (\lim\limits_{n\to\infty} a_n)\cdot(\lim\limits_{n\to\infty} b_n) = L \cdot M$.

(4) Quotient Rule： $\lim\limits_{n\to\infty}(\frac{a_n}{b_n}) = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n} = \frac{L}{M}$, $M \neq 0$.

(5) Constant Multiple Rule： $\lim\limits_{n\to\infty}(ka_n) = k\lim\limits_{n\to\infty} a_n = kL$, k is a constant.

When the general term of an infinite sequence is a fraction whose numerator and denominator are both polynomials:

(1) If the degree of the numerator is less than or equal to the degree of the denominator, the sequence *converges*.

(2) If the degree of the numerator is greater than the degree of the denominator, the sequence *diverges*.

The Squeeze/Sandwich Theorem for Sequences

If $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$ and there is an integer N for which $a_n \le b_n \le c_n$ for all $n>N$,

then $\lim\limits_{n\to\infty} b_n = L$

[Continuous compounding and the number e]

Assuming that the principal is $P = 1$ and the annual interest rate is 100%, if the interest is compounded n times a year, and the interest rate is $100\% \div n = \frac{1}{n}$, the balance after one year is $(1+\frac{1}{n})^n$. As n gets bigger and bigger, - it means the compounding occurs for more times. Thus, the balance also gets bigger. Use a calculator to substitute 10, 100, 1000, 10000, 100000, and 1000000 for n into $(1+\frac{1}{n})^n$, and the following results will be obtained:

n	10	100	1000	10000	100000	1000000
$(1+\frac{1}{n})^n$	2.59374	2.70481	2.71692	2.71815	2.71827	2.71828

The table strongly suggests that such a number exists. As n is getting bigger and bigger, there are more frequencies of compounding, and the balance is also becoming bigger, but its value will not approach infinity. In fact, as n becomes bigger and bigger, $(1+\frac{1}{n})^n$ will approach a fixed value, denoted as e, that is, $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$, $e \approx 2.718281828459045235$ is a constant.

運算問題的講解

例題一

說明：判斷無窮數列是否收斂？若收斂，求其極限。

（英文）Determine whether the following infinite sequences converge. If convergent, find the limit. (1) $<(1.001)^n>$ (2) $<\frac{(-3)^n}{5^n}>$

（中文）判斷下列無窮數列 (1) $<(1.001)^n>$、(2) $<\frac{(-3)^n}{5^n}>$ 是否收斂？若收斂，求其極限。

Teacher: Use the calculator and plug in the numbers. Let's start with n =1, and then continue with bigger values. Now what can we see?

Student: In part (1), we can see that as n approaches infinity, $<(1.001)^n>$ gets bigger without bound. In part (2), as n approaches infinity, $<\frac{(-3)^n}{5^n}>$ gets closer and closer to 0.

Teacher: What do we find out?

Student: $<(1.001)^n>$ is a divergent sequence. The limit of $<(1.001)^n>$ doesn't exist. $<\frac{(-3)^n}{5^n}>$ converges and its limit is 0.

Teacher: In part (1), because the common ratio r=1.001 is greater than 1, it diverges. In part (2), because the common ratio $r=\frac{-3}{5}$ is between -1 and 1, it converges.

老師：　當 n 的值越大，利用計算機計算其值。

學生：　第一小題，當 n 的值越大，$<(1.001)^n>$ 的值趨向無限大。第一小題，當 n 的值越大，$<\frac{(-3)^n}{5^n}>$的值趨向 0。

老師：　此題的答案為？

學生：　$<(1.001)^n>$ 為發散數列，極限不存在。$<\frac{(-3)^n}{5^n}>$為收斂數列，極限為 0。

老師：　第一小題，因為公比為 1.001，公比大於 1，所以此數列發散。第二小題，因為公比為 $\frac{-3}{5}$，公比介於 1 和 -1 之間，所以此數列收斂。

例題二

說明：求有理函數的極限。

（英文）Evaluate (1) $\lim_{n\to\infty}\frac{2n^2+1}{n^2-n+2}$ (2) $\lim_{n\to\infty}\frac{n+1}{3n^2+2}$

（中文）試求下列各式的極限 (1) $\lim_{n\to\infty}\frac{2n^2+1}{n^2-n+2}$ (2) $\lim_{n\to\infty}\frac{n+1}{3n^2+2}$

Teacher:　Now observe a rational function. In part (1), the numerator and denominator are $2n^2+1$ and n^2-n-2 respectively. Are they convergent or divergent?

Student:　They are both divergent.

Teacher:　Divide the numerator and denominator by the highest power term n^2.

Student:　We get the following expression: $\frac{2n^2+1}{n^2-n+2}=\frac{2+\frac{1}{n^2}}{1-\frac{1}{n}+\frac{2}{n^2}}$.

Teacher:　Find the limits of $2+\frac{1}{n^2}$ and $1-\frac{1}{n}+\frac{2}{n^2}$

Student:　The limits are 2 and 1.

Teacher:　We can then use the quotient rule of limits.

$$\lim_{n\to\infty}\frac{2n^2+1}{n^2-n+2}=\lim_{n\to\infty}\frac{2+\frac{1}{n^2}}{1-\frac{1}{n}+\frac{2}{n^2}}=\frac{\lim\limits_{n\to\infty}(2+\frac{1}{n^2})}{\lim\limits_{n\to\infty}(1-\frac{1}{n}+\frac{2}{n^2})}=\frac{2}{1}=2$$

Teacher: In part (2), the numerator $n+1$ and denominator $3n^2+2$ are both divergent sequences. Divide the numerator and denominator by the highest power term n^2.

Student: It follows that:

$$\frac{n+1}{3n^2+2}=\frac{\frac{1}{n}+\frac{1}{n^2}}{3+\frac{2}{n^2}}$$

Teacher: Find the limits of $\frac{1}{n}+\frac{1}{n^2}$ and $3+\frac{2}{n^2}$

Student: The limits are 0 and 3.

Teacher: We can also use the quotient rule of limits.

$$\lim_{n\to\infty}\frac{n+1}{3n^2+2}=\lim_{n\to\infty}\frac{\frac{1}{n}+\frac{1}{n^2}}{3+\frac{2}{n^2}}=\frac{\lim\limits_{n\to\infty}(\frac{1}{n}+\frac{1}{n^2})}{\lim\limits_{n\to\infty}(3+\frac{2}{n^2})}=\frac{0}{3}=0$$

老師：我們觀察這些有理函數，第一小題的分子分母分別是$2n^2+1$和n^2-n+2。他們是收斂或發散？

學生：他們皆發散。

老師：將分子、分母分別除以最高次方 n^2。

學生：

可得式子：$\frac{2n^2+1}{n^2-n+2}=\frac{2+\frac{1}{n^2}}{1-\frac{1}{n}+\frac{2}{n^2}}$

老師：$2+\frac{1}{n^2}$ 和 $1-\frac{1}{n}+\frac{2}{n^2}$ 極限各為多少？

學生：極限分別為 2 和 1。

老師：我們可使用極限的商數性質。得到

$$\lim_{n\to\infty}\frac{2n^2+1}{n^2-n+2}=\lim_{n\to\infty}\frac{2+\frac{1}{n^2}}{1-\frac{1}{n}+\frac{2}{n^2}}=\frac{\lim\limits_{n\to\infty}(2+\frac{1}{n^2})}{\lim\limits_{n\to\infty}(1-\frac{1}{n}+\frac{2}{n^2})}=\frac{2}{1}=2$$

老師：第二小題的分子分母分別是 $n+1$ 和 $3n^2+2$，他們是發散數列。分別將分子分母分別除以最高次方 n^2。

學生：可得式子：

$$\frac{n+1}{3n^2+2}=\frac{\frac{1}{n}+\frac{1}{n^2}}{3+\frac{2}{n^2}}$$

老師： $\frac{1}{n}+\frac{1}{n^2}$ 和 $3+\frac{2}{n^2}$ 極限各為多少？

學生： 極限分別為 0 和 3。

老師： 我們可使用極限的商數性質。得到

$$\lim_{n\to\infty}\frac{n+1}{3n^2+2}=\lim_{n\to\infty}\frac{\frac{1}{n}+\frac{1}{n^2}}{3+\frac{2}{n^2}}=\frac{\lim\limits_{n\to\infty}(\frac{1}{n}+\frac{1}{n^2})}{\lim\limits_{n\to\infty}(3+\frac{2}{n^2})}=\frac{0}{3}=0$$

例題三

說明：利用夾擠定理求極限。

（英文）Show that the sequence $<\frac{\cos n}{n}>$ converges, and find its limit.

（中文）試說明 $<\frac{\cos n}{n}>$ 收斂，求其極限。

Teacher: Let's first figure out the range of $\cos n$.

Student: The maximum of $\cos n$ is 1 and the minimum of $\cos n$ is -1, that is: $-1\le\cos n\le 1$.

Teacher: Divide each term by n.

Student: Now, we can get the following expression: $\frac{-1}{n}\le\frac{\cos n}{n}\le\frac{1}{n}$

Teacher: Which theorem should we apply?

Student: We can apply the squeeze/sandwich theorem.

Teacher: Then: $\lim\limits_{n\to\infty}(\frac{-1}{n})=\lim\limits_{n\to\infty}(\frac{1}{n})=0$. What is the result?

Student: Because $\lim\limits_{n\to\infty}(\frac{-1}{n})=\lim\limits_{n\to\infty}(\frac{1}{n})=0$, we can get $\lim\limits_{n\to\infty}(\frac{\cos n}{n})=0$. Thus, the sequence $<\frac{\cos n}{n}>$ converges.

老師： 我們先找出 $\cos n$ 的值域。

學生： $\cos n$ 的最大值為 1 最小值為-1。也就是 $\frac{-1}{n}\le\frac{\cos n}{n}\le\frac{1}{n}$。

老師： 你們可以使用甚麼定理？

學生： 夾擠定理。

老師： $\frac{-1}{n}$ 和 $\frac{1}{n}$ 的極限為 0。我們可以有何種結論？

學生： 因為 $\frac{-1}{n}$ 和 $\frac{1}{n}$ 的極限為 0，所以$<\frac{\cos n}{n}>$收斂且 $\frac{\cos n}{n}$ 的極限為 0。

應用問題 / 學測指考題

例題一

說明：二元一次不等式解區域內的格子點與極限的運算

（英文）The points (x, y) whose x-coordinate and y-coordinate are both integers, are called *lattice points*. Let n be a positive integer and T_n be the triangular area (including the boundary) enclosed by the line $y = \frac{-1}{2n}x + 3$, the x-axis, and the y-axis. Let a_n be the number of all the lattice points on T_n. Find $\lim_{n\to\infty}\frac{a_n}{n}$ =?

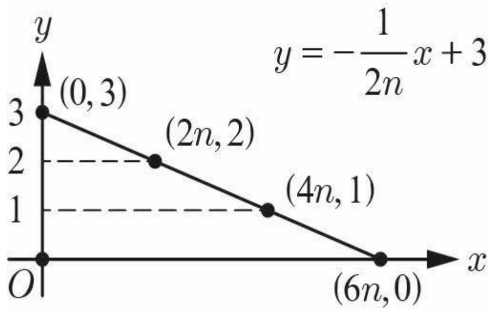
（中文）坐標平面上，x 坐標與 y 坐標均為整數的點稱為格子點。令 n 為正整數，T_n 為平面上以直線 $y = \frac{-1}{2n}x + 3$，以及 x 軸、y 軸所圍成的三角形區域（包含邊界），而 a_n 為 T_n 上的格子點數目，則 $\lim_{n\to\infty}\frac{a_n}{n} = ?$

（106 年指考數甲）

Teacher: Let's find the x-intercept and y-intercept of $y = \frac{-1}{2n}x + 3$

Student: Replace x with 0. We get $y = 3$. So, the y intercept is $(0, 3)$.
Set $y = 0$ then solve for $x = 6n$. So, the x intercept is $(6n, 0)$.

Teacher: Let's graph the linear function $y = \frac{-1}{2n}x + 3$ and list the lattice points on T_n.



Student: We can find the lattice points at $y = 0, 1, 2, 3$.

Teacher: According to the graph above, fill in the following table with the sum of the numbers of lattice points. Then, sum up the four numbers to find a_n, the number of lattice points on T_n.

y	3	2	1	0
x				
the number of lattice points				

Student: It follows:

y	3	2	1	0
x	0	$0,1,\cdots,2n$	$0,1,\cdots,4n$	$0,1,\cdots,6n$
the number of lattice points	1	$2n+1$	$4n+1$	$6n+1$

The sum is $a_n = 1 + (2n+1) + (4n+1) + (6n+1) = 12n + 4$.

Teacher: Next, find $\lim_{n\to\infty} \frac{a_n}{n} =?$

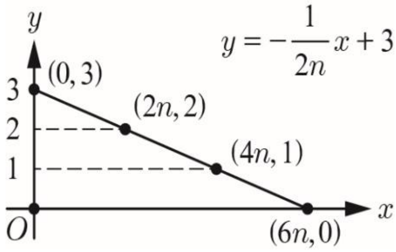
Student: We get $\frac{a_n}{n} = \frac{12n+4}{n} = 12 + \frac{4}{n}$ The limit of $\frac{4}{n}$ is 0 as n approaches infinity. So,

$\lim_{n\to\infty} \frac{a_n}{n} = 12$

老師： 先找出直線 $y = \frac{-1}{2n}x + 3$ 的 x 軸截距與 y 軸截距。

學生： 令 $x = 0$，可得到 $y = 3$，因此 y 軸截距為 3。令 $y = 0$，解得 $x = 6n$，可得 x 軸截距為 $6n$。

老師： 畫出直線 $y = \frac{-1}{2n}x + 3$ 的圖形，並列出在 T_n 上格子點。



學生：　我們可以分別對 $y=0, 1, 2, 3$ 找出其格子點。

老師：　完成下列表格，並求格子點個數的總和 $a_n=?$ 。

y	3	2	1	0
x				
格子點數				

學生：　可得下列結果，格子點個數的總和為 $a_n=12n+4$ 。

x	0	$0,1,\cdots,2n$	$0,1,\cdots,4n$	$0,1,\cdots,6n$
格子點數	1	$2n+1$	$4n+1$	$6n+1$

老師：　找出 $\frac{a_n}{n}$ 的極限。

學生：　我們可以得 $\frac{a_n}{n}=\frac{12n+4}{n}=12+\frac{4}{n}$。當 n 趨近無限大，$\frac{4}{n}$ 的極限為 0。

因此 $\lim\limits_{n\to\infty}\frac{a_n}{n}=12$。

例題二

說明：這題利用常數 e 解決日常生活問題。

（英文）If a principal P is invested at an annual rate r and compounded n times in a year, then the amount of A in the account after t years is given by $A = P(1+\frac{r}{n})^{nt}$.

This is known as the *continuous compound interest formula*.

Let's assume that \$1,000 is deposited in an account that earns compound interest at an annual rate of 8% for 2 years. If n is the number of compounding periods per year, the amount in the account at the end of 2 years is given by: $A(n)$

(1) Find $A(4) = ?$ with a calculator (compounding quarterly).

(2) Use the continuous compound interest formula: $\lim_{n\to\infty} P(1+\frac{r}{n})^{nt} = Pe^{rt}$

Find $\lim_{n\to\infty} A(n) =$ ____ with a calculator. Compute the answer to the nearest cent.

（中文）如果存入的本金是 P 且年利率為 r，若一年複利計息 n 次，則 t 年後的本利和為，$A = P(1+\frac{r}{n})^{nt}$。這就是連續複利公式。

假設存入 1000 元且年利率為8%，為期 2 年。如果一年複利 n 次，則2年後的本利和為 $A(n)$

(1) 利用計算機求 $A(4) = ?$

(2) 利用連續複利公式求 $A(n)$ 的極限？答案近似到小數點後第二位。

Teacher: In part (1), plug $P = 1000$, $r = 0.08$, $n = 4$, $t = 2$ into the formula $A = P(1+\frac{r}{n})^{nt}$. Let's find the answer of $A(4)$ with a calculator.

Student: The answer is \$1171.66.

Teacher: For part (2), let's represent $A(n)$ as a function of n.

Student: Since $r = 0.08$, $t = 2$, we can apply the formula $A = P(1+\frac{r}{n})^{nt}$ and write the function of $A(n) = 1000(1+\frac{0.08}{n})^{2n}$

Teacher: In fact, it appears that $A(n)=1000(1+\frac{0.08}{n})^{2n}$ might be approaching a value as n increases without bound. Let's find the answer of $\lim_{n\to\infty} A(n)=1000(1+\frac{0.08}{n})^{2n}=?$

Student: Based on the continuous compound interest formula, we can get: $\lim_{n\to\infty} A(n)=1000(1+\frac{0.08}{n})^{2n}=1000e^{0.08\times 2}=1173.51$ In part (2), the answer is \$1173.51.

老師： 第一小題，我們直接把 $A=P(1+\frac{r}{n})^{nt}$ 。利用計算機找到答案 $A(4)$。

學生： 答案是 1171.66。

老師： 試著表示，$A(n)$ 是 n 的函數。

學生： 因為我們直接把 $P=1000$、$r=0.08$、$n=4$、$t=2$ 直接帶入公式 $A=P(1+\frac{r}{n})^{nt}$，我們就可寫下$A(n)=1000(1+\frac{0.08}{n})^{2n}$

老師： 當 n 越來越大，$A(n)=1000(1+\frac{0.08}{n})^{2n}$似乎會趨近於定值。求 $A(n)=1000(1+\frac{0.08}{n})^{2n}$ 的極限。

學生： 我們利用連續複利的公式 $\lim_{n\to\infty} A(n)=1000(1+\frac{0.08}{n})^{2n}=1000e^{0.08\times 2}=1173.51$。第二小題的答案為 1173.51。

單元二　無窮等比級數
Infinite Geometric Series

臺北市中正高中　鄧宇凱老師

■ 前言 Introduction

本單元將介紹介紹一個符號「Σ」，讀作 sigma，介紹有限級數的運算性質、級數和與探討無窮級數的收斂與發散，進而介紹無窮等比級數求和的方法與收斂級數的和與差。最後運用無窮等比級數求和將循環小數化成分數。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
finite series	**有限級數**	**infinite series**	**無窮級數**
arithmetic series	等差級數	infinite geometric series	無窮等比級數
convergent series	**收斂級數**	**divergent series**	**發散級數**
repeating decimal	循環小數	irreducible fraction	最簡分數

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ Write _______ by using _______.

例句：**Write** the following series **by using** summation notation.

試將下列級數用 Σ 符號表示。

❷ Express _______ as _______.

例句：**Express** the repeating decimal **as** an irreducible fraction.

將這循環小數化為最簡分數。

❸ _______ is read as "_______."

例句：The notation $\sum_{i=1}^{n} a_n$ **is read as** "a summation of a_n for all integers of i from 1 to n."

$\sum_{i=1}^{n} a_n$ 可讀作數列 a_n 的總和。

■ 問題講解 Explanation of Problems

說明

[Definition of a finite series]

The sum of the first n terms of the sequence is called a **finite series** and is denoted by

$$S_n = a_1 + a_2 + \cdots + a_n$$

[Definition of Σ (sigma) notation]

This section will introduce the idea of **sigma notation**, which is also known as **summation notation**. A finite series $a_1 + a_2 + \cdots + a_n$ can be denoted as $\sum_{k=1}^{n} a_k$, where $k = 1$ means a_1 and $k = n$ means a_n, i.e.,

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

[Properties of Σ notation]

There are a couple of formulas for summation notation.

(1) $\sum_{k=1}^{n}(a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k$. Break up a summation across a sum or difference.

(2) $\sum_{k=1}^{n} c a_k = c \sum_{k=1}^{n} a_k$, where c is a constant that can be factorized.

[Definition of an infinite series]

An infinite series is an expression of the form $a_1 + a_2 + \cdots + a_n + \cdots$ denoted as:

$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + ... + a_n + ...$ Let $S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$ be a sequence $< S_n >$.

If the sequence $< S_n >$ has a limit S as $n \to \infty$, the series **converges** to the sum S. We can then write $a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n = S$. Otherwise, the series **diverges**.

[Infinite geometric series]

Consider an infinite geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{k=1}^{\infty} ar^{k-1}$, where a and $r (r \neq 0)$ are the first term and common ratio respectively.

The sum of the finite geometric series is $S_n = a + ar + ar^2 + \cdots + ar^{n-1} = \begin{cases} \frac{a(1-r^n)}{1-r}, & r \neq 1 \\ na, & r = 1 \end{cases}$

(1) If $-1 < r < 1$, $r \neq 0$, it shows that $\lim_{n\to\infty} r^n = 0$. Hence, the sequence $< S_n >$ is convergent and $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}\lim_{n\to\infty}(1-r^n) = \frac{a}{1-r}$, that is, $\sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \frac{a}{1-r}$.

(2) If $r = 1$, $S_n = na$ and the sequence $< S_n >$ is divergent. Hence, the sum of the series $\sum_{k=1}^{\infty} ar^{k-1}$ doesn't exist.

(3) If $r>1$ or $r \leq -1$, $S_n = \frac{a(1-r^n)}{1-r}$ and the sequence $< S_n >$ is divergent. Hence, the sum of the series $\sum_{k=1}^{\infty} ar^{k-1}$ doesn't exist.

運算問題的講解

例題一

說明：將 Σ 符號寫成級數的形式或將級數用 Σ 符號表示

（英文）(1) Write $\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k$ in a series form.

(2) Write the series $-4-1+2+5+8+\ldots+71$ by using summation notation.

（中文）(1)將 $\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k$ 寫成級數的形式。

(2)將級數 $-4-1+2+5+8+\ldots+71$ 用 Σ 符號表示。

Teacher: In part (1), we use the definition of sigma notation to expand the expression.

Student: In part (1), we substitute the values of 1, 2, 3, …, n, to infinity for k, then add up the results as: $\sum_{k=1}^{\infty}(\frac{1}{2})^k = (\frac{1}{2}) + (\frac{1}{2})^2 + (\frac{1}{2})^3 + \ldots + (\frac{1}{2})^n + \ldots$

Teacher: In part (2), let's first observe the series. What is the series called? Find the nth-term of the sequence.

Student: It is an arithmetic sequence with the first term $a_1 = -4$ and the common difference $d = 3$. We can use the nth- term formula of the arithmetic sequence: $a_n = a_1 + (n-1)\cdot d$. Hence, $a_k = -4 + (k-1)\cdot 3 = 3k - 7$.

Teacher: How many terms are added up in the series?

Student: The last term is $a_n = 71 = -4 + (n-1)\cdot 3$ And we find $n = 26$, which means it contains 26 terms. Therefore, the series can be written as: $\sum_{k=1}^{26} a_k = \sum_{k=1}^{26}(3k-7)$.

老師： 第一小題中，我們可以用 sigma 的定義展開。

學生： 第一小題中，我們將此數列 k 由 $1, 2, 3, \ldots, n,$ 至無限大帶入，以加號連接起來可得: $\sum_{k=1}^{\infty}(\frac{1}{2})^k = (\frac{1}{2}) + (\frac{1}{2})^2 + (\frac{1}{2})^3 + \ldots + (\frac{1}{2})^n + \ldots$

老師： 此級數有多少項相加？

學生： 這是等差數列。此數列的首項 $a_1 = -4$，公差 $d = 3$。使用一般項公式 $a_n = a_1 + (n-1)\cdot d$，可知第 k 項 $a_k = -4 + (k-1)\cdot 3 = 3k - 7$。

老師： 第二小題中，觀察此級數是何種級數？試著找出此數列的第 n 項。

學生： 末項是 $a_n = 71 = -4 + (n-1)\cdot 3$，可解得 $n = 26$，可知此級數共有 26 項。故此數列可表示成 $\sum_{k=1}^{26} a_k = \sum_{k=1}^{26}(3k-7)$。

例題二

說明：求特定級數的和。

（英文）Find the sum of the series: $1\times 3 + 2\times 4 + 3\times 5 + \cdots + n(n+2)$.

（中文）求級數 $1\times 3 + 2\times 4 + 3\times 5 + \cdots + n(n+2)$ 的和。

Teacher: First, let’s list the first 3 terms and the nth term of the sequence. Write the series by using summation notation.

Student: It shows that $a_1 = 1\times 3, a_2 = 2\times 4, a_3 = 3\times 5$, and $a_k = k(k+2)$. Hence, we can write the series as: $1\times 3 + 2\times 4 + 3\times 5 + \cdots + n(n+2) = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} k(k+2)$

Teacher: Use the properties of summation notation, and then the following formulas can be applied: $1 + 2 + 3 + \ldots + n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ and

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Student: The expression can be written as:

$$1\times 3 + 2\times 4 + 3\times 5 + \cdots + n(n+2) = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} k(k+2)$$

We break up a summation across a sum. It follows:

$$\sum_{k=1}^{n}(k^2+2k)==\sum_{k=1}^{n}k^2+2\sum_{k=1}^{n}k$$

$$=\frac{n(n+1)(2n+1)}{6}+2\times\frac{3n(n+1)}{6}$$ (Get the common denominator of 6.)

$$=\frac{n(n+1)(2n+1)+6n(n+1)}{6}$$ (Add up the fractions.)

$$=\frac{n(n+1)(2n+7)}{6}$$ (Simplify the numerator.)

老師： 列出數列的前三項和第 k 項，並將此級數用 Σ 表示

學生： 此數列的第 1 項 $a_1=1\times3$，第 2 項 $a_2=2\times4$，第 3 項 $a_3=3\times5$，一般項 $a_k=k(k+2)$。因此級數可用 $\sum$ 表示如下：

$$1\times3+2\times4+3\times5+\cdots+n(n+2)=\sum_{k=1}^{n}k(k+2)$$

老師： 利用 sigma 性質及常見級數的求和公式，求此級數和。

學生： 我們將此級數 $1\times3+2\times4+3\times5+\cdots+n(n+2)=\sum_{k=1}^{n}a_k=\sum_{k=1}^{n}k(k+2)$ 拆成兩個級數的和，經過通分母、合併與化簡後，可得式子如下：

$$\sum_{k=1}^{n}(k^2+2k)==\sum_{k=1}^{n}k^2+2\sum_{k=1}^{n}k$$

$$=\frac{n(n+1)(2n+1)}{6}+2\times\frac{3n(n+1)}{6}$$

$$=\frac{n(n+1)(2n+1)+6n(n+1)}{6}$$

$$=\frac{n(n+1)(2n+7)}{6}$$

例題三

說明：判別無窮等比級數是否收斂？若無窮等比級數收斂，求出它的和。

（英文）Determine whether the series $-\frac{3}{4}+\frac{9}{16}+\cdots+(-\frac{3}{4})^n+\cdots$is convergent or divergent. If it is convergent, give its sum.

（中文）判別下列無窮等比級數是否收斂？若無窮等比級數收斂，求出它的和。

$$-\frac{3}{4}+\frac{9}{16}+\cdots+(-\frac{3}{4})^n+\cdots$$

Teacher: Let's figure out the first term and the common ratio of the infinite geometric series.

Student: The first term is: $a=-\frac{3}{4}$ and the common ratio is: $r=\frac{9}{16}\div\frac{-3}{4}=\frac{-3}{4}$

Teacher: Now determine whether the series is convergent or divergent.

Student: The geometric series is convergent due to $r=-\frac{3}{4}$ which lies between -1 and 1.

Teacher: Since the geometric series is convergent, obtain its sum.

Student: According to the formula $\sum_{k=1}^{\infty}ar^{k-1}=a+ar+ar^2+\cdots+ar^{n-1}+\cdots=\frac{a}{1-r}$, replace a with $-\frac{3}{4}$, and r with $-\frac{3}{4}$ respectively. The series converges to: $\frac{-\frac{3}{4}}{1-(-\frac{3}{4})}=-\frac{3}{7}$

Hence, its sum is: $\frac{-3}{7}$.

老師：我們先找出此無窮等比級數的首項與公比。

學生：首項 $a=-\frac{3}{4}$ 與公比 $r=\frac{9}{16}\div\frac{-3}{4}=\frac{-3}{4}$

老師：試判斷此無窮級數是否收斂？

學生：因為 $r=\frac{-3}{4}$，介於 1 與 -1 之間，所以此無窮等比級數收斂。

老師：已知此級數收斂，則求出它的和。

學生：依據無窮等比級數收斂其和的公式為 $\frac{a}{1-r}$，分別將 $a=\frac{-3}{4}$ 和 $r=\frac{-3}{4}$ 帶入，可知此級數的和收斂至 $\frac{-\frac{3}{4}}{1-(-\frac{3}{4})}=-\frac{3}{7}$。因此，其和為 $\frac{-3}{7}$。

例題四

說明：利用無窮級數和的方式將循環小數化為最簡分數。

（英文）Express the repeating decimal $0.\overline{7}$ as an irreducible fraction.

（中文）將循環小數 $0.\overline{7}$ 化為最簡分數。

Teacher: In *Book One*, we understood by intuition that repeating decimals are rational numbers. Describe what a rational number is.

Student: A rational number can be expressed as $\frac{p}{q}$, where p and q are both integers and q ≠ 0.

Teacher: I will show how to convert a repeating decimal to its irreducible fractions in proper steps. First, rewrite $0.\overline{7}$ as a series.

Student: $0.\overline{7}$ can be written as: $0.\overline{7} = 0.777... = \frac{7}{10} + \frac{7}{10^2} + \frac{7}{10^3} + ... + \frac{7}{10^n} + ...$

Teacher: Apparently, it is an infinite geometric series. Figure out the first term a and the common ratio r of the geometric series. Then give its sum.

Student: The first term is: $a = \frac{7}{10}$ and the common is ratio: $r = \frac{1}{10}$. Plug $a = \frac{7}{10}$ and $r = \frac{1}{10}$ into the formula $\frac{a}{1-r}$. The series is convergent, so: $0.\overline{7} = \frac{a}{1-r} = \frac{\frac{7}{10}}{1-\frac{1}{10}} = \frac{7}{9}$

老師：在第一冊我們用直觀的方式來認識循環小數是有理數。你們描述甚麼是有理數嗎？

學生：有理數可以表達為兩個整數比的數 $\frac{p}{q}$，其中 q 不為 0。

老師：我們用嚴謹的方式來說明如何將循環小數化為最簡分數 $\frac{p}{q}$ 的形式。先將 $0.\overline{7}$ 改寫成級數的形式。

學生：$0.\overline{7}$ 可以寫成 $0.\overline{7} = 0.777... = \frac{7}{10} + \frac{7}{10^2} + \frac{7}{10^3} + ... + \frac{7}{10^n} + ...$。

老師：很明顯，此級數為等比級數，找出此無窮等比級數的首項與公比並算出和。

學生：首項是 $a = \frac{7}{10}$、公比是 $r = \frac{1}{10}$，分別將 $a = \frac{7}{10}$和 $r = \frac{1}{10}$帶入公式 $\frac{a}{1-r}$，可知此級數的和收斂至 $\frac{\frac{7}{10}}{1-\frac{1}{10}} = \frac{7}{9}$

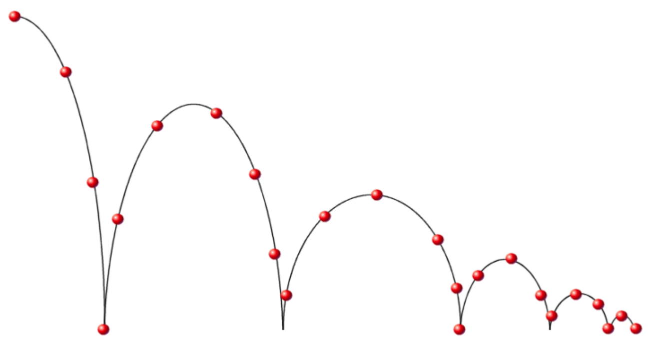
應用問題 / 學測指考題

例題一

說明：這題利用無窮等比級數解決日常生活問題。

（英文）A ball is dropped from a height of 4 m. Each time it strikes the ground after falling a height of h m, it rebounds to a height of $0.6h$ m. Find the total distance of the ball bouncing until it stops.

（中文）一顆球從 4 公尺高處落下，若球每次從 h 公尺高處落下，碰到地面後會反彈至 $0.6\,h$公尺的高度。求此球上下過程中所行經之距離。



(圖片來源：http://mskaysartworld.weebly.com/uploads/1/3/7/3/13735102/1285714_orig.png)

Teacher: The diagram suggests that the first term a_1 of the series is 4 when the ball is dropped from a height of 4 m; the second term a_2 is the distance of the ball after the first bounce. Write the second term $a_2 = ?$

Student: The second term is: $a_2 = 2\times 4\times (0.6)$. It contains the distance of the ball traveling up and down.

Teacher: Next, write the third term $a_3 = ?$

Student: The third term is: $a_3 = 2\times 4\times (0.6)^2$, which is the distance of the ball bouncing up and down during that bounce.

Teacher: The total distance is the sum of $a_1 + a_2 + a_3 + \ldots$ Find its sum.

Student: The expression is:

$$a_1 + a_2 + a_3 + ... = 4 + 2\times4\times(0.6) + 2\times4\times(0.6)^2 + 2\times4\times(0.6)^3 + ...$$

Starting with the second term, it is a geometric series with the first term $a = 2\times4\times(0.6)$ and the common ratio $r = 0.6$.

Hence, the total distance is: $4 + \dfrac{2\times4\times0.6}{1-0.6} = 16$

老師：圖形顯示，當球從 4 公尺高處落下，此級數的第一項為 4。第二項包含球反彈後的往上和向下的距離，寫出第二項的值。

學生：第二項包含球反彈後的往上和向下的距離所以 $a_2 = 2\times4\times(0.6)$。

老師：再繼續寫出第三項的值。

學生：第三項為 $a_2 = 2\times4\times(0.6)^2$，同時也包含球反彈後的往上和向下的距離。

老師：球所行經之距離為 $a_1 + a_2 + a_3 + ...$，計算總其總和。

學生：行經之距離為 $a_1 + a_2 + a_3 + ... = 4 + 2\times4\times(0.6) + 2\times4\times(0.6)^2 + 2\times4\times(0.6)^3 + ...$。

從第二項開始，這些項的和是等比級數，其首項為 $a = 2\times4\times(0.6)$、公比為 $r = 0.6$。因此，球上下過程中所行經之距離為 $4 + \dfrac{2\times4\times0.6}{1-0.6} = 16$ 公尺。

例題二

說明：這題利用等比級數解決日常生活問題。

（英文）There are 2 red balls and 4 black balls in a bag. A and B take turns to withdraw one ball at a time, and return it to the bag after each turn. It is agreed that whoever gets the red ball first wins. Assume that A takes the ball first and find the probability that B wins the game in the end.

（中文）袋中有 2 個紅球，4 個黑球，甲、乙兩人輪流取球，每次取一球，球取出後均再放回，約定先取到紅球者獲勝。現由甲先取球，試求最後是乙獲勝的機率。

Teacher: Let p represent the probability that a red ball is withdrawn at each time. Write the value of p. What does $1-p$ stand for?

Student: Obviously, $p=\frac{2}{2+4}=\frac{1}{3}$. So $1-p=\frac{2}{3}$ stands for the probability of getting a black ball each time.

Teacher: Also, $(1-p)p$ means that A withdraws a black ball, and then B gets a red ball to win the game in the end.

What does $(1-p)\times(1-p)\times(1-p)\times p=(1-\mathrm{p})^3 p$ mean?

Student: It means A takes out a black ball first, B takes out a black ball again, then A takes out a black ball, and then B takes out a red ball to win the game in the end.

Teacher: If A takes out a black ball first, B takes out a black ball again, then A takes out a black ball, B takes out a black ball again, and the turns are repeated until B takes out a red ball at the end to win the game. The expressions can be written as: $(1-p)p$, $(1-p)^3p, (1-p)^5p$, … Add up these terms and give the sum.

Student: The corresponding series is an infinite geometric series: $(1-p)p+(1-p)^3p+(1-p)^5p+\ldots$ with the first term $(1-p)p$ and the common ratio $(1-p)^2$. The sum is: $\frac{(1-p)p}{1-(1-p)^2}=\frac{\frac{2}{3}\times\frac{1}{3}}{1-(\frac{2}{3})^2}=\frac{2}{5}$ Hence the probability that B wins the game is $\frac{2}{5}$

老師： p 代表取到紅球的機率，p 值為何？$1-p$又代表甚麼？

學生： 顯而易見 $p=\frac{2}{2+4}=\frac{1}{3}$，$1-p=\frac{2}{3}$ 代表取到黑球的機率。

老師： $(1-p)p$ 代表甲先取到黑球乙再取到黑球的機率。

接著$(1-p)\times(1-p)\times(1-p)\times p=(1-\mathrm{p})^3 p$ 如何解釋呢？

學生： 可解釋成甲先取到黑球、乙再取到黑球，接著甲又取到黑球，乙最後取到紅球贏得比賽的機率。

老師： 如果甲先取到黑球、乙再取到黑球，接著甲又取到黑球，乙再取到黑球，…，依序輪流取球，乙最後取到紅球贏得比賽。我們分別寫下算式為 $(1-p)p$，$(1-p)^3p, (1-p)^5p$,…將這些項相加並求其和？

學生： 對應的級數是無窮等比級數，其首項是$(1-p)p$、公比是$(1-p)^2$。

其和為 $\frac{(1-p)p}{1-(1-p)^2}=\frac{\frac{2}{3}\times\frac{1}{3}}{1-(\frac{2}{3})^2}=\frac{2}{5}$，因此最後乙獲勝的機率是 $\frac{2}{5}$。

單元三　函數的概念
The Concept of Functions

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■ 前言 Introduction

在前幾冊中我們學過許多函數，例如多項式函數、指數與對數函數、三角函數等，接下來我們會先複習函數的概念後將函數以集合的形式定義，並介紹函數的四則運算與合成、反函數、奇偶函數和一些常見函數的圖形。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
function	**函數**	**independent variable**	**自變數**
element	元素	**dependent variable**	**應變數**
domain	**定義域**	implicit function	隱函數
corresponding domain / codomain	**對應域**	**composite function**	**合成函數**
range	**值域**	**inverse function**	**反函數**
Gaussian function	高斯函數	piecewise function	分段函數
real-valued function	實數值函數	natural exponential function	標準指數函數

odd function	奇函數	natural logarithm function	自然對數函數
even function	偶函數	concavity	凹向性

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ think of __(something)__ as __(something else)__.

例句：We often **think of** a function **as** a machine.

我們經常將函數想像成一台機器。

❷ _______ is called _______.

例句：The new function **is called** a composite function.

新的函數稱為合成函數。

❸ _______ associates with _______.

例句：A function is a relation that uniquely **associates** members of one set **with** members of another set.

函數是一種將一個集合的元素與另一個集合的元素唯一對應的關係。

❹ _______ is written as _______.

例句：The inverse of $f(x)$ **is written as** $f^{-1}(x)$.

$f(x)$ 的反函數寫成 $f^{-1}(x)$。

❺ That's the reason why _______.

例句：**That's the reason why** inverse functions only exist for one-to-one functions.

這就是為什麼反函數僅存在於一對一函數的原因。

❻ _______ is/am/are neither _______ nor _______.

例句：In fact, most functions **are neither** odd **nor** even.

事實上，大多數函數既不是奇函數也不是偶函數。

■ 問題講解 Explanation of Problems

☙ 說明 ❧

We are going to review the concept of functions and develop a deeper understanding of functions and their special properties.

Why do we study functions?

Many situations in the real world can be modeled as functions. For example, the number of bacteria present after the start of an experiment or the temperature of a hot drink as it cools over time.

What is a function?

We often think of a function as a machine. We can input a variable x, and then we get an output which is a variable y. x is called an **independent variable** and y is called a **dependent variable.**

A **function** is a relation that uniquely associates members of one set with members of another set. Functions can either be one-to-one or many-to-one.

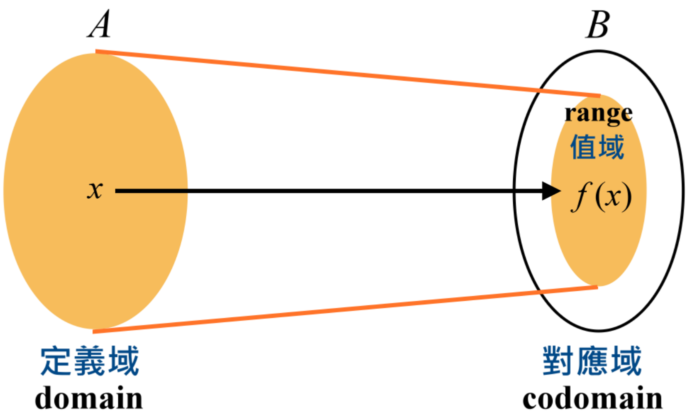
The **domain** is the set of all possible inputs for a function. The **codomain** is the set of all possible outputs for a function. The **range** is the set of all outputs for a function.

The codomain and range are both on the output side, but are a little different.

The codomain is the set of values that could **possibly** come out. And the range is the set of values that **do** come out.

So, the range is a subset of the codomain.

You can see this in the diagram below.



If A and B are sets, then a function f from A to B, written $f: A \to B$, is a rule that associates each element x in A with a unique element denoted $f(x)$ in B. A is the domain of f, B is the codomain of f, and the set $\{f(x) | x \in A\}$ is the range of f.

Function Laws

It is possible to combine functions in several different ways.

For example, if $f(x) = x + 1$ and $g(x) = x - 3$, then we could write

$f(x) + g(x) = x + 1 + x - 3 = 2x - 2$.

In this example, two functions are added.

Similarly, if $f(x) = x + 1$ and $g(x) = x^2$, then $f(x) \cdot g(x) = (x + 1) \cdot x^2 = x^3 + x^2$.

In this example, two functions are multiplied.

In fact, we can add, subtract, multiply, and divide functions. The domains are the **intersection** of their domains.

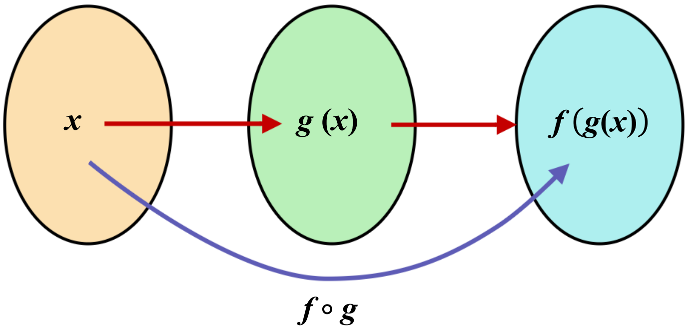
Let $f(x)$ and $g(x)$ be real-valued functions. Then each of the following statements holds:

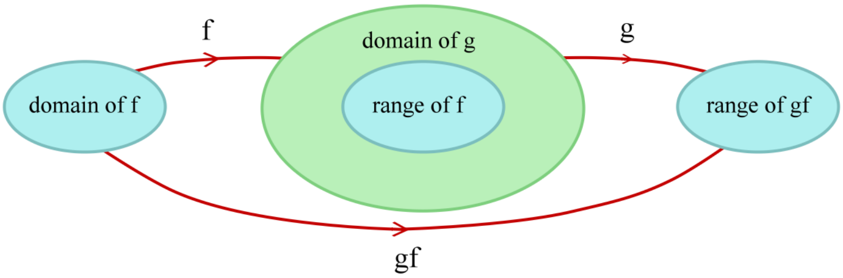
- Sum law for functions: $(f + g)(x) = f(x) + g(x)$, the domains are the intersection of $f(x)$ and $g(x)$.
- Difference law for functions: $(f - g)(x) = f(x) - g(x)$, the domains are the intersection of $f(x)$ and $g(x)$.
- Product law for functions: $(fg)(x) = f(x)g(x)$, the domains are the intersection of $f(x)$ and $g(x)$.
- Quotient law for functions: $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$, the domains are the intersection of $f(x)$ and $g(x)$ but $g(x) \neq 0$.

Composite functions

Two or more functions can be combined to make a new function. The new function is called a **composite function**. It is also called a function of a function. The symbol for composition is a small circle. We can see the key points of composition functions below.

- $(f \circ g)(x)$ means apply g first, then apply f.
- $(f \circ g)(x) = f(g(x))$
- $f(g(x))$ is read as "f of g of x."



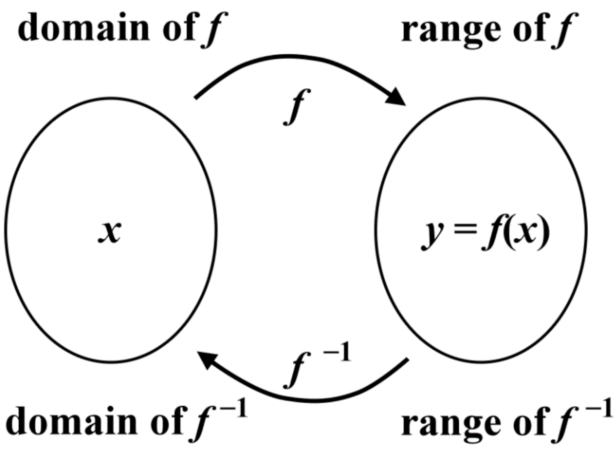


Inverse functions

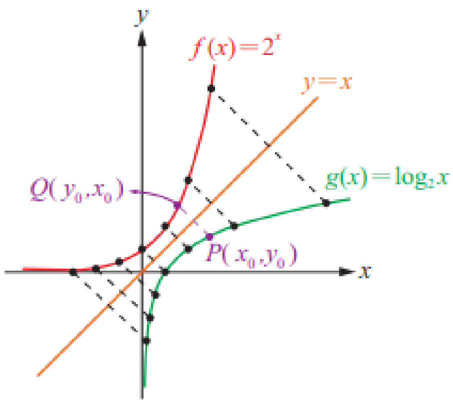
The inverse of a function $f(x)$ is the function that undoes what $f(x)$ has done. In other words, the inverse of a function performs the opposite operation to the original function. It takes the elements in the range of the original function and maps them back into elements of the domain of the original function. That's why inverse functions only exist for one-to-one functions.

The inverse of $f(x)$ is written as: $f^{-1}(x)$. This is read as the "f inverse of x."

- $ff^{-1}(x) = f^{-1}f(x) = x$
- The graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of each other about the line $y = x$.



For example, $f(x) = 2^x$ has an inverse function $g(x) = \log_2 x$. The graph of $g(x)$ is a reflection of the graph of $f(x)$ about the line $y = x$.



The graph of functions

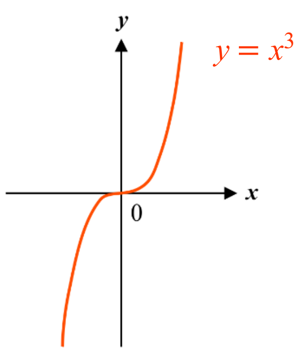
Vertical line test

Given a function f, every vertical line that may be drawn intersects the graph of f no more than once. If any vertical line intersects a set of points more than once, the set of points does not represent a function.

Common functions

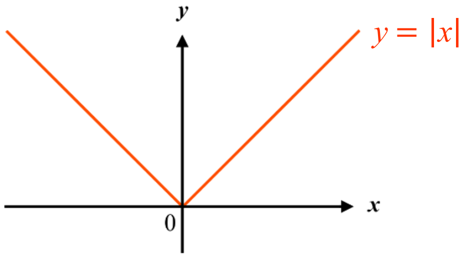
- Cubic function: $f(x) = x^3$

  Its domain is all the real numbers. Its range is also the real numbers.



- Absolute value function: $f(x) = |x|$

  Its domain is all the real numbers. Its range is: $\{y \mid y \in R, y \geq 0\}$



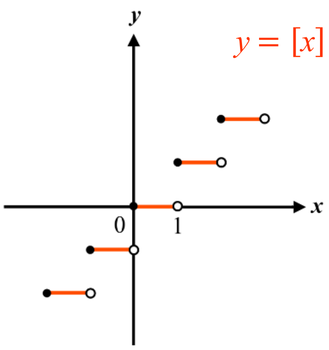
Absolute value function can also be expressed by the following formula:

$$f(x) = |x| = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$$

- Gaussian function/floor function: $f(x) = [x]$

  $[x]$ is the greatest integer that is less than or equal to x.

  Its domain is all the real numbers. Its range is all the integers.



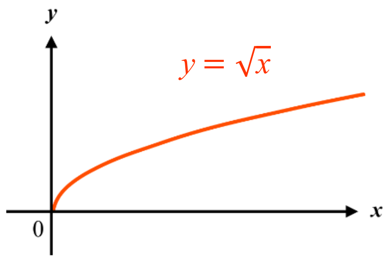
The Gaussian function can also be expressed by the following formula:

$$f(x) = |x| = \begin{cases} \vdots \\ 2, 2 \leq x < 3 \\ 1, 1 \leq x < 2 \\ 0, 0 \leq x < 1 \\ -1, 1 \leq x < 0 \\ -2, \ -2 \leq x < -1 \\ \vdots \end{cases}$$

The absolute value function and the Gaussian function are both piecewise functions.

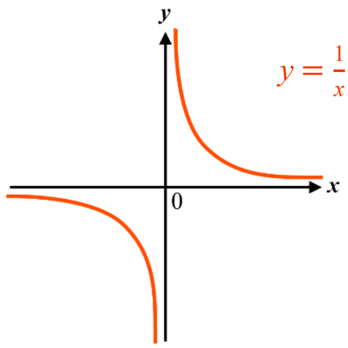
- Square root function: $f(x) = \sqrt{x}$

  Its domain is: $\{x \mid x \in R, x \geq 0\}$ Its range is: $\{y \mid y \in R, y \geq 0\}$



- Reciprocal function: $f(x) = \frac{1}{x}$

  Its domain is: $\{x \mid x \in R, x \neq 0\}$ Its range is: $\{y \mid y \in R, y \neq 0\}$

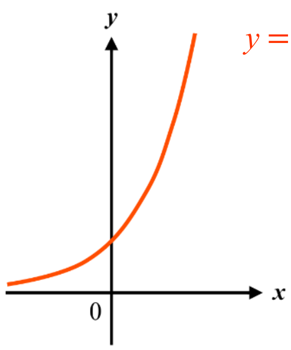


- Natural exponential function: $f(x) = e^x$

  e is "Euler's Number." The value of ***e*** is **about** 2.718281828459.

  The number ***e*** is one of the most important numbers in mathematics.

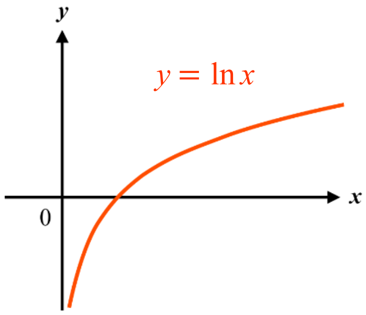
  The domain of natural exponential function is all the real numbers. Its range is: $\{y \mid y \in R, y > 0\}$



- Natural logarithm function: $f(x) = \log_e x = \ln x$

  Natural logarithm function is the logarithm function using base e which is Euler's Number.

  Its domain is: $\{x|\, x \in R, x > 0\}$ Its range is all the real numbers.



Odd and even functions

A function is called "odd" when $f(-x) = -f(x)$ for all x. The graph of the function has origin symmetry.

For example, the cubic function $f(x) = x^3$ is an odd function. The graph of the function has origin symmetry.

A function is called "even" when $f(-x) = f(x)$ for all x. The graph of the function has symmetry about the y-axis.

For example, the absolute value function $f(x) = |x|$ is an even function. The graph of the function has symmetry about the y-axis.

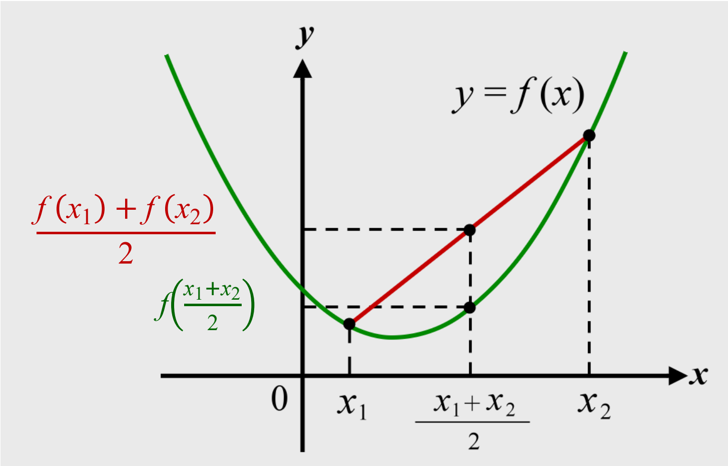
Don't be misled by the names "odd" and "even". They are just **names** and a function does **not have to be** even or odd. In fact, most functions are neither odd nor even.

Concavity

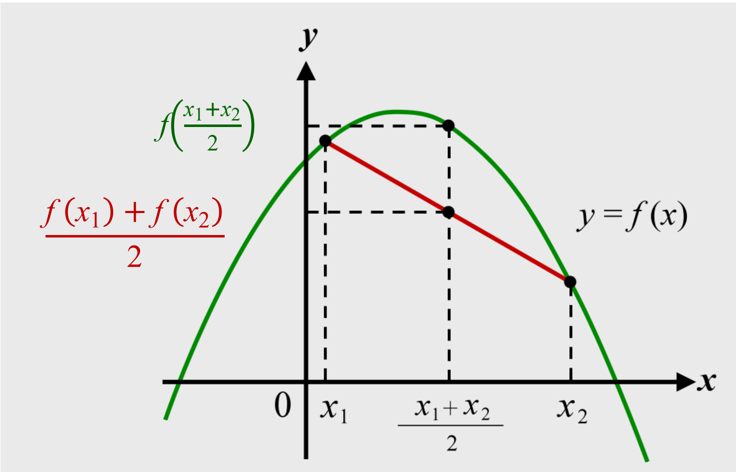
Let's review the concavity of function graphs discussed before:

Let $f(x)$ be a continuous function, and x_1, x_2 are any two numbers of the domain of f.

The graph of f is concave up if $\frac{f(x_1)+f(x_2)}{2} \geq f\left(\frac{x_1+x_2}{2}\right)$



The graph of f is concave down if $\frac{f(x_1)+f(x_2)}{2} \leq f\left(\frac{x_1+x_2}{2}\right)$



運算問題的講解

例題一

說明：找出函數的定義域與值域。

（英文）Find the largest possible domain for each function.

(i) $f(x) = \frac{1}{x}$

(ii) $f(x) = \sqrt{1-x}$

（中文）找到下列函數的定義域。

(i) $f(x) = \frac{1}{x}$

(ii)$f(x) = \sqrt{1-x}$

Teacher: What is the restriction of the denominator of a fraction?

Student: The denominator of a fraction is not zero.

Teacher: So, do you know what the domain is for the first function?

Student: x is a real number but is not zero.

Teacher: For the second function, what are the restrictions on the numbers inside the root sign?

Student: They are nonnegative.

Teacher: We now get $1-x$ is nonnegative, which means $1-x \geq 0$. So, what is the range of x?

Student: We get: $x \leq 1$.

Teacher: So, do you know what the domain is for the second function?

Student: x is a real number but x is no greater than 1.

老師： 分數的分母有什麼限制條件？

學生： 分母不為零。

老師： 所以你知道第一個函數的定義域為何嗎？

學生： x 是實數但不為零。

老師： 觀察第二個函數，根號中的數字有何限制條件嗎？

學生： 根號內要是非負的數字。

老師： 我們可知 $1-x$ 是非負，也就是說 $1-x \geq 0$。x 的範圍為何呢？

學生： 我得到 $x \leq 1$。

老師：　所以你知道第二個函數的定義域為何嗎？
學生：　x 是實數但 x 不大於 1。

例題二

說明：合成函數的運算。

（英文）The functions f and g are defined by $f(x) = 2x + 1$ and $g(x) = x^2$. Find:
(i) the function $(g \circ f)(x)$
(ii) the function $(f \circ g)(x)$
（中文）設函數 $f(x) = 2x + 1$，$g(x) = x^2$。求：
(i) $(g \circ f)(x)$
(ii) $(f \circ g)(x)$

Teacher:　In the first function $(g \circ f)(x)$, what is the first function to act?
Student:　$f(x)$.
Teacher:　The composite function $(g \circ f)(x)$ should be performed from right to left: start with *x*, then apply *f*, and then *g*. So, we first get $2x + 1$ then g acts on the result. We get:
$(g \circ f)(x) = g(2x + 1) = (2x + 1)^2$
Now simplify the answer.
Student:　$4x^2 + 4x + 1$
Teacher:　Next, we see the second composite function. What is the first function to act?
Student:　$g(x)$
Teacher:　Please try to solve the function $(f \circ g)(x)$ now.
Teacher:　What is the result?
Student:　$2x^2 + 1$
Teacher:　Compare the two functions of this question, do you think $(g \circ f)(x) = (f \circ g)(x)$?
Student:　No.
Teacher:　Clearly, the order of functions can cause different results. This tells us, in general terms, that the order in which we compose functions matters.

老師：在第一個函數 $(g \circ f)(x)$ 中，哪一個函數先起作用呢？

學生：$f(x)$。

老師：合成函數 $(g \circ f)(x)$ 都是由左至右作用，先將 x 放在 f 中再作用於 g 。我們先得到 $2x+1$，再將 $2x+1$ 作用於 g。
得 $(g \circ f)(x) = g(2x+1) = (2x+1)^2$。
請化簡答案。

學生：$4x^2+4x+1$。

老師：接著我們來看第二個合成函數。哪一個函數先起作用呢？

學生：$g(x)$。

老師：現在請嘗試解出 $(f \circ g)(x)$。

老師：請問結果是什麼？

學生：$2x^2+1$。

老師：比較一下這個問題中的兩個函數，請問是否 $(g \circ f)(x) = (f \circ g)(x)$？

學生：不是。

老師：我們可以知道合成的順序可以得到不同的結果。一般來說，這告訴我們，合成函數的順序是很重要的。

應用問題 / 學測指考題

例題一

說明：用合成函數的概念找出關係式。

（英文）A new mother is bathing her baby for the first time. She takes the temperature of the bathwater with a thermometer which reads in Celsius, but then has to convert the temperature to degrees Fahrenheit to apply the rule that her mother taught her:

"At one o five,
he'll cook alive,
but ninety-four,
is rather raw."

Write down the two functions that are involved, and apply them to readings of:

(1)30℃ (2)36℃ (3)46℃ .

（中文）一位新手媽媽第一次幫她的寶寶洗澡。她用攝氏溫度的溫度計測量了洗澡水的溫度，但隨後必須將溫度轉換為華氏溫度，以便應用她母親教給她的規則：

在 105 度，
他會活活被煮熟。
但在 94 度，
還是相當生的。

請寫出所涉及的兩個函數，並應用它們來解讀以下的溫度

(1)30℃ (2)36℃ (3) 46℃。

Teacher: What is the function that converts the Celsius temperature C into a Fahrenheit temperature F?

Student: $F=\frac{9}{5}C+32$.

Teacher: Good job! The second function maps Fahrenheit temperatures onto the state of the bath. What can we get from this rule?

Student: The Fahrenheit temperature $F\geq 105$ is too hot. The baby will be "cooked alive." $F\leq 94$ is too cold for the baby.

Teacher: Excellent. Our best temperature for bathing babies is 94 to 104 degrees Fahrenheit. We get another function, we may call it $g(F)$. We write $g(F)$ as a piecewise function as follows:

$$g(F)=\begin{cases}\text{too hot if } F \geq 105°F \\ \text{all right if } 94°F < F < 105°F \\ \text{too cold if } F \leq 94°F\end{cases}$$

In this case the composite function would be (to the nearest degree):

$$g(C)=\begin{cases}\text{too hot if } C \geq 41°C \\ \text{all right if } 34°C < C < 41°C \\ \text{too cold if } C \leq 34°C\end{cases}$$

Which temperature is most suitable for bathing a baby? (1)30℃, (2)36℃ and (3)46℃ ?

Student: 30℃ is too cold. 36℃ is all right. 46℃ is too hot for a baby.

老師： 將攝氏溫度 C 轉換為華氏溫度 F 的函數是什麼？

學生： $F=\frac{9}{5}C+32.$

老師： 很好。第二個函數將華氏溫度轉換到浴缸的狀態。根據這條規則我們能得到什麼呢？

學生： 華氏溫度 $F \geq 105$ 太熱。寶寶會活活被煮熟。$F \leq 94$ 對寶寶來說太冷了。

老師： 太棒了。我們給嬰兒洗澡的最佳溫度是華氏 94 至 104 度。我們得到另一個函數，我們可以稱之為 $g(F)$。我們將 $g(F)$ 寫為分段函數，如下所示：

$$g(F)=\begin{cases}\text{太熱，當 } F \geq 105°F \\ \text{剛剛好，當 } 94°F < F < 105°F \\ \text{太冷，當 } F \leq 94°F\end{cases}$$

在這種情況下，合成函數將是（取最接近的整數度數）

$$g(C)=\begin{cases}\text{太熱，當 } C \geq 41°C \\ \text{剛剛好，當 } 34°C < C < 41°C \\ \text{太冷，當 } C \leq 34°C\end{cases}$$

哪個溫度最適合給寶寶洗澡？ (1)30℃、(2)36℃、(3)46℃

學生： 30℃ 太冷。36℃ 剛剛好。46℃ 太熱了。

例題二

說明：分段函數的解讀與應用。

（英文）The taxi fare in a specific city is as follows: the fare is NT$85 for a distance not exceeding 1.25 kilometers, and NT$5 is charged for every 0.2 kilometers beyond 1.25 kilometers, and there is no charge for a distance less than 0.2 kilometers. Now, use the function $g(x)$ to express the fare required to travel x kilometers.

Expressed as follows:

$$g(x)=\begin{cases}85, x\le 1.25\\ a+b\left[\frac{x-1.25}{0.2}\right], x>1.25\end{cases}$$

(1) Find the values of a, b.

(2) How much should Jenny pay if the distance by taxi is 4.8 kilometers?

（中文）某城市的計程車車資計費方式為：不超過 1.25 公里，車資一律為 85 元，超過 1.25 公里的部分，每 0.2 公里加收 5 元，不足 0.2 公里部分不計費。今以函數 $g(x)$ 表示搭乘 x 公里所需支付的車資，表示如下：

$$g(x)=\begin{cases}85, x\le 1.25\\ a+b\left[\frac{x-1.25}{0.2}\right], x>1.25\end{cases}$$

(1) 請寫出常數 a、b 之值。

(2) 若 Jenny 搭乘計程車的路程為 4.5 公里，則他應該付多少車資呢？

Teacher: In this city, we know that the taxi fare is NT$85 for a distance no greater than 1.25 kilometers, so we get the first part of $g(x)$. $g(x)$ is a piecewise function. We get $g(x)=85$ when $x\le 1.25$.

Teacher: We also notice that NT$5 is charged for every 0.2 kilometers beyond 1.25 kilometers, and there is no charge for the distance less than 0.2 kilometers. What can we get from this description? What is the value of a in $g(x)$?

Student: a is 85.

Teacher: That's right. What is the value of b?

Student: b is 5.

Teacher: Correct. For the second question, if Jenny travels 4.5 kilometers by taxi, how much should she pay?

Student: You only need to put $a = 85, b = 5, x = 4.5$ into the $g(x)$.

Then you get: $g(4.5) = 85 + 5\left[\frac{4.5-25}{0.2}\right] = 85 + 5 \times [16.25] = 165$. The answer is 165.

Teacher: Good job! So, Jenny should pay NT$165 for this ride.

老師： 在這個城市，我們知道距離不超過 1.25 公里時，計程車費用為 85 元，因此我們得到 $g(x)$ 的第一部分。$g(x)$ 是分段函數。當 $x \leq 1.25$ 時，我們得到 $g(x) = 85$。

老師： 我們還注意到，超過 1.25 公里的距離，每 0.2 公里加收 5 元，0.2 公里以內的距離不收費。從這個描述中我們能得到什麼？具體來說，$g(x)$ 中 a 的值是多少？

學生： a 是 85。

老師： 沒錯。b 的值是多少？

學生： b 是 5。

老師： 正確。針對第二個問題，如果珍妮乘坐計程車行駛了 4.5 公里，她應該付多少錢呢？

學生： 我把 $a = 85$、$b = 5$、$x = 4.5$ 代入 $g(x)$。我得到 $g(4.5) = 85 + 5\left[\frac{4.5-25}{0.2}\right] = 85 + 5 \times [16.25] = 165$。答案是 165。

老師： 做得好。所以 Jenny 這次搭車應該付 165 元。

單元四　函數的極限
The Limit of a Function

臺灣師範大學附屬高級中學　林佳葦老師

■ 前言 Introduction

函數的極限是微積分學的核心概念之一。在這一節中，我們將透過函數的極限來定義連續函數，認識極限的運算性質，接著介紹連續函數的介值定理，及其應用的勘根定理，最後介紹夾擠定理。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
approach	趨近	left limit	左極限
limit	極限	right limit	右極限
continuity	**連續**	approximate	估計
continuous function	連續函數	discontinuity	不連續
intermediate value theorem	**介質定理（中間值定理）**	squeeze/sandwich theorem	夾擠定理
determination of roots	**勘根定理**	**Bolzano's theorem intermediate / zero theorem**	**零點定理**

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ ________ either ________ or ________

例句：We often use **either** graphical **or** numerical methods to approximate the value.
我們經常使用圖形或數值方法來估計該值。

❷ _______ be as _______ as _______.

例句：In other words, the difference between $\frac{x^2-x-2}{x-2}$ and 6 can **be as** close **as** you want.

換句話說，$\frac{x^2-x-2}{x-2}$ 和 6 之間的差異可以盡可能接近。

❸ It can be deduced that_______.

例句：**It can be deduced that** $\lim_{x \to a} x^n = a^n$ is true for all positive integers n.

可以推斷，對於所有正整數 n，$\lim_{x \to a} x^n = a^n$ 成立。

❹ It is trivial that _______.

例句：**It is trivial that** this statement is true.
這句話的真實性是容易解決的。

❺ According to _______.

例句：**According to** the factor theorem, what relation equation can we get?
根據因式定理，我們可以得到什麼關係式？

❻ _______ whether _______ or not

例句：We are unable to determine **whether** $\lim_{x \to 0}(f(x)+1)\frac{x}{|x|}$ exists **or not.**

我們無法判斷是否 $\lim_{x \to 0}(f(x)+1)\frac{x}{|x|}$ 存在或不存在。

■ 問題講解 Explanation of Problems

說明

An introduction to limits

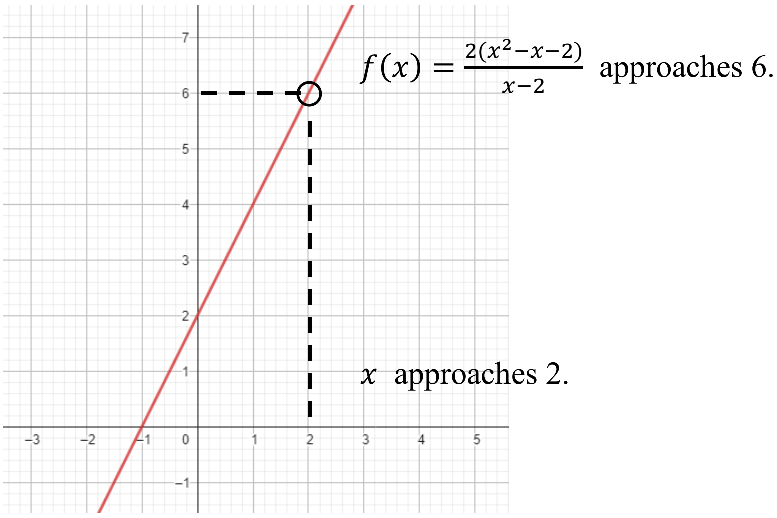
We begin our study of limits by considering examples that demonstrate key concepts that will be explained as we progress.

Consider the function $f(x)=\frac{2(x^2-x-2)}{x-2}$. When x is near the value 2, what value (if any) is $f(x)$ near?

Although our question is not precisely formed (what “near the value 2” is), the answer is easy to find.

First, we look at the graph of this function to approximate the appropriate $f(x)$ values.

Let $y=f(x)$. Consider Figure 1, where $y=\frac{2(x^2-x-2)}{x-2}$ is graphed. We also use a table to list the values of the function for the given values of x. See Table 1. We often use either graphical or numerical methods to approximate the value.



x	$y=\frac{2(x^2-x-2)}{x-2}=2(x+1)$ if $x\neq 2$
1.9	5.8
1.99	5.98
1.999	5.998
1.9999	5.9998
2	Undefined

2.0001	6.0002
2.001	6.002
2.01	6.02
2.1	6.2

We know that when x is near the value 2, $y=\frac{2(x^2-x-2)}{x-2}$ is near the value 6. To be more precise, we want the value of $y=\frac{2(x^2-x-2)}{x-2}$ to be closer to 6, so there is a way to limit the range of x, so that for x within this range, the difference between y and 6 will be determined within the selected error range. For example,

1. For the desired error of 0.01, you can choose 0.005, so that when $0<|x-2|<0.005$, $\left|\frac{x^2-x-2}{x-2}-6\right|<0.01$.
2. For the desired error of 0.001, you can choose 0.0005, so that when $0<|x-2|<0.0005$, $\left|\frac{x^2-x-2}{x-2}-6\right|<0.001$.
3. For the desired error of 0.0001, you can choose 0.00005, so that when $0<|x-2|<0.00005$, $\left|\frac{x^2-x-2}{x-2}-6\right|<0.0001$.

In other words, the difference between $\frac{x^2-x-2}{x-2}$ and 6 can be as close as you want. Mathematically, we say that the limit of $f(x)$ as x approaches 2 is 6. We express this limit in symbols as:

$\lim_{x\to 2} f(x)=6$.

Definition of limits

Let $f(x)$ be a function defined at all values in an open interval containing a, and let L be a real number. If all values of the function $f(x)$ approach the real number L as the values of x $(x\neq a)$ approach the number a, then we say that the limit of $f(x)$ as x approaches a is L.

(More concisely, as x gets closer to a, $f(x)$ gets closer and stays close to L.)

We express this idea as: $\lim_{x\to a} f(x)=L$.

Below are some ideas you need to know.

- "x approaches a" means "x is getting closer and closer to a", not "$x = a$". "x approaches a" is written as: $x \to a$
- For the limit of a function to exist at a point, the function values must approach a single real number value at that point.
- If the function values do not approach a single value, then the limit does not exist.
- Not every function has a limit as x approaches a.
- The value $f(a)$ does not necessarily exist.

One-sided limits

Sometimes, the limit of a function exists from one side or the other (or both) even though the limit does not exist. Since it is useful to talk about this situation, we introduce the concept of a one-sided limit:

Call L the right limit of a function $f(x)$ as x approaches a, written as $\lim_{x \to a^+} f(x) = L$, if $f(x)$ approaches L as x approaches a for values of x larger than a.

Call L the left limit of a function $f(x)$ as x approaches a, written as $\lim_{x \to a^-} f(x) = L$, if $f(x)$ approaches L as x approaches a for values of x smaller than a.

Let's talk about the relationship between the limit of a function at a point and the limits from the right and left at that point. Clearly, if the limit from the right and the limit from the left have a common value, then that common value is the limit of the function at that point. Similarly, if the limit from the left and the limit from the right take on different values, the limit of the function doesn't exist. These conclusions are summarized in the following part.

The condition for the existence of limits

The limit of a function exists **if and only if** both its right limit and left limit exist and are equal:

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x)$$

We now take a look at the limit laws, the individual properties of limits.

Limit laws

Suppose that the limits of $f(x)$, $g(x)$ exist. Let $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$, and k be a constant. Then each of the following statements holds:

(1) Constant law

$$\lim_{x\to a} kf(x) = kL$$

(2) Sum law and difference law

$$\lim_{x\to a} (f(x) \pm g(x)) = L \pm M$$

(3) Product law

$$\lim_{x\to a} (f(x)g(x)) = LM$$

(4) Quotient law

If $M \neq 0$ and $g(x) \neq 0$ for all $x \neq a$ on an open interval I containing a, then $\lim_{x\to a} \left(\frac{f(x)}{g(x)}\right) = \frac{L}{M}$

(5) Root law

If $L \geq 0$ and $f(x) \geq 0$ for all $x \neq a$ on an open interval I containing a, then $\lim_{x\to a} \sqrt{f(x)} = \sqrt{L}$

Suppose k is a constant. It is obvious that $\lim_{x\to a} k = k$ and $\lim_{x\to a} x = a$. Using the above properties (3) product law and mathematical induction, it can be deduced that $\lim_{x\to a} x^n = a^n$ is true for all positive integers n. Therefore, if $f(x) = k_n x^n + k_{n-1}x^{n-1} + \cdots + k_0$ is a polynomial function, then

$$\begin{aligned}
\lim_{x\to a} f(x) &= \lim_{x\to a} (k_n x^n + k_{n-1}x^{n-1} + \cdots + k_0) \\
&= \lim_{x\to a} k_n x^n + \lim_{x\to a} k_{n-1}x^{n-1} + \cdots + \lim_{x\to a} k_0 \\
&= k_n \lim_{x\to a} x^n + k_{n-1} \lim_{x\to a} x^{n-1} + \cdots + k_0 \\
&= k_n a^n + k_{n-1}a^{n-1} + \cdots + k_0 \\
&= f(a)
\end{aligned}$$

Generally speaking, when finding the limit of a polynomial function, you can just substitute the value directly. Using the above properties (4) quotient law, we can also substitute the value directly when finding the limit of a rational function (the denominator must be non-zero).

We rewrite them as follows:

Limits of polynomial and rational functions

Let $f(x)$, $g(x)$ be polynomials and a a real number. Then

1. $\lim_{x \to a} f(x) = f(a)$

2. $\lim_{x \to a} \left(\frac{f(x)}{g(x)}\right) = \frac{f(a)}{g(a)}$, when $g(a) \neq 0$

Next, we introduce an important theorem.

The Squeeze/Sandwich Theorem

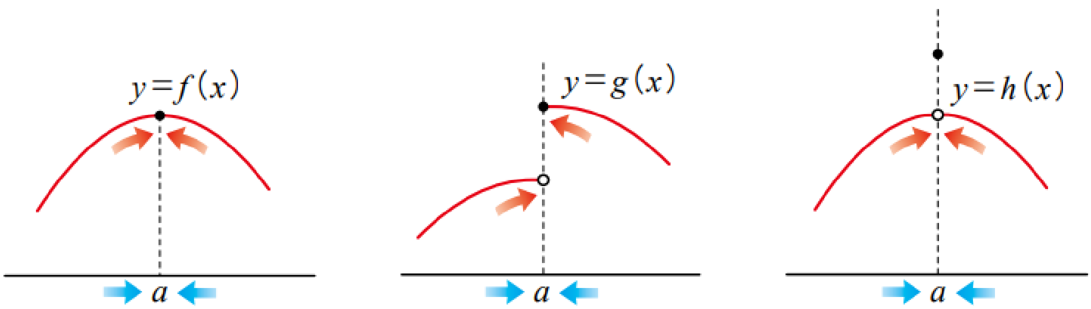
Suppose that $g(x) \leq f(x) \leq h(x)$ for all x close to a but not necessarily equal to a. If $\lim_{x \to a} g(x) = L = \lim_{x \to a} h(x)$, then $\lim_{x \to a} f(x) = L$ where L is a real number.

Intuitively, the squeeze (or sandwich) theorem says that if one function is "squeezed" between two functions approaching the same limit, then the function in the middle must also approach that limit.

Continuity

Informally, a function is continuous if you can "draw it" without "lifting your pencil." For example, linear functions, quadratic functions, exponential functions, logarithmic functions, sine functions, and cosine functions are all continuous functions.

Look at the following graphs:



(1) (2) (3)

We say that the function in Figure 1 is continuous at $x=a$, but the function in Figures 2 and 3 are discontinuous at $x=a$. Observe Figure 1; the function is continuous at $x=a$. It means that no matter when x approaches a from the left or the right, the function must have a limit, and the limit value of $f(x)$ at $x=a$ is just the function value $f(a)$.

We now give the formal definition of continuity.

Continuity at a point

Suppose a is in the domain of f. A function f is continuous at a point a if $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} f(x) = f(a)$.

A useful way to establish whether or not a function f is continuous at a is to verify the following three things:

1. $\lim_{x\to a} f(x)$ exists
2. $f(a)$ is defined
3. $\lim_{x\to a} f(x) = f(a)$

Continuous functions

A function f is continuous on an interval if it is continuous at every point in the interval.

For a closed interval $I=$ [a, b], a function f is continuous on I if it is continuous on the open interval (a,b) and if $\lim_{x\to a^+} f(x) = f(a)$ (i.e. f is right continuous at $x=a$) and $\lim_{x\to b^-} f(x) = f(b)$ (i.e. f is left continuous at $x=b$).

A function is discontinuous at a point if it is not continuous there.

A continuous function is continuous over its entire domain.

Properties of continuous functions

If $f(x)$, $g(x)$ are continuous at $x = a$, then each of the following statements holds:

(1) $f(x) + g(x)$ is continuous at $x = a$.

(2) $f(x) - g(x)$ is continuous at $x = a$.

(3) $f(x)g(x)$ is continuous at $x = a$.

(4) $\frac{f(x)}{g(x)}$ is continuous at $x = a$, where $g(x) \neq 0$ for all x in an open interval containing a.

Continuity of polynomials and rational functions

Polynomials and rational functions are continuous at every point in their domains.

It is trivial that this statement is true. Previously, we showed that if $f(x)$, $g(x)$ are polynomials and a is a real number. Then:

1. $\lim\limits_{x \to a} f(x) = f(a)$

2. $\lim\limits_{x \to a} \left(\frac{f(x)}{g(x)}\right) = \frac{f(a)}{g(a)}$, when $g(a) \neq 0$

Therefore, polynomials and rational functions are continuous in their domains.

Composite Function Theorem

If $f(x)$ is continuous at $x = a$ and $g(x)$ is continuous at $x = f(a)$, then $(g \circ f)(x)$ is continuous at $x = a$.

If a ball falls from the balcony on the 7th floor to the ground on the 1st floor, it must pass by the 4th floor at some point. The temperature was 20°C at 9:00 am and rose to 30°C by 14:00 pm. Since the temperature changes continuously, there will be at least one moment when the temperature is exactly 25°C during the period from 9:00 to 14:00. These everyday examples help us understand the Intermediate Value Theorem.

Intermediate Value Theorem

If $f(x)$ is a continuous function for all x in the closed interval $[a, b]$ and d is between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ such that $f(c) = d$. Note that this theorem only guarantees that c must be found (there may be more than one), but it does not tell us how to find it.

An important application of the Intermediate Value Theorem is root finding. Given a function f, we are often interested in finding values of x where $f(x) = 0$. These roots may be very difficult to find exactly. Good approximations can be found by successive applications of this theorem.

Suppose by direct computation, we find that $f(a) < 0$ and $f(b)>0$, where $a < b$. The Intermediate Value Theorem states at least one c in (a, b) such that $f(c) = 0$. The theorem does not give us any clue about where to find such a value in the interval (a, b), only that there is at least one such value. The above process brings us to the following theorem:

Bolzano's Theorem

If $f(x)$ is continuous such that $f(a)$ and $f(b)$ have opposite signs, then $f(x) = 0$ has at least one real solution in $[a, b]$.

If a continuous function has values of opposite signs inside an interval, then it has a root in that interval. This theorem is a tool to approximate the root of an unsolvable equation or to show that it exists.

運算問題的講解

例題一

說明：計算函數的極限。

（英文）Evaluate the following limits.

(i) $\lim_{x\to 2}(x^2-3x+1)(x+1)$

(ii) $\lim_{x\to -1}[x+1]$

（中文）計算下列各極限。

(i) $\lim_{x\to 2}(x^2-3x+1)(x+1)$

(ii) $\lim_{x\to -1}[x+1]$

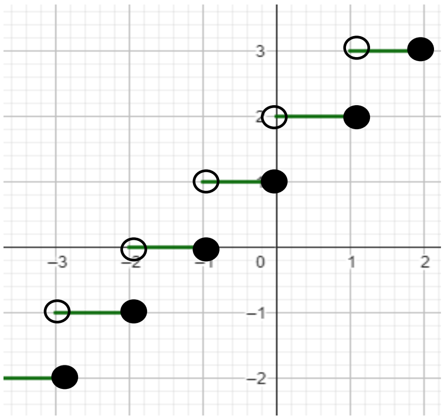
Teacher: Let's calculate the first limit. Since it combines two polynomials, we can just use the limit laws to solve the limit. Please use the limit laws to simplify the limit.

Student: $\lim_{x\to 2}(x^2-3x+1)(x+1)=\lim_{x\to 2}(x^2-3x+1)\cdot\lim_{x\to 2}(x+1)=-1\cdot 3=-3.$

Teacher: Good. Now let's see the second limit. Can we just put the number 2 directly into the limit?

Student: No. It is not a polynomial.

Teacher: Let's roughly draw the graph of $f(x)=[x+1]$. We can easily see the left limit and the right limit of $f(x)$. What are the two limits?



Student: The left limit $\lim_{x\to -1^-}[x+1]$ is 0. The right limit $\lim_{x\to -1^+}[x+1]$ is 1.

Teacher: Great. Then what is the limit of $f(x)$ at $x = -1$?

Student: The limit of $f(x)$ at $x = -1$ doesn't exist.

Teacher: Well done.

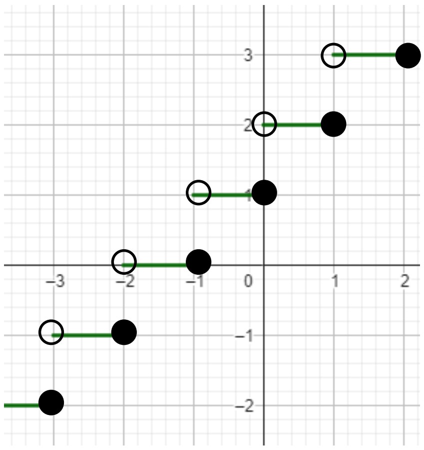
老師： 我們來計算第一個極限。由於它結合了兩個多項式，因此我們可以使用極限定律來求解極限。請使用極限定律來化簡極限。

學生： $\lim_{x\to 2}(x^2 - 3x + 1)(x + 1) = \lim_{x\to 2}(x^2 - 3x + 1)\cdot\lim_{x\to 2}(x + 1) = -1\cdot 3 = -3$。

老師： 好的。現在讓我們看看第二個極限。我們可以直接將數字 2 放入極限嗎？

學生： 不可以，它不是多項式。

老師： 我們粗略地畫出 $f(x) = [x + 1]$ 的圖。我們可以很容易地看出 $f(x)$ 的左極限和右極限。兩個極限是什麼？



學生： 左極限 $\lim_{x\to -1^-}[x + 1]$ 為 0。右極限 $\lim_{x\to -1^+}[x + 1]$ 為 1。

老師： 很好。那麼 $x = -1$ 時，$f(x)$ 的極限是多少？

學生： $x = -1$ 時，$f(x)$ 的極限不存在。

老師： 做得好。

例題二

說明：利用函數極限的性質解未知數。

（英文）Suppose $\lim_{x\to 2}\frac{ax^2+bx+2}{x^2-x-2}=1$ where a and b are real numbers. Find the values of a and b.

（中文）假設 $\lim_{x\to 2}\frac{ax^2+bx+2}{x^2-x-2}=1$ ，其中 a 和 b 是實數。求 a 和 b 的值 。

Teacher: Since $\lim_{x\to 2}\frac{ax^2+bx+2}{x^2-x-2}=1$, and $x^2-x-2=0$ at $x=2$, then we know $x-2$ must divide ax^2+bx+2. According to the factor theorem, what relation equation can we get?

Student: Let $x=2$. $a\cdot 2^2+b\cdot 2+2=0 \Rightarrow 4a+2b+2=0 \Rightarrow b=-2a-1$.

Teacher: Substitute $b=-2a-1$, we get

$ax^2+bx+2=ax^2+(-2a-1)x+2=(ax-1)(x-2)$.

Use this result in $\lim_{x\to 2}\frac{ax^2+bx+2}{x^2-x-2}=1$.

Student: $\lim_{x\to 2}\frac{ax^2+bx+2}{x^2-x-2}=1 \Rightarrow \lim_{x\to 2}\frac{(ax-1)(x-2)}{(x-1)(x-2)}=1 \Rightarrow \lim_{x\to 2}\frac{(ax-1)}{(x-1)}=1$

$\Rightarrow \lim_{x\to 2}\frac{(a\cdot 2-1)}{(2-1)}=1 \Rightarrow 2a-1=1 \Rightarrow a=1$.

Teacher: Good job. Now calculate b.

Student: $b=-2a-1=-2\cdot 1-1=-3$.

Teacher: Yes, $a=1$ and $b=-3$.

老師：由於 $\lim_{x\to 2}\frac{ax^2+bx+2}{x^2-x-2}=1$ ，在 $x=2$ 時，$x^2-x-2=0$，那麼我們知道 $x-2$ 必須整除 ax^2+bx+2。根據因式定理，我們可以得到什麼關係式？

學生：令 $x=2$。$a\cdot 2^2+b\cdot 2+2=0 \Rightarrow 4a+2b+2=0 \Rightarrow b=-2a-1$。

老師：代入$b=-2a-1$，得

$ax^2+bx+2=ax^2+(-2a-1)x+2=(ax-1)(x-2)$。

將此結果代入 $\lim_{x\to 2}\frac{ax^2+bx+2}{x^2-x-2}=1$。

學生：$\lim_{x\to 2}\frac{ax^2+bx+2}{x^2-x-2}=1 \Rightarrow \lim_{x\to 2}\frac{(ax-1)(x-2)}{(x-1)(x-2)}=1 \Rightarrow \lim_{x\to 2}\frac{(ax-1)}{(x-1)}=1$

$\Rightarrow \lim_{x\to 2}\frac{(a\cdot 2-1)}{(2-1)} = 1 \Rightarrow 2a-1=1 \Rightarrow a=1$。

老師： 很好。現在請計算 b。

學生： $b=-2a-1=-2\cdot 1-1=-3$。

老師： 是的，$a=1$ 和 $b=-3$。

例題三

說明：利用介值定理來解題。

（英文）Use the intermediate value theorem and function $f(x)=x^2$ to illustrate that there is a positive real number whose square is equal to 3(that is, the existence of $\sqrt{3}$).

（中文）試利用介值定理與函數 $f(x)=x^2$，來說明存在一正實數，其平方等於 3（即 $\sqrt{3}$ 的存在性）。

Teacher: Is the polynomial function $f(x)=x^2$ continuous?

Student: Yes, $f(x)=x^2$ is continuous.

Teacher: Great. Calculate $f(1)$ and $f(2)$.

Student: $f(1)=1$ and $f(2)=4$.

Teacher: Good job. So, can we apply the intermediate value theorem?

Student: Yes, because $f(x)=x^2$ is continuous, $f(1)=1$, $f(2)=4$, and 3 is between 1 and 4, therefore, according to the intermediate value theorem, there must be a positive real number c in the interval (1, 2), satisfying $f(c)=c^2=3$.

Teacher: Excellent.

老師： 多項式函數 $f(x)=x^2$ 是連續的嗎？

學生： 是的，$f(x)=x^2$ 是連續的。

老師： 很好。計算 $f(1)$ 和 $f(2)$。

學生： $f(1)=1$ 和 $f(2)=4$。

老師： 做得好。所以我們可以運用介質定理了嗎？

學生： 是的，因為 $f(x)=x^2$ 是連續的，$f(1)=1$，$f(2)=4$，和 3 介在 1 和 4 之間，

所以根據介質定理，在區間 $(1, 2)$ 中必存在一正實數 c，滿足 $f(c) = c^2 = 3$。
老師：太棒了。

應用問題 / 學測指考題

例題一

說明：分段函數的極限的存在性與四則運算性質。

（英文）Consider two functions $f(x)=\begin{cases}1+x, & x \le 1\\ 1, & x > 1\end{cases}$, $g(x)=\begin{cases}1, & x \le 1\\ 3-x, & x > 1\end{cases}$. Regarding the limit of the function, choose the correct option.

(1) $\lim\limits_{x\to 1} f(x)$ exists, $\lim\limits_{x\to 1} g(x)$ exists, $\lim\limits_{x\to 1} (f(x)+g(x))$ exists.

(2) $\lim\limits_{x\to 1} f(x)$ exists, $\lim\limits_{x\to 1} g(x)$ doesn't exist, $\lim\limits_{x\to 1} (f(x)+g(x))$ doesn't exist.

(3) $\lim\limits_{x\to 1} f(x)$ doesn't exists, $\lim\limits_{x\to 1} g(x)$ exists, $\lim\limits_{x\to 1} (f(x)+g(x))$ doesn't exist.

(4) $\lim\limits_{x\to 1} f(x)$ doesn't exist, $\lim\limits_{x\to 1} g(x)$ doesn't exist, $\lim\limits_{x\to 1} (f(x)+g(x))$ exists.

(5) $\lim\limits_{x\to 1} f(x)$ doesn't exist, $\lim\limits_{x\to 1} g(x)$ doesn't exist, $\lim\limits_{x\to 1} (f(x)+g(x))$ doesn't exist.

（中文）考慮兩個函數 $f(x)=\begin{cases}1+x, & x \le 1\\ 1, & x > 1\end{cases}$、$g(x)=\begin{cases}1, & x \le 1\\ 3-x, & x > 1\end{cases}$。關於函數的極限，試選出正確的選項。

(1) $\lim\limits_{x\to 1} f(x)$ 存在、$\lim\limits_{x\to 1} g(x)$ 存在、$\lim\limits_{x\to 1} (f(x)+g(x))$ 存在

(2) $\lim\limits_{x\to 1} f(x)$ 存在、$\lim\limits_{x\to 1} g(x)$ 不存在、$\lim\limits_{x\to 1} (f(x)+g(x))$ 不存在

(3) $\lim\limits_{x\to 1} f(x)$ 不存在、$\lim\limits_{x\to 1} g(x)$ 存在、$\lim\limits_{x\to 1} (f(x)+g(x))$ 不存在

(4) $\lim\limits_{x\to 1} f(x)$ 不存在、$\lim\limits_{x\to 1} g(x)$ 不存在、$\lim\limits_{x\to 1} (f(x)+g(x))$ 存在

(5) $\lim\limits_{x\to 1} f(x)$ 不存在、$\lim\limits_{x\to 1} g(x)$ 不存在、$\lim\limits_{x\to 1} (f(x)+g(x))$ 不存在

（109 指考數甲補考 試卷第 1 題）

Teacher: First, we compute the $\lim_{x\to1} f(x)$. It is obvious that $f(x)$ is a piecewise function.

We need to compute the two limits $\lim_{x\to1^+} f(x)$ and $\lim_{x\to1^-} f(x)$. Compute $\lim_{x\to1^+} f(x)$.

What do we get?

Student: $\lim_{x\to1^+} f(x) = 1$.

Teacher: Good. Now compute $\lim_{x\to1^-} f(x)$. What do we get?

Student: $\lim_{x\to1^-} f(x) = \lim_{x\to1^-} (1 + x) = 2$.

Teacher: Good. Are the right limit and the left limit of $f(x)$ at $x = 1$ the same?

Student: No, they are different.

Teacher: So, we know $\lim_{x\to1} f(x)$ does not exist. Second, we compute the $\lim_{x\to1} g(x)$.

Similarly, we compute $\lim_{x\to1^+} g(x)$ and $\lim_{x\to1^-} g(x)$. What do we get?

Student: $\lim_{x\to1^+} g(x) = \lim_{x\to1^+} (3 - x) = 2$, and $\lim_{x\to1^-} g(x) = 1$.

Teacher: Good job. Compare the right limit and the left limit of $g(x)$ at $x = 1$. Are they the same?

Student: No, they are different.

Teacher: Ok, so we know $\lim_{x\to1} g(x)$ does not exist. Last, we evaluate $\lim_{x\to1} (f(x) + g(x))$.

What is $f(x) + g(x)$?

Student: $f(x) + g(x) = \begin{cases} (1 + x) + 1, x \le 1 \\ 1 + (3 - x), x > 1 \end{cases} = \begin{cases} 2 + x, x \le 1 \\ 4 - x, x > 1 \end{cases}$.

Teacher: Excellent. Now what is the right limit of $f(x) + g(x)$ at $x = 1$? That is, what value is $\lim_{x\to1^+} (f(x)+g(x))$?

Student: $\lim_{x\to1^+} (f(x)+g(x)) = \lim_{x\to1^+} (2+x) = 3$.

Teacher: Yes. And what is the left limit of $f(x)+g(x)$ at $x = 1$?

Student: $\lim_{x\to1^-} (f(x)+g(x)) = \lim_{x\to1^-} (4 - x) = 3$.

Teacher: So, the limit of $f(x)+g(x)$ at $x=1$ is 3.

In conclusion, we get $\lim_{x\to 1} f(x)$ doesn't exist, $\lim_{x\to 1} g(x)$ doesn't exist,

$\lim_{x\to 1}(f(x)+g(x))=3$ exists. We choose option (4).

老師： 首先，我們計算 $\lim_{x\to 1} f(x)$。已知 $f(x)$ 是分段函數。我們需要計算兩個極限 $\lim_{x\to 1^+} f(x)$ 和 $\lim_{x\to 1^-} f(x)$。計算 $\lim_{x\to 1^+} f(x)$。我們能得到什麼？

學生： $\lim_{x\to 1^+} f(x)=1$。

老師： 很好。在 $x=1$，$f(x)$ 的右極限和左極限是否相同？

學生： 不，他們不同。

老師： 所以我們知道 $\lim_{x\to 1} f(x)$ 不存在。其次，我們計算 $\lim_{x\to 1} g(x)$。類似地，我們計算 $\lim_{x\to 1^+} g(x)$ 和 $\lim_{x\to 1^-} g(x)$。我們能得到什麼？

學生： $\lim_{x\to 1^+} g(x)=\lim_{x\to 1^+}(3-x)=2,$ 以及 $\lim_{x\to 1^-} g(x)=1$。

老師： 很棒。比較在 $x=1$ 時，$g(x)$ 的右極限和左極限，它們是否相同？

學生： 不，它們不同。

老師： 好的，所以我們知道 $\lim_{x\to 1} g(x)$ 不存在。

最後，我們計算 $\lim_{x\to 1}(f(x)+g(x))$。$f(x)+g(x)$ 是什麼？

學生： $f(x)+g(x)=\begin{cases}(1+x)+1, x\le 1\\ 1+(3-x), x>1\end{cases}=\begin{cases}2+x, x\le 1\\ 4-x, x>1\end{cases}$。

老師： 很棒。現在，在 $x=1$ 時，$f(x)+g(x)$ 的右極限是多少？

即 $\lim_{x\to 1^+}(f(x)+g(x))$是多少？

學生： $\lim_{x\to 1^+}(f(x)+g(x))=\lim_{x\to 1^+}(2+x)=3$。

老師： 是的。在 $x=1$ 時，$f(x)+g(x)$ 的左極限是多少？

學生： $\lim_{x\to 1^-}(f(x)+g(x))=\lim_{x\to 1^-}(4-x)=3$。

老師： 所以在 $x = 1$ 時，$f(x)+g(x)$ 的極限是 3。

綜合以上，我們得到 $\lim_{x\to1} f(x)$ 不存在、$\lim_{x\to1} g(x)$ 不存在、$\lim_{x\to1}(f(x)+g(x))$ 存在。我們選擇選項(4)。

例題二

說明：利用函數的極限的存在性來推理。

（英文）Let $f(x)$ be a real-valued function defined on nonzero real numbers. $\lim_{x\to0} f(x)\frac{|x|}{x}$ is known to exist. Try to choose the correct options.

(1) $\lim_{x\to0}\left(\frac{x}{|x|}\right)^2$ exists.

(2) $\lim_{x\to0} f(x)\frac{x}{|x|}$ exists.

(3) $\lim_{x\to0}(f(x)+1)\frac{x}{|x|}$ exists.

(4) $\lim_{x\to0} f(x)$ exists.

(5) $\lim_{x\to0} f(x)^2$ exists.

（中文）設 $f(x)$ 為一定義在非零實數上的實數值函數。已知極限 $\lim_{x\to0} f(x)\frac{|x|}{x}$ 存在。試選出正確的選項。

(1) $\lim_{x\to0}\left(\frac{x}{|x|}\right)^2$ 存在

(2) $\lim_{x\to0} f(x)\frac{x}{|x|}$ 存在

(3) $\lim_{x\to0}(f(x)+1)\frac{x}{|x|}$ 存在

(4) $\lim_{x\to0} f(x)$ 存在

(5) $\lim_{x\to0} f(x)^2$ 存在

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Teacher: The question says that it is known $\lim_{x\to 0} f(x)\frac{|x|}{x}$ exists. What can we get from this description?

Student: x must be a factor of $f(x)$.

Teacher: That's right. Suppose that $f(x) = xg(x)$ where $g(x)$ is a real-valued function defined on nonzero real numbers. Since it is known $\lim_{x\to 0} f(x)\frac{|x|}{x}$ exists, we get

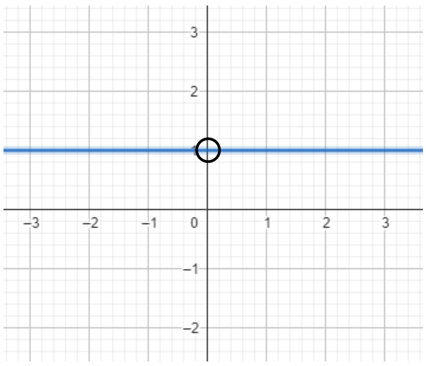
$\lim_{x\to 0} f(x)\frac{|x|}{x} = \lim_{x\to 0}(xg(x)\cdot\frac{|x|}{x}) = \lim_{x\to 0}(|x|\cdot g(x))$ exists.

Next, we see the first option of this question. Does $\lim_{x\to 0}\left(\frac{x}{|x|}\right)^2$ exist?

Let $h(x) = \left(\frac{x}{|x|}\right)^2$. Please use a piecewise function to describe $h(x)$.

Student: $$h(x) = \left(\frac{x}{|x|}\right)^2 = \begin{cases} \left(\frac{x}{x}\right)^2, & x > 0 \\ \text{undefined}, & x = 0 \\ \left(\frac{x}{-x}\right)^2, & x < 0 \end{cases} = \begin{cases} 1, & x > 0 \\ \text{undefined}, & x = 0. \\ 1, & x < 0 \end{cases}$$

Teacher: Good job. We can use the graphical method to observe the limit. Let's first roughly draw the graph of $h(x)$.



We can easily find that the limit $\lim_{x\to 0}\left(\frac{x}{|x|}\right)^2$ is 1. We now see the second option. How do we compute $\lim_{x\to 0} f(x)\frac{x}{|x|}$?

Student: We compare the right limit and the left limit of $f(x)\frac{x}{|x|}$ at $x = 0$.

Teacher: Yes. Now compute these two limits.

Student: $\lim_{x\to 0^+} f(x)\frac{x}{|x|} = \lim_{x\to 0^+}(xg(x)\cdot\frac{x}{|x|}) = \lim_{x\to 0^+}(g(x)\cdot\frac{x^2}{|x|}) = \lim_{x\to 0^+}(g(x)\cdot\frac{|x|^2}{|x|})$

$= \lim_{x\to 0^+}(g(x)\cdot|x|)$.

$\lim\limits_{x\to0^-} f(x)\dfrac{x}{|x|} = \lim\limits_{x\to0^-}(xg(x)\cdot\dfrac{x}{|x|}) = \lim\limits_{x\to0^-}(g(x)\cdot\dfrac{x^2}{|x|}) = \lim\limits_{x\to0^-}(g(x)\cdot\dfrac{|x|^2}{|x|})$

$= \lim\limits_{x\to0^-}(g(x)\cdot|x|)$. Since $\lim\limits_{x\to0}(|x|\cdot g(x))$ exists, then $\lim\limits_{x\to0^-} f(x)\frac{x}{|x|}$ exists.

Teacher: Excellent. Let's move on to the third option. Does $\lim\limits_{x\to0}(f(x)+1)\frac{x}{|x|}$ exist? We simplify the limit. $\lim\limits_{x\to0}(f(x)+1)\frac{x}{|x|} = \lim\limits_{x\to0}\left(f(x)\frac{x}{|x|}+\frac{x}{|x|}\right)$. Since the limit is combined with two parts $\lim\limits_{x\to0}\left(f(x)\frac{x}{|x|}\right)$ and $\lim\limits_{x\to0}\left(\frac{x}{|x|}\right)$. According to the second option, we know that $\lim\limits_{x\to0}\left(f(x)\frac{x}{|x|}\right)$ exists. Next, we need to see whether $\lim\limits_{x\to0}\left(\frac{x}{|x|}\right)$ exists or not. Please use a piecewise function to describe $\frac{x}{|x|}$.

Student: $\frac{x}{|x|} = \begin{cases} \frac{x}{x}, & x>0 \\ \text{undefined}, & x=0 \\ \frac{x}{-x}, & x<0 \end{cases} = \begin{cases} 1, & x>0 \\ \text{undefined}, & x=0 \\ -1, & x<0 \end{cases}$.

Teacher: Nice. Does $\lim\limits_{x\to0}\left(\frac{x}{|x|}\right)$ exist?

Student: According to the piecewise function, we know that $\lim\limits_{x\to0}\left(\frac{x}{|x|}\right)$ does not exist.

Teacher: So, we are unable to determine whether $\lim\limits_{x\to0}(f(x)+1)\frac{x}{|x|}$ exists or not. We can guess that the limit doesn't exist. What should we do if we want to show that the sentence is wrong?

Student: We need to give a counterexample.

Teacher: Yes. Please give a counterexample now.

Student: Ok. Let $f(x)=1, x\neq0$. $\lim\limits_{x\to0}(f(x)+1)\frac{x}{|x|} = \lim\limits_{x\to0}(1+1)\frac{x}{|x|} = \lim\limits_{x\to0}\frac{2x}{|x|}$. It is easy to check that $\lim\limits_{x\to0}\frac{2x}{|x|}$ does not exist.

Teacher: Very good. Therefore, the third option is false. Let's look at the fourth option. Does $\lim\limits_{x\to0^-} f(x)$ exist?

Student: Not necessarily.

Teacher: Can you give a counterexample?

Student: I'll try. Let $f(x)=\frac{x}{|x|}$, $x\neq 0$. Previously, we knew that $\lim\limits_{x\to 0}\left(\frac{x}{|x|}\right)$ does not exist. The fourth option is false.

Teacher: Look at the last option. Does $\lim\limits_{x\to 0} f(x)^2$ exist?

Student: I don't know.

Teacher: Let's think about it together.

We can try to use the above options. Sometimes we need to create something out of nothing. We compute $\lim\limits_{x\to 0} f(x)^2=\lim\limits_{x\to 0}(f(x)^2\cdot\frac{x}{|x|}\cdot\frac{|x|}{x})=\lim\limits_{x\to 0}(f(x)\frac{x}{|x|}\cdot f(x)\frac{|x|}{x})$ $=\lim\limits_{x\to 0}(f(x)\frac{x}{|x|})\cdot\lim\limits_{x\to 0}(f(x)\frac{|x|}{x})$ exists. This is the hardest option to judge in this question.

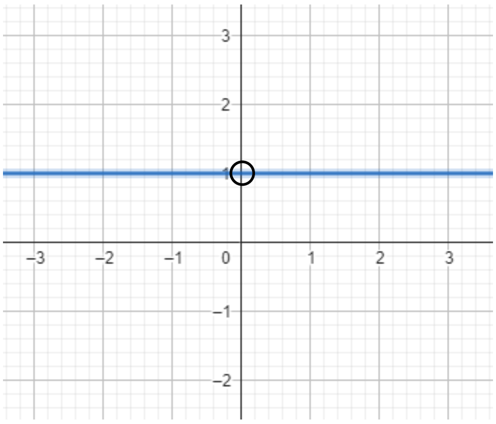
老師： 這個問題說已知存在 $\lim\limits_{x\to 0} f(x)\frac{|x|}{x}$。從這個敘述中我們可以得到什麼？

學生： x 必是 $f(x)$ 的一個因式。

老師： 這是正確的。假設 $f(x)=xg(x)$，其中 $g(x)$ 是在非零實數上定義的實值函數。由於已知 $\lim\limits_{x\to 0} f(x)\frac{|x|}{x}$ 存在，因此我們得到 $\lim\limits_{x\to 0} f(x)\frac{|x|}{x}=\lim\limits_{x\to 0}(xg(x)\cdot\frac{|x|}{x})=\lim\limits_{x\to 0}(|x|\cdot g(x))$ 存在。接下來我們看這道題的第一個選項。$\lim\limits_{x\to 0}\left(\frac{x}{|x|}\right)^2$ 存在嗎？令 $h(x)=\left(\frac{x}{|x|}\right)^2$。請使用分段函數來描述 $h(x)$。

學生： $h(x)=\left(\frac{x}{|x|}\right)^2=\begin{cases}\left(\frac{x}{x}\right)^2, & x>0\\ \text{未定義}, & x=0\\ \left(\frac{x}{-x}\right)^2, & x<0\end{cases}=\begin{cases}1, & x>0\\ \text{未定義}, & x=0\\ 1, & x<0\end{cases}$。

老師：　好棒。我們可以用圖解的方法來觀察極限。我們先粗略地畫一下 $h(x)$ 的圖。



我們很容易發現極限 $\lim\limits_{x\to 0}\left(\frac{x}{|x|}\right)^2$ 是 1。現在我們看第二個選項。我們該如何計算 $\lim\limits_{x\to 0} f(x)\frac{x}{|x|}$？

學生：　我們比較 $f(x)\frac{x}{|x|}$ 在 $x=0$ 處的右極限和左極限。

老師：　是的。現在計算這兩個極限。

學生：　$\lim\limits_{x\to 0^+} f(x)\frac{x}{|x|} = \lim\limits_{x\to 0^+}(xg(x)\cdot\frac{x}{|x|}) = \lim\limits_{x\to 0^+}(g(x)\cdot\frac{x^2}{|x|}) = \lim\limits_{x\to 0^+}(g(x)\cdot\frac{|x|^2}{|x|})$

$= \lim\limits_{x\to 0^+}(g(x)\cdot|x|)$。

$\lim\limits_{x\to 0^-} f(x)\frac{x}{|x|} = \lim\limits_{x\to 0^-}(xg(x)\cdot\frac{x}{|x|}) = \lim\limits_{x\to 0^-}(g(x)\cdot\frac{x^2}{|x|}) = \lim\limits_{x\to 0^-}(g(x)\cdot\frac{|x|^2}{|x|})$

$= \lim\limits_{x\to 0^-}(g(x)\cdot|x|)$。

因為 $\lim\limits_{x\to 0}(|x|\cdot g(x))$ 存在，則 $\lim\limits_{x\to 0} f(x)\frac{x}{|x|}$ 存在。

老師：　太棒了。讓我們繼續第三個選項。 $\lim\limits_{x\to 0}(f(x)+1)\frac{x}{|x|}$ 存在嗎？我們化簡極限。

$\lim\limits_{x\to 0}(f(x)+1)\frac{x}{|x|} = \lim\limits_{x\to 0}\left(f(x)\frac{x}{|x|}+\frac{x}{|x|}\right)$。由於極限由兩部分 $\lim\limits_{x\to 0}\left(f(x)\frac{x}{|x|}\right)$ 和 $\lim\limits_{x\to 0}\left(\frac{x}{|x|}\right)$ 組合而成。藉由第二個選項，我們知道 $\lim\limits_{x\to 0}\left(f(x)\frac{x}{|x|}\right)$ 存在。接下來，我們需要看看 $\lim\limits_{x\to 0}\left(\frac{x}{|x|}\right)$ 是否存在。請用分段函數來描述 $\frac{x}{|x|}$。

學生：$\frac{x}{|x|} = \begin{cases} \frac{x}{x}, & x > 0 \\ 未定義, & x = 0 \\ \frac{x}{-x}, & x < 0 \end{cases} = \begin{cases} 1, & x > 0 \\ 未定義, & x = 0 \\ -1, & x < 0 \end{cases}$。

老師：很好。$\lim_{x \to 0} \left(\frac{x}{|x|}\right)$ 存在嗎？

學生：根據分段函數，我們知道 $\lim_{x \to 0} \left(\frac{x}{|x|}\right)$ 不存在。

老師：所以我們無法判斷是否 $\lim_{x \to 0} (f(x) + 1)\frac{x}{|x|}$ 存在或不存在。我們可以猜測，也許這個極限並不存在。如果我們想證明這句話是錯誤的，該怎麼辦呢？

學生：我們需要舉一個反例。

老師：是的。請現在舉一個反例。

學生：好的。令 $f(x) = 1, x \neq 0$。$\lim_{x \to 0} (f(x) + 1)\frac{x}{|x|} = \lim_{x \to 0} (1 + 1)\frac{x}{|x|} = \lim_{x \to 0} \frac{2x}{|x|}$。我們很容易可以驗證 $\lim_{x \to 0} \frac{2x}{|x|}$ 不存在。

老師：非常好。因此，第三個選項是錯誤的。

我們來看看第四個選項。$\lim_{x \to 0} f(x)$ 存在嗎？

學生：不一定。

老師：請試著舉一個反例。

學生：我試試看。令 $f(x) = \frac{x}{|x|}, x \neq 0$。之前，我們知道 $\lim_{x \to 0} \left(\frac{x}{|x|}\right)$ 不存在。

第四個選項是錯誤的。

老師：看最後一個選項。$\lim_{x \to 0} f(x)^2$ 存在嗎？

學生：我不知道。

老師：我們一起想一想。我們可以嘗試使用上面的選項。有時我們需要無中生有創造一些東西。我們計算 $\lim_{x \to 0} f(x)^2 = \lim_{x \to 0} \left(f(x)^2 \cdot \frac{x}{|x|} \cdot \frac{|x|}{x}\right) = \lim_{x \to 0} \left(f(x)\frac{x}{|x|} \cdot f(x)\frac{|x|}{x}\right) = \lim_{x \to 0} \left(f(x)\frac{x}{|x|}\right) \cdot \lim_{x \to 0} \left(f(x)\frac{|x|}{x}\right)$ 存在。這是這個問題中最難判斷的選項。

單元五　微分
Differentiation

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■ 前言 Introduction

導數是微積分學的重要概念，它可以用來表示曲線在一點處的切線斜率及函數的變化，在這個單元，我們將會學到導數與導函數的極限定義、切線與導數、多項式函數及簡單代數函數之導函數，微分基本公式及係數積和加減性質，以及微分乘法律、除法律和連鎖律，以便應用於生活中更複雜的概念。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
approximate	估計	velocity	速度
tangency	相切	instantaneous	瞬間
approach	逼近	acceleration	加速度
infinity	無窮	**continuity**	連續性
vertical	鉛直	generalize	一般化
derivative	導數	**domain**	定義域
differentiation	微分	**derivative of a function**	導函數
differentiable function	可微分函數		

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ Find ________ of ________at________.

例句：**Find** the slope **of** the graph of $f(x)$ **at** the point $(1, 1)$.

在 $f(x)$ 的圖形上，求以點 $(1, 1)$ 為切點的斜率。

❷ Applying ________, we have ________.

例句：**Applying** the definition of the derivative, **we have** $f'(x) = 3x^2$.

由導數的定義，得 $f'(x) = 3x^2$。

❸ We obtain ________ by ________.

例句：**We obtain** a velocity function **by** differentiating a position function.

我們將位置函數微分，即得到速度函數。

❹ ________ can be represented by ________.

例句：The position of a free-falling object under the influence of gravity **can be represented by** the equation: $s(t) = \frac{1}{2}gt^2 + v_0t + s_0$.

在重力影響下，自由落體的位置函數可被表示為 $s(t) = \frac{1}{2}gt^2 + v_0t + s_0$。

■ 問題講解 Explanation of Problems

說明

1. The slope of the tangent line

We can approximate the slope using a secant line through the point of tangency and a second point on the curve, as shown in Figure 1.

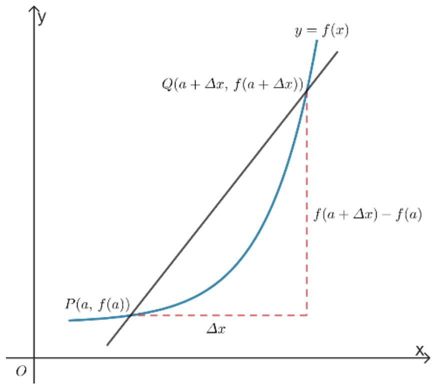

Figure 1

In Figure 2, if $A(a, f(a))$ is the point of tangency and $B(x, f(x))$ is a second point on the graph of f, the slope of the secant line $m_{\sec}$ through the two points using the slope formula is

$$m_{\sec}=\frac{y_2-y_1}{x_2-x_1}=\frac{f(x)-f(a)}{x-a}.$$

This equation is a difference quotient. When the denominator x approaches a, the slope $m=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ exists. It defines that the tangent line of the function at point $(a, f(a))$ with the slope $m=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$.

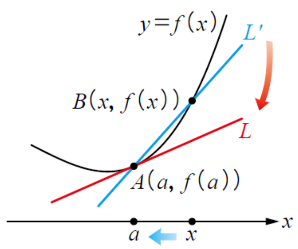

Figure 2

Moreover, if the limit approaches infinity, the tangent line of the function at the point is the vertical line.

2. The Derivative of a Function

The limit used to define the slope of a tangent line is also used to define one of the two fundamental operations of calculus－**differentiation**.

The derivative of the function f at the point $x=a$ is the limit

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

provided the limit exists.

Consider $x=a+h$. When x approaches a, h becomes $x-a$, which leads to h approaching 0. The alternative limit form of the derivative is shown below. The derivative of f at c is

$$f'(c)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

provided this limit exists.

3. Rates of Change

(1) Average Velocity and Instantaneous Velocity

The derivative can also be used to determine the rate of change. A common use for the rate of change is to describe the motion of an object moving in a straight line.

The function s that gives the position of an object as a function of time t is called a position function. If, over a period of time t, the object changes its position by the amount $s(t+h)-s(t)$, then, by the familiar formula:

$$Rate=\frac{Distance}{time},$$

The *average velocity* is

$$\frac{s(t+h)-s(t)}{h}.$$

When h approaches 0, the *instantaneous velocity* of an object at time t is the velocity of the object at that exact moment, that is

$$\lim_{h\to 0}\frac{s(t+h)-s(t)}{h}.$$

(2) A Falling Object

The position of a free-falling object under the influence of gravity can be represented by the equation:

$$s(t) = -\frac{1}{2}gt^2 + v_0 t + s_0$$

where s_0 is the initial height of the object, v_0 is the initial velocity of the object, and g is the acceleration due to gravity.

4. Differentiability Implies Continuity

A function being continuous at point $x=a$ does not guarantee that the function is differentiable at $x=a$. A continuous function might have a *discontinuity*, a *corner*, a *cusp*, or a *vertical tangent line*, and hence not be differentiable at a given point. Conversely, a function being differentiable at point $x=a$ implies that the function is continuous at point $x=a$.

5. The Derivative of a Function

We have learned the derivative of a function at a particular point. Now generalize the concept and consider the function $f(x)=x^3+2x$ as an example. As shown below,

applying the definition of the derivative of a function, where a is a real number, we have

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{(x^3+2x)-(a^3+2a)}{x-a}=\lim_{x\to a}\frac{(x-a)(x^2+ax+a^2+2)}{x-a}$$

$$=\lim_{x\to a}(x^2+ax+a^2+2)=a^2+a\cdot a+a^2+2=3a^2+2.$$

That means, for every real number a in the domain, we have the derivative of function $f'(a)$. Hence, the relationship where a corresponds to $f'(a) = 3a^2 + 2$ represents a function, just as the *derivative of a function*[17] f, denoted as $f'(x) = 3x^2 + 2$, with respect to x.

The process of finding the derivative of a function is called *differentiation*. Given a function $f(x)$ is *differentiable* at x when its derivative $f'(x)$ exists at x and is differentiable on an open interval I when it is differentiable at every point in the interval, denoted $f(x)$, is a *differentiable function*.

In addition to $f'(x)$, other notations are used to denote the derivative of $y = f(x)$. The most common are: y', $f'(x)$, $\left(f(x)\right)'$ or $\frac{d}{dx}f(x)$

For instance, the derivative function of $f(x) = x^2$ is $f'(x) = 2x$, is also denoted as

$$\frac{d}{dx}x^2 = 2x \text{ or } \left(x^2\right)' = 2x.$$

The notation $\frac{d}{dx}$ is read as "*the derivative of y with respect to x*" or "*dy/ dx*."

6. Basic Differentiation Rules

We have learned to use the limit definition to find derivatives. In the next section, we will explore several differentiation rules that enable us to find derivatives without directly applying the limit definition.

(1) The Constant Rule

The derivative of a constant function is 0. That is, if c is a real number, then

$$(c)' = 0.$$

(2) The Power Rule

If n is a natural number, then the function $f(x) = x^n$ is differentiable and

$$\left(x^n\right)' = nx^{n-1}$$

(3) The Constant Multiple Rule

If f is a differentiable function and c is a real number, then cf is also differentiable and

$$\left(cf(x)\right)' = cf'(x).$$

(4) The Sum and Difference Rules

The sum (or difference) of two differentiable functions f and g is itself differentiable. Moreover, the derivative of $f + g$ (or $f - g$) is the sum (or difference) of the derivatives of f and g,

$$\left(f(x)+g(x)\right)' = f'(x)+g'(x),\ \left(f(x)-g(x)\right)' = f'(x)-g'(x).$$

(5) The Product Rule

The product of two differentiable functions f and g is itself differentiable. Moreover, the derivative of fg is the first function times the derivative of the second, plus the second function times the derivative of the first. That is,

$$\frac{d}{dx}\left[f(x)g(x)\right]=f(x)g'(x)+f'(x)g(x)$$

(6) The Quotient Rule

The quotient $\frac{f}{g}$ of two differentiable functions f and g is itself differentiable at all values of x for which $g(x)\neq 0$. Moreover, the derivative of $\frac{f}{g}$ is given by the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)f'(x)-f(x)g'(x)}{\left[g(x)\right]^2},\ g(x)\neq 0$$

(7) The Chain Rule

If $f(x)$ is a differentiable function of $g(x)$ and $g(x)$ is a differentiable function of x, then $g\circ f(x)$ is a differentiable function of x and

$$(g\circ f)'(x)=\frac{d}{dx}\left[g\circ f(x)\right]=f'(g(x))g'(x).$$

運算問題的講解

例題一

說明：本題是練習導數與切線方程式。

（英文）Find $f'(2)$ for $f(x) = x^2$. Then find the slopes of the graph of $f(x)$ at the points $P(2, 4)$.

（中文）求函數 $f(x) = x^2$ 的導數 $f'(2)$。在 $f(x)$的圖形上，求以點 $P(2, 4)$ 為切點的切線方程式。

Teacher: Applying the definition of the derivative, we have $f'(2) = \lim\limits_{x\to 2}\frac{f(x)-f(2)}{x-2}$.

Please simplify the equation.

Student: $f'(2) = \lim\limits_{x\to 2}\frac{f(x)-f(2)}{x-2} = \lim\limits_{x\to 2}\frac{x^2-4}{x-2} = \lim\limits_{x\to 2}(x+2) = 4$

Teacher: So, we know that the slope of the tangent line is $f'(2) = 4$.

Using the point-slope formula, what will be the result?

Student: We obtain the equation of the tangent line, which is $y - 4 = 4(x-2)$

Teacher: Simplify the result.

Student: $y = 4x - 4$

Teacher: Well done, that is the answer.

老師：由導數的定義，得 $f'(2) = \lim\limits_{x\to 2}\frac{f(x)-f(2)}{x-2}$，請化簡算式。

學生：$f'(2) = \lim\limits_{x\to 2}\frac{f(x)-f(2)}{x-2} = \lim\limits_{x\to 2}\frac{x^2-4}{x-2} = \lim\limits_{x\to 2}(x+2) = 4$。

老師：因此，所求切線的斜率等於 $f'(2) = 4$，我們利用直線的點斜式，可以得切線方程式為

學生：我們可以知道切線方程式為 $y - 4 = 4(x-2)$。

老師：請化簡。

學生：$y = 4x - 4$。

老師：很好，這就是答案。

例題二

說明：本題是探討每個函數在其定義域中的導數。

（英文）Does the derivative of $f(x)=|x|$ at $x=0$ exist?

（中文）函數 $f(x)=|x|$ 在 $x=0$ 的導數是否存在？

Teacher: Applying the definition of the derivative, we have $f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$. Please simplify the equation.

Student: $f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{|x|-|0|}{x-0}=\lim_{x\to 0}\frac{|x|}{x}$.

Teacher: We verify the existence of the left-hand derivative

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{|x|-0}{x-0}=\frac{-x}{x}=-1.$$

Please verify the existence of the right-hand derivative.

Student: The right-hand derivative is $\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{|x|-0}{x-0}=\frac{x}{x}=1$.

Teacher: Now we get the left-hand derivative is not equal to the right-hand derivative. Does the derivative of $f(x)=|x|$ at $x=0$ exist?

Student: Since the limit $\lim_{x\to 0}\frac{|x|}{x}$ does not exist, the derivative of $f(x)=|x|$ at $x=0$ does not exist.

Teacher: Excellent.

老師： 由導數的定義，得 $f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$。請化簡算式。

學生： $f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{|x|-|0|}{x-0}=\lim_{x\to 0}\frac{|x|}{x}$。

老師： 我們可知左極限為 $\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{|x|-0}{x-0}=\frac{-x}{x}=-1$，請找出右極限。

學生： 右極限為 $\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{|x|-0}{x-0}=\frac{x}{x}=1$。

老師： 所以我們發現左極限不等於右極限，所以函數 $f(x)=|x|$ 在 $x=0$ 的導數是否存在？

學生： 因為極限 $\lim\limits_{x \to 0} \frac{|x|}{x}$ 存在，故函數 $f(x) = |x|$ 在 $x = 0$ 的導數不存在。

老師： 很棒。

例題三

說明：本題是練習利用導數的定義，求導函數。

（英文）Find the derivative of $f(x) = \sqrt{x}$. $(x > 0)$

（中文）求函數 $f(x) = \sqrt{x}$（$x > 0$）的導函數。

Teacher: Setting a as a positive real number and applying the definition of the derivative, we have $f'(a) = \lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a}$. Please simplify the equation.

Student: $f'(a) = \lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a} = \lim\limits_{x \to a} \frac{\sqrt{x}-\sqrt{a}}{x-a}$.

Teacher: To simplify this radical, we need to rationalize the numerator.

Multiply both the numerator and the denominator by the conjugate of the numerator, which is $\sqrt{x} + \sqrt{a}$. What will it be?

Student: $f'(a) = \lim\limits_{x \to a} \frac{\sqrt{x}-\sqrt{a}}{x-a} \times \frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}} = \lim\limits_{x \to a} \frac{x-a}{(x-a)(\sqrt{x}+\sqrt{a})} = \lim\limits_{x \to a} \frac{1}{\sqrt{x}+\sqrt{a}} = \frac{1}{2\sqrt{a}}$.

Teacher: Splendid! So the derivative of $f(x)$ is $f'(x) = \frac{1}{2\sqrt{x}}$

老師： 設 a 為正實數，由導數的定義，計算 $f'(a) = \lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a}$。請化簡算式。

學生： $f'(a) = \lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a} = \lim\limits_{x \to a} \frac{\sqrt{x}-\sqrt{a}}{x-a}$。

老師： 為了要化簡此繁根式，我們可以有理化分子。將分子分母都乘上分子的共軛數 $\sqrt{x} + \sqrt{a}$。就會變成？

學生： $f'(a) = \lim\limits_{x \to a} \frac{\sqrt{x}-\sqrt{a}}{x-a} \times \frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}} = \lim\limits_{x \to a} \frac{x-a}{(x-a)(\sqrt{x}+\sqrt{a})} = \lim\limits_{x \to a} \frac{1}{\sqrt{x}+\sqrt{a}} = \frac{1}{2\sqrt{a}}$。

老師： 很好，故 $f(x)$ 的導函數為 $f'(x) = \frac{1}{2\sqrt{x}}$。

例題四

說明：本題是練習使用連鎖律。

（英文）Given $f(x) = x^2 + 3x + 4$ and $g(x) = x^7$, find the derivative of the composite function $(g \circ f)(x)$.

（中文）已知函數 $f(x) = x^2 + 3x + 4$，$g(x) = x^7$，求合成函數 $(g \circ f)(x)$ 的導函數。

Teacher: By the chain rule, we get the derivative of $(g \circ f)(x)$ is $(g \circ f)'(x) = g'(f(x))f'(x)$. Also $g'(x) = 7x^6$, then what will we get?

Student: $g'(f(x)) = 7(x^2 + 3x + 4)^6$.

Teacher: And $f'(x) = 2x + 3$. Put them all together, yielding.

Student: $(g \circ f)'(x) = g'(f(x))f'(x) = 7(x^2 + 3x + 4)^6(2x + 3)$.

Teacher: Great job!

老師： 利用連鎖律，得 $(g \circ f)(x)$ 的導函數 $(g \circ f)'(x) = g'(f(x))f'(x)$。因為 $g'(x) = 7x^6$，我們會得到？

學生： $g'(f(x)) = 7(x^2 + 3x + 4)^6$。

老師： 又因為 $f'(x) = 2x + 3$，把所有已知放進連鎖律，我們會得到？

學生： $(g \circ f)'(x) = g'(f(x))f'(x) = 7(x^2 + 3x + 4)^6(2x + 3)$。

老師： 很棒！

例題五

說明：本題是練習使用連鎖律。

（英文）Find the derivative of the function $h(x)=(x^2+x-1)(x+2)$.

（中文）求函數 $h(x)=(x^2+x-1)(x+2)$ 的導函數。

Teacher: Apply the product rule. What is the product rule of $h'(x)$?

Student: $h'(x)=f'(x)g(x)+f(x)g'(x)$.

Teacher: Perfect! Let $f(x)=x^2+x-1$ and $g(x)=x+2$. What can we get?

Student: $h'(x)=f'(x)g(x)+f(x)g'(x)=(2x+1)(x+2)+(x^2+x-1)(1)$.

Teacher: Great! After doing the math, you'll get the answer, which is?

Student: $h'(x)=3x^2+6x+1$

老師： 這題要利用乘積的導函數，其公式為？

學生： $h'(x)=f'(x)g(x)+f(x)g'(x)$ 。

老師： 很好，所以我們令 $f(x)=x^2+x-1$，$g(x)=x+2$ 會得到？

學生： $h'(x)=f'(x)g(x)+f(x)g'(x)=(2x+1)(x+2)+(x^2+x-1)(1)$

老師： 非常好，然後展開後就會得到答案。

學生： $h'(x)=3x^2+6x+1$

應用問題 / 學測指考題

例題一

說明：關於運動學的例題。

（英文）For an object moving along a straight line, the object's position s at time t is the position function $s\,(t) = 50t - 4.9t^2$ where s is measured in meters and t is measured in seconds. Find the instantaneous velocity and acceleration at $t = 2$.

（中文）已知某質點作直線運動時，時刻 t 的位置函數為 $s\,(t) = 50t - 4.9t^2$，其中 s 的單位為公尺、t 的單位為秒，求此質點在時刻 $t = 2$ 的瞬時速度及瞬時加速度。

Teacher: Find the derivative and the second derivative of the function $s(t)$.

Student: $s'(t) = 50 - 9.8t,\ s''(t) = -9.8$.

Teacher: We know the instantaneous velocity is the derivative of the function $s'(t) = 50 - 9.8t$ and the instantaneous acceleration is the second derivative of the function $s''(t) = -9.8$. Plug in $t = 2$, and we get the answer.

Student: Instantaneous velocity: $s'(2) = 50 - 9.8 \times 2 = 30.4$ (m/s).
Instantaneous acceleration: $s''(2) = -9.8$ (m/s^2).

Teacher: Terrific!

老師：函數 $s(t)$ 的導函數與二階導函數分別為何？

學生：$s'(t) = 50 - 9.8t$、$s''(t) = -9.8$。

老師：我們知道函數 $s(t)$ 的導函數為瞬時速度且二階導函數為瞬時加速度。故將$t = 2$代入即得答案。

學生：瞬時速度為：$s'(2) = 50 - 9.8 \times 2 = 30.4$（公尺／秒）
瞬時加速度為：$s''(2) = -9.8$（公尺／秒平方）

老師：很好！

例題二

說明：關於地球與月球的重力加速度。

（英文）An astronaut standing on the moon throws a rock upward. The height of the rock is $s(t) = -0.8t^2 + 30t + 2$ where s is measured in meters and t is measured in seconds. How does the acceleration of the rock compare with the acceleration due to gravity on Earth?

（中文）一位太空人在月球上拋擲一枚石子，石子離地面的高度和時間可用方程式 $s(t) = -0.8t^2 + 30t + 2$ 表示，其中 s 單位為公尺，t 的單位為秒。請問月球對於石子的加速度與地球上的重力加速度之比值為何？

Teacher: Find the second derivative of the function $s(t)$.

Student: $s'(t) = -1.62t + 30,\ s''(t) = -1.62.$

Teacher: We know the acceleration is the second derivative of the function, which is $s''(t) = -1.62.$

Student: Acceleration: $s''(t) = -1.62\ (\text{m}/s^2).$

Teacher: So, the acceleration due to gravity on the moon is -1.62 meters per second squared. What is the acceleration due to gravity on Earth?

Student: The acceleration due to gravity on Earth is -9.8 meters per second squared.

Teacher: Right! The ratio of Earth's gravitational force to the moon's gravitational force is going to be…

Student: $\frac{\text{Earth's gravitational force}}{\text{Moon's gravitational force}} = \frac{-9.8}{-1.62} \approx 6.0.$

Teacher: Fantastic!

老師： 函數 $s(t)$ 的二階導函數為何？

學生： $s'(t) = -1.62t + 30$、$s''(t) = -1.62$。

老師： 我們知道函數 $s(t)$ 的二階導函數為瞬時加速度。

學生： 加速度為：$s''(t) = -1.62$（公尺／秒平方）

老師：　所以，在月球上的重力加速度為 -1.62 公尺每平方秒，那在地球上的重力加速度為何？

學生：　在地球上的重力加速度為 -9.8 公尺每平方秒。

老師：　沒錯，所以地球的重力與月球的重力比值為何？

學生：　$\frac{\text{地球重力}}{\text{月球重力}}=\frac{-9.8}{-1.62}\approx 6.0$

老師：　做得好！

單元六　微分的應用
Applications of Differentiation

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■ 前言 Introduction

在這個單元，我們將會學習如何以微分作為工具，判定函數的單調性及凹向性，並求解函數遞增遞減，以求解函數的極值，觀察函數的一次近似，以及解決生活中基本的最佳化問題。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
strictly	嚴格地	**minimum**	極小
increasing	遞增	**extrema**	極值
interval	區間	**derivative**	導數
decreasing	遞減	horizontal	水平
concavity	凹向性	tangent	切線
point of inflection	反曲點	domain	定義域
relative	相對的	polynomial	多項式

maximum	極大值	**differentiable**	可微分的
endpoint	端點	**extreme candidate**	極值候選點
absolute	絕對的	**local linear approximation**	一次近似法

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ **Discuss ________ of ________.**

例句：**Discuss** the concavity **of** the graph of $f(x) = x^4 - 6x^2 + 5$.

討論函數 $f(x) = x^4 - 6x^2 + 5$ 的凹向性。

❷ **________ occur at ________.**

例句：The extrema of a polynomial function f can **occur at** points where its derivative is zero.

多項式函數的極值會發生在導數為 0 的點。

❸ **________ neither ________ nor ________.**

例句：$f(c)$ is **neither** a relative minimum **nor** a relative maximum.

$f(c)$不是極小值也不是極大值。

❹ **Find ________ of ________ at ________.**

例句：**Find** the local linear approximation **of** $f(x)$ **at** $x = 1$

求函數 $f(x)$ 在 $x = 1$ 附近的一次估計。

■ 問題講解 Explanation of Problems

說明

1. Decreasing and Increasing

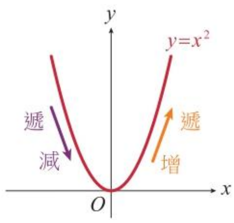


Figure 1

See Figure 1, and consider a function $y = x^2$.

(1) When $x \geq 0$, as the graph moves to the right, its graph moves up. That is, when x increases, the corresponding y value also increases.

We define that the function $y = x^2$ is **strictly increasing** on the **interval** $[0, \infty)$.

(2) When $x \leq 0$, as the graph moves to the right, its graph moves down. That is, when x increases, the corresponding y value decreases.

We define that the function $y = x^2$ is **strictly decreasing** on the interval $-\infty, 0]$.

[Definitions of Increasing and Decreasing Functions]

(1) A function f is **strictly increasing** on an interval I if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

(2) A function f is **increasing** on an interval I if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) \leq f(x_2)$.

(3) A function f is **strictly decreasing** on an interval I if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

(4) A function f is **decreasing** on an interval I if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) \geq f(x_2)$.

In fact, the sign of the derivative will indicate negative when the function is decreasing and indicate positive when the function is increasing.

Test for Increasing and Decreasing Functions

Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).

(1) If $f'(x) > 0$ for all x in (a, b), then f is **strictly increasing** on $[a, b]$.

(2) If $f'(x) < 0$ for all x in (a, b), then f is **strictly decreasing** on $[a, b]$.

(3) If $f'(x) = 0$ for all x in (a, b), then f is constant on $[a, b]$.

2. Concavity

[Definition of Concavity]

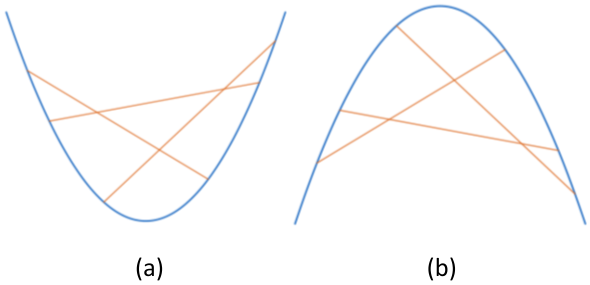


Figure 2

A function f is concave upward if every line segment joining two points on its graph lies above the graph at any point, see figure 2(a). Symmetrically, a function f is concave downward if every line segment joining two points on its graph lies below the graph at any point, see figure 2(b).

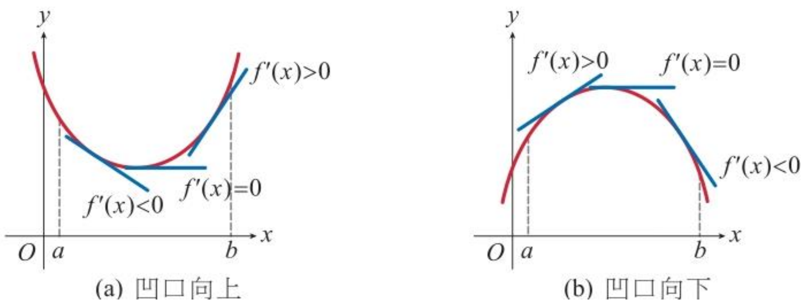


Figure 3

(1) See Figure 3(a), let f be differentiable on an open interval. If the graph of f is concave upward, then as the point of tangency moves to the right, the slope of the tangent line increases. That is, f' is increasing on the interval. Thus, the derivative of $f'(x)$ is positive, which is $\left(f'(x)\right)' > 0$ or $f''(x) > 0$.

(2) See Figure 3(b), let f be differentiable on an open interval. If the graph of f is concave downward, then as the point of tangency moves to the right, the slope of the tangent line decreases. That is, f' is decreasing on the interval. Thus, the derivative of $f'(x)$ is negative, which is $\left(f'(x)\right)' < 0$ or $f''(x) < 0$.

Test for Concavity

Let f be a function whose second derivative exists on an open interval I.

(1) If $f''(x) > 0$ for all x in I, then the graph of f is concave **upward** on I.

(2) If $f''(x) < 0$ for all x in I, then the graph of f is concave **downward** on I.

3. Point of Inflection

The graph in Figure 4 has one point at which the concavity changes. Suppose the tangent line to the graph exists at such a point, which is **a point of inflection (or an inflection point)**.

There are three types of points of inflection are shown.

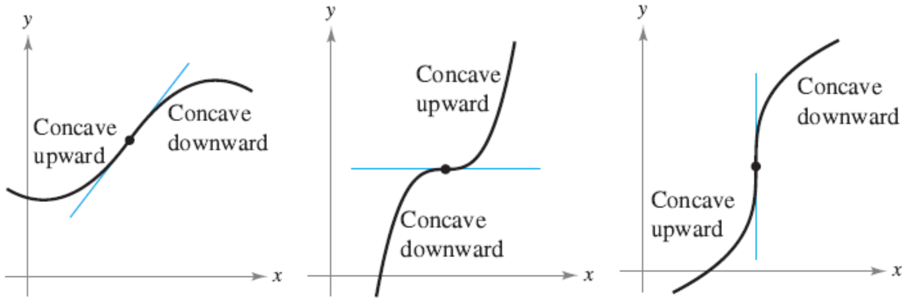


Figure 4

[Definition of Inflection Points]

Let f be a function that is continuous on an open interval, and let a be a point in the interval. If the graph of f has a tangent line at this point $(a, f(a))$, then this point is a **point of inflection** of the graph of f if the concavity of f changes from upward to downward (or downward to upward) at the point.

4. Extrema

Let a function f be defined on an interval I containing a, b, c, and d.

(1) See Figure 5; point *B*, which is higher than any other points nearby, is called a **relative maximum**. Point *D* is an **endpoint** higher than any other point around point *B*. So, point *D* is called **endpoint maximum**. The y-coordinates $f(b)$ and $f(d)$ are the maximums of f on I. Moreover, point *D* is the highest point on I, so $f(d)$ is called the **absolute maximum** of f on I.

(2) Point *C*, which is lower than any other point nearby, is called a **relative minimum**. Point *A* is an endpoint and is lower than any other point around point *A*. So, point *A* is called **endpoint minimum**. The y-coordinates $f(a)$ and $f(c)$ are a minimum of f on I. Moreover, point *C* is the lowest point on I, so $f(c)$ is called the **absolute minimum** of f on I.

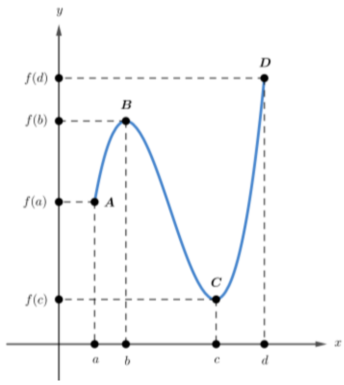


Figure 5

[Definition of Extrema]

Let a function f be defined on an interval I containing a, b, c, and d.

(1) $f(a)$ is the **maximum** of f on I when $f(a) \geq f(x)$ for x around a in I.

$f(b)$ is the **absolute maximum** of f on I when $f(b) \geq f(x)$ for all x in I.

(2) $f(c)$ is the **minimum** of f on I when $f(c) \leq f(x)$ for x around c in I.

$f(d)$ is the **absolute minimum** of f on I when $f(d) \leq f(x)$ for all x in I.

5. The Derivative Test

See Figure 6 for a continuous function, a maximum as occurring on a "hill" on the graph (like points C and F) and a minimum as occurring in a "valley" on the graph (like points *E* and *G*). The graph has a **horizontal tangent** line at the high point (or low point). Hence, the common characteristic of these points is that they have horizontal tangent lines. And the derivative of the function at these points is equal to zero.

In general, the **domain** of the **polynomial** function is all real numbers, which is an open interval. The extrema of a polynomial function can occur only at the numbers where the derivative of x is zero of the function. But, when it changes the domain to a closed interval $[a,b]$, the extrema can occur at the endpoints (like points *A* and *B*), where f is not **differentiable** at.

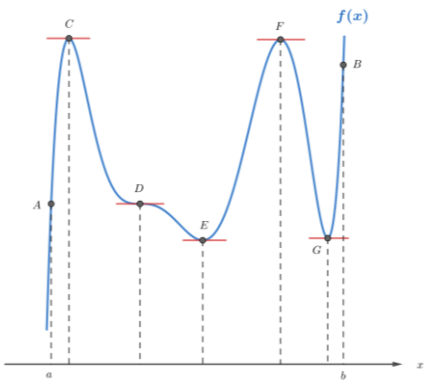


Figure 6

[Points Where Extrema Might Occur]

(1) The extrema (maxima and minima) of a polynomial function f can occur at points where its derivative is zero.

(2) There are two types of points where extrema might occur for a polynomial function f on a closed interval $[a,b]$.

(i) The points where the function's derivative is zero, denoted by $f'(x)=0$.

(ii) The endpoints of the closed interval $[a,b]$.

Note that point D in Figure 6 has a horizontal tangent line, while point D is not an extremum. Thus, the point where the derivative of the function is zero is just an **extreme candidate**, which could potentially be an extremum but isn't guaranteed to be one.

After identifying all the extreme candidates, we determine whether they are extrema by analyzing the intervals on which a function is increasing or decreasing. Furthermore, the behavior of increase or decrease in a differentiable function can be determined by examining the sign of its derivative.

[The Derivative Test]

Let c be the number of point C where $f'(c)=0$ for a polynomial function f. The value $f(c)$ can be classified as follows.

(1) If $f'(x)$ changes from positive to negative at c, then f has a **relative maximum** at $(c, f(c))$, as shown in Figure 7(a).

(2) If $f'(x)$ changes from negative to positive at c, then f has a **relative minimum** at $(c, f(c))$, as shown in Figure 7(b).

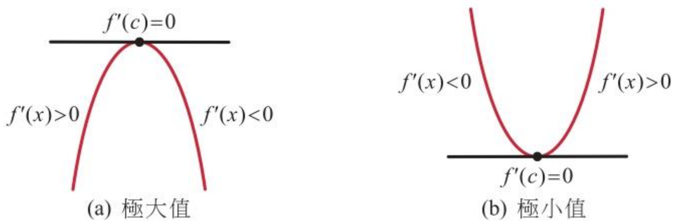


Figure 7

(3) Note that if $f'(x)$ is positive on both sides of c or negative on both sides of c, then $f(c)$ is neither a relative minimum nor a relative maximum, as shown in Figure 8.

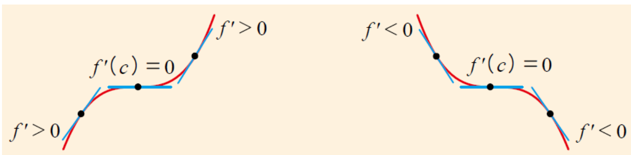


Figure 8

6. Local Linear Approximation

If a function f is differentiable at a, then a magnified portion of the graph of f centered at the point $P\left(a, f\left(a\right)\right)$ takes on the appearance of a straight line segment, shown in Figure 9. For this reason, a function that is differentiable at a is sometimes said to be locally linear at a.

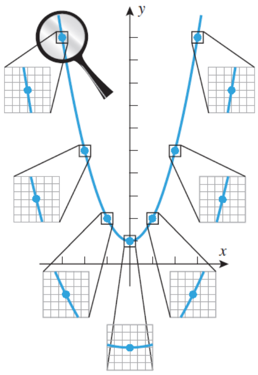


Figure 9

The line that best approximates the graph of f in the **vicinity** of $P\left(a, f\left(a\right)\right)$ is the tangent line to the graph of f at a, given by the equation:

$$y = f\left(a\right) + f'\left(a\right)\left(x-a\right).$$

Thus, for values of x near a we can approximate values of $f\left(x\right)$ by:

$$f\left(x\right) \approx f\left(a\right) + f'\left(a\right)\left(x-a\right)$$

This is called the local linear approximation of f at a.

[The Relationship between Local Linear Approximation and Tangent lines]

The local linear approximation of a polynomial function f at $x=a$ is represented as the tangent line of the function at $\left(a, f\left(a\right)\right)$.

運算問題的講解

例題一

說明：本題是判斷可微分函數在哪個區間遞增減。

（英文）Find the open intervals on which $f(x) = x^3 - 3x + 2$ is increasing or decreasing.

（中文）討論函數 $f(x) = x^3 - 3x + 2$ 的遞增區間及遞減區間。

Teacher: Differentiate f and set $f'(x)$ equals 0, and then factorize it.

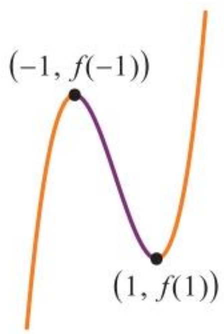
Student: $f(x) = x^3 - 3x + 2.\ f'(x) = 3x^2 - 3 = 0.\ f'(x) = 3(x+1)(x-1) = 0$

Teacher: So, we use the roots of $f'(x) = 0$, which is $x = 1 \vee -1$. Fill in the table that summarizes the three intervals determined by these two numbers.

Student:

Interval	$(-\infty, -1)$	-1	$(-1, 1)$	1	$(1, \infty)$
Sign of $f'(x)$	$+$	0	$-$	0	$+$
Conclusion	increasing		decreasing		increasing

Teacher: Well done. So, f is increasing on the intervals $(-\infty, -1)$ and $(1, \infty)$ and decreasing on the interval $(-1, 1)$, as shown below.



老師： 求出 f 的導函數，並將其因式分解。

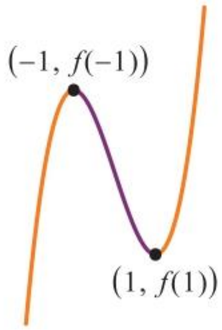
學生： $f(x) = x^3 - 3x + 2$。$f'(x) = 3x^2 - 3 = 0$。$f'(x) = 3(x+1)(x-1) = 0$。

老師： 用方程式$f'(x) = 0$的實根將數線分成數個區間。請完成表格。

學生：

區間	$(-\infty, -1)$	-1	$(-1, 1)$	1	$(1, \infty)$
$f'(x)$的正負	$+$	0	$-$	0	$+$
結論	遞增		遞減		遞增

老師：很好，根據函數遞增與遞減的判定，得 f 在區間 $(-\infty, -1)$ 與 $(1, \infty)$ 上為遞增函數，在區間 $(-1, 1)$ 上為遞減函數。電腦繪圖如下。



例題二

說明：本題是討論函數凹向及求反曲點。

（英文）Determine the points of inflection and discuss the concavity of the graph of:

$f(x) = x^4 - 6x^2 + 5.$

（中文）討論函數 $f(x) = x^4 - 6x^2 + 5$ 圖形的凹向，並求其反曲點。

Teacher: Differentiating f twice produces f' and f'', and then factorize f''.

Student: $f(x) = x^4 - 6x^2 + 5$. $f'(x) = 4x^3 - 12x$. $f''(x) = 12x^2 - 12 = 12(x+1)(x-1)$

Teacher: Setting $f''(x) = 0$, we can determine that the possible points of inflection occur at $x = 1$ and $x = -1$. Fill in the table that summarizes the three intervals determined by these two numbers.

Student:

Interval	$(-\infty, -1)$	-1	$(-1, 1)$	1	$(1, \infty)$
Sign of $f''(x)$	$+$	0	$-$	0	$+$
Conclusion	Concave upward		Concave downward		Concave upward

Teacher: Fantastic! So, the graph of f is concave upward on the intervals $(-\infty, -1)$ and $(1, \infty)$ and is concave downward on the interval $(-1, 1)$. Furthermore, the concavity changes both $x = -1$ and $x = 1$. So $(-1, 0)$ and $(1, 0)$ are inflection points, as shown below.

老師： 求出 f 的導函數及二階導函數，並將二階導函數因式分解。

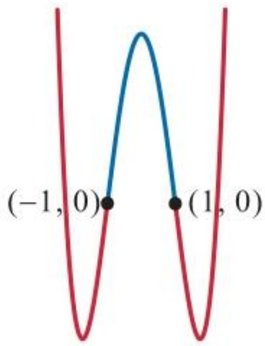
學生： $f(x)=x^4-6x^2+5.\quad f'(x)=4x^3-12x.\quad f''(x)=12x^2-12=12(x+1)(x-1)$

老師： 用方程式 $f''(x)$ 的實根將數線分成數個區間。請完成表格。

學生：

區間	$(-\infty,-1)$	-1	$(-1,1)$	1	$(1,\infty)$
$f''(x)$ 的正負	+	0	−	0	+
結論	凹向上		凹向下		凹向上

老師： 很好，根據函數遞增與遞減的判定，得 f 在區間 $(-\infty,-1)$ 與 $(1,\infty)$ 的圖形是凹口向上，在區間 $(-1,1)$ 的圖形是凹口向下。又因 f 圖形的凹向在 $x=1$ 和 $x=-1$ 兩處改變，所以點 $(-1,0)$ 和 $(1,0)$ 都是反曲點。圖形見下圖。



例題三

說明：本題透過函數極值的判定法求極值

（英文）Find the relative extrema of $f(x)=x^4-6x^2+5$.

（中文）求函數 $f(x)=x^4-6x^2+5$ 的極值。

Teacher: Differentiating f produces f', and then factorize it.

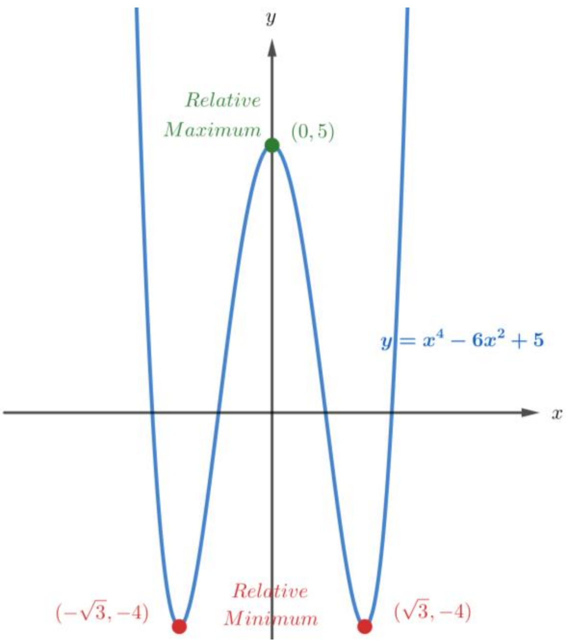
Student: $f(x)=x^4-6x^2+5$. $f'(x)=4x^3-12x=4x\left(x-\sqrt{3}\right)\left(x+\sqrt{3}\right)$

Teacher: Setting $f'(x)=0$, we get the extreme candidates at $x=-\sqrt{3}, 0$ or $\sqrt{3}$. Fill in the table that summarizes the four intervals determined by these two numbers.

Student:

Interval	$(-\infty,-\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3},0)$	0	$(0,\sqrt{3})$	$\sqrt{3}$	$(\sqrt{3},\infty)$
Sign of $f'(x)$	$-$	0	$+$	0	$-$	0	$+$
Conclusion $f(x)$	Decreasing	-4	Increasing	5	Decreasing	-4	Increasing

Teacher: Fantastic! By applying the Derivative Test, we can conclude that f has a relative minimum (or absolute minimum) at the point $(-\sqrt{3},-4)$, and another relative minimum at the point $(\sqrt{3},-4)$, and a relative maximum at the point $(0,5)$, as shown below



老師： 求出 f 的導函數，並將質因式分解。

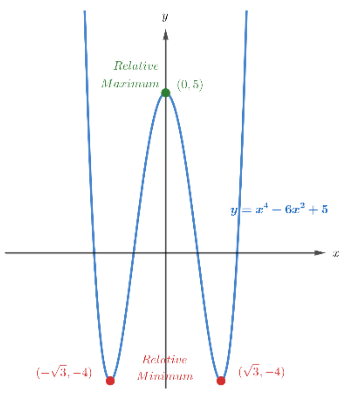
學生： $f(x)=x^4-6x^2+5$. $f'(x)=4x^3-12x=4x\left(x-\sqrt{3}\right)\left(x+\sqrt{3}\right)$.

老師： 用解 $f'(x)=0$，得 $x=-\sqrt{3},0$ 或 $\sqrt{3}$。這三個實根將數線分成數個區間。請完成表格。

學生：

區間	$\left(-\infty,-\sqrt{3}\right)$	$-\sqrt{3}$	$\left(-\sqrt{3},0\right)$	0	$\left(0,\sqrt{3}\right)$	$\sqrt{3}$	$\left(\sqrt{3},\infty\right)$
$f'(x)$ 的正負	−	0	+	0	−	0	+
總結 $f(x)$	遞減	−4	遞增	5	遞減	−4	遞增

老師： 很好，利用極值的檢定法，得極小值為 $\left(-\sqrt{3},-4\right)$ 與 $\left(\sqrt{3},-4\right)$，極大值為 $(0,5)$。圖形見下圖。



例題四

說明：本題是練習使用一次估計公式求近似值。

（英文）A function $f(x) = x^3 - 10x^2 + 24x + 1$

(a) Find the local linear approximation of $f(x)$ at $x = 1$.

(b) Use the local linear approximation obtained in part (a) to approximate $f(1.01)$.

(c) Compare the approximation to the result produced directly by a calculator.

（中文）設函數 $f(x) = x^3 - 10x^2 + 24x + 1$ 。

(a) 求函數 $f(x)$ 在 $x = 1$ 附近的一次估計。

(b) 使用(a)，求 $f(1.01)$ 的近似值。

(c) 使用計算機計算 $f(1.01)$ 的精確值。

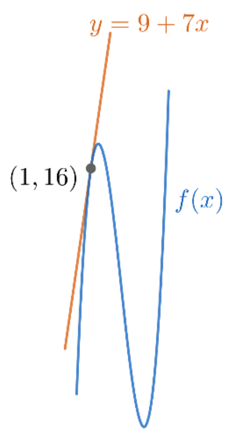
Teacher: For exercise (a), differentiate f .

Student: $f(x) = x^3 - 10x^2 + 24x + 1$. $f'(x) = 3x^2 - 20x + 24$

Teacher: We follow the local linear approximation formula $f(x) \approx f(a) + f'(a)(x-a)$ at $x = 1$. What do we get?

Student: The local linear approximation at a point $x = 1$ is $f(x) \approx f(1) + f'(1)(x-1) = 16 + 7(x-1) = 9 + 7x$.

Teacher: Fantastic! The graphs of $f(x) = x^3 - 10x^2 + 24x + 1$ and the local linear approximation $y = 9 + 7x$ are shown in the following figure:



Teacher: For exercise (b), applying $f(x) \approx 9 + 7x$ with $x = 1.01$, what does it give?

Student: $f(1.01) \approx 9 + 7 \times 1.01 = 16.07$

Teacher: Awesome! For the last exercise, use a calculator to find the accurate value of $f(1.01)$.

Student: $f(1.01) = (1.01)^3 - 10\times(1.01)^2 + 24\times(1.01) + 1 = 16.069301$

Teacher: Outstanding!

So, the approximation error from part (b) is $16.07 - 16.069301 = 0.000699$.

老師： 求出 f 的導函數。

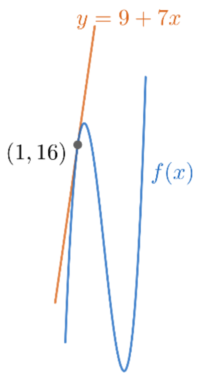
學生： $f(x) = x^3 - 10x^2 + 24x + 1$。 $f'(x) = 3x^2 - 20x + 24$。

老師： 利用一次估計公式，得 $f(x)$ 在 $x = 1$ 附近的一次估計為何？

學生： 一次估計公式，得 $f(x)$ 在 $x = 1$為

$f(x) \approx f(1) + f'(1)(x-1) = 16 + 7(x-1) = 9 + 7x$。

老師： 很好，函數 $f(x) = x^3 - 10x^2 + 24x + 1$ 及其一次近似的圖形如下。



老師： 練習題(b)，將 $x = 1.01$ 代入函數 $f(x) \approx 9 + 7x$ 得？

學生： $f(1.01) \approx 9 + 7\times1.01 = 16.07$

老師： 很好，最後一題請用計算機找 $f(1.01)$ 的精確值。

學生： $f(1.01) = (1.01)^3 - 10\times(1.01)^2 + 24\times(1.01) + 1 = 16.069301$

老師： 非常好，所以與(b)求得的近似值相差 $16.07 - 16.069301 = 0.000699$。

應用問題 / 學測指考題

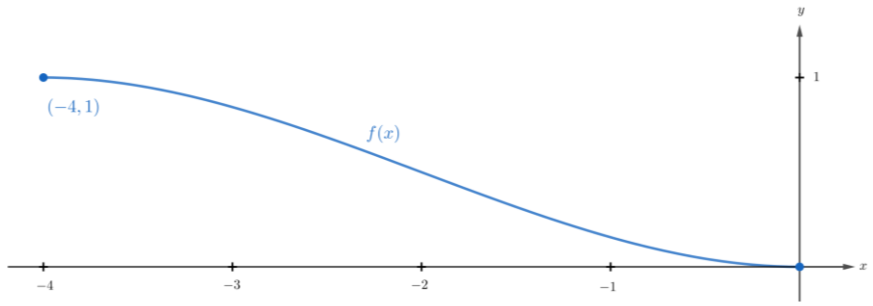
例題一

說明：關於飛機降落的應用問題。

（英文）An airplane begins its descent 4 miles off the runway from the touchdown point and at an altitude of 1 mile, as the figure below shows.

(a) Find the cubic function on the interval $[-4, 0]$ that describes a smooth glide path for the airplane.

(b) Assume that the plane follows your model function in part (a). When would the plane be descending at the greatest rate?



（中文）一架飛機從距離落地 4 英哩時開始下降，高度為 1 英哩。

(a) 在閉區間 $[-4, 0]$ 中，找一個三次函數符合平滑的飛行路徑。

(b) 從(a)的路徑中，求飛機高度變化率的最大值。

Teacher: Set the cubic function as $f(x) = ax^3 + bx^2 + cx + d$ and the leading coefficient a is not equal to 0. Find the derivative of $f(x)$.

Student: $f'(x) = 3ax^2 + 2bx + c$

Teacher: The graph of the function $f(x)$ passes through $(0, 0)$ and $(-4, 1)$, and has a horizontal tangent line at these two points. List all the known values to form the system of simultaneous equations.

Student:
$$\begin{cases} f(0) = d = 0 \\ f(-4) = -64a + 16b - 4c + d = 1 \\ f'(0) = c = 0 \\ f'(-4) = 48a - 8b + c = 0 \end{cases}$$

Teacher: Perfect! Now solve it.

Student: $a = \frac{1}{32},\ b = \frac{3}{16},\ c = 0,\ d = 0.$

Teacher: Brilliant! We can conclude that $f(x) = \frac{1}{32}x^3 + \frac{3}{16}x^2$ is the function we are looking for.

Teacher: For exercise (b), the descending rate of the plane is the derivative function $f(x)$ on the interval $[-4, 0]$. Find the function $f'(x)$ and $f''(x)$, and factorize $f''(x)$.

Student: $f(x) = \frac{1}{32}x^3 + \frac{3}{16}x^2 . f'(x) = \frac{3}{32}x^2 + \frac{3}{8}x . f''(x) = \frac{3}{16}x + \frac{3}{8} = \frac{3}{16}(x+2).$

Teacher: Set $f''(x) = 0$ and fill in the table that summarizes the two intervals determined by $x = -2$ within the interval $[-4, 0]$.

Student:

Interval	$(-4,-2)$	-2	$(-2,0)$
Sign of $f''(x)$	$-$	0	$+$
Conclusion of $f'(x)$	decreasing	$-\frac{3}{8}$	increasing

Teacher: Great! $f'(x)$ achieves its absolute minimum at $x = -2$. In other words, the plane is descending at the greatest rate of $-\frac{3}{8}$ when it is 2 miles from the runway, where its altitude is $f(-2) = \frac{1}{2}$ mile.

老師： 設三次函數為 $f(x) = ax^3 + bx^2 + cx + d$，其中 a 不為 0。找 $f(x)$ 的導函數。

學生： $f'(x) = 3ax^2 + 2bx + c$

老師： 因為 $f(x)$ 的圖形通過 $(0, 0)$ 及 $(-4, 1)$，且此兩點有水平切線，所以

學生：
$$\begin{cases} f(0) = d = 0 \\ f(-4) = -64a + 16b - 4c + d = 1 \\ f'(0) = c = 0 \\ f'(-4) = 48a - 8b + c = 0 \end{cases}$$

老師： 很好，請解聯立方程式。

學生：$a=\frac{1}{32}$、$b=\frac{3}{16}$、$c=0$、$d=0$。

老師：所以函數為$f(x)=\frac{1}{32}x^3+\frac{3}{16}x^2$。

老師：練習題(b)，視導函數$f'(x)$為飛機高度的變化率函數。求出$f'(x)$ 與$f''(x)$，並將$f''(x)$ 因式分解。

學生：$f(x)=\frac{1}{32}x^3+\frac{3}{16}x^2$。$f'(x)=\frac{3}{32}x^2+\frac{3}{8}x$。$f''(x)=\frac{3}{16}x+\frac{3}{8}=\frac{3}{16}(x+2)$

老師：接著解 $f''(x)=0$ ，在$[-4,0]$ 範圍內的實根將數線分成二個區間，請完成表格。

學生：

區間	$(-4,-2)$	-2	$(-2,0)$
$f''(x)$的正負	$-$	0	$+$
$f'(x)$的結論	遞減	$-\frac{3}{8}$	遞增

老師：很好，$f'(x)$ 在區間 $[-4,0]$ 上的最小值為$f'(-2)=-\frac{3}{8}$。也就是說，飛機高度變化率的最大值為$-\frac{3}{8}$，此時飛機的坐標為$\left(-2,\frac{1}{2}\right)$。

例題二

說明：關於導數及向量的應用題（111 分科測驗）。

（英文）Consider two vectors, $\overrightarrow{a}$, $\overrightarrow{b}$, given that $\left|\overrightarrow{a}\right|+\left|\overrightarrow{b}\right|=9$ and $\left|\overrightarrow{a}-\overrightarrow{b}\right|=7$. Let $\left|\overrightarrow{a}\right|=x$, where $1<x<8$, and the angle of vectors $\overrightarrow{a}$, $\overrightarrow{b}$ is θ . A triangle is formed by vectors $\overrightarrow{a}$ 、 $\overrightarrow{b}$ 、 $\overrightarrow{a}-\overrightarrow{b}$. $\cos\theta$ can be represented as: $\frac{c}{9x-x^2}+d$, where c and d are constants and $c>0$. Let the expression be $f(x)$.

(a) Find $f(x)$ and its derivative.

(b) Analyze the intervals on which $f(x)$ is decreasing or increasing. Find the value of x at which the maximum angle between vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ occur.

(c) Apply the local linear approximation of $f(x)$ at $x=4.96$ to round the value of $\cos\theta$.

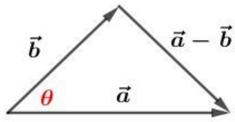
（中文）考慮坐標平面上之向量 $\overrightarrow{a}$ 、 $\overrightarrow{b}$ 滿足 $\left|\overrightarrow{a}\right|+\left|\overrightarrow{b}\right|=9$ 以及 $\left|\overrightarrow{a}-\overrightarrow{b}\right|=7$ 。若令 $\left|\overrightarrow{a}\right|=x$ ，其中 $1<x<8$ ，且令 $\overrightarrow{a}$ 、 $\overrightarrow{b}$ 的夾角為 θ ，則利用向量 $\overrightarrow{a}$ 、 $\overrightarrow{b}$ 、 $\overrightarrow{a}-\overrightarrow{b}$ 所形成的三角形，可將 $\cos\theta$ 以 x 表示成 $\frac{c}{9x-x^2}+d$ ，其中 c 、 d 為常數且 $c>0$ 。令此表示式為 $f(x)$ ，

(a) 求 $f(x)$ 及其導函數。

(b) 說明 $f(x)$ 在定義域中遞增、遞減的情況。並說明 x 為多少時 $\overrightarrow{a}$ 、 $\overrightarrow{b}$ 的夾角 θ 最大。

(c) 利用 $f(x)$ 的一次估計（一次近似），求當 $x=4.96$ 時， $\cos\theta$ 約為多少？

Teacher: For exercise (a). We know that the triangle looks like the figure below. What is the formula of the law of $\cos\theta$?



Student:
$$\cos\theta=\frac{\left|\overrightarrow{a}\right|^2+\left|\overrightarrow{b}\right|^2-\left|\overrightarrow{a}-\overrightarrow{b}\right|^2}{2\left|\overrightarrow{a}\right|\left|\overrightarrow{b}\right|}$$

Teacher: Great! Let's plug in the given values and simplify it.

Student:
$$\cos\theta=\frac{\left|\overrightarrow{a}\right|^2+\left|\overrightarrow{b}\right|^2-\left|\overrightarrow{a}-\overrightarrow{b}\right|^2}{2\left|\overrightarrow{a}\right|\left|\overrightarrow{b}\right|}=\frac{x^2+(9-x)^2-7^2}{2x(9-x)}=\frac{x^2-9x+16}{9x-x^2}=\frac{16}{9x-x^2}-1$$

Teacher: Perfect! So, $f(x)=\frac{16}{9x-x^2}-1$. Find the derivative of it.

Student:
$$f(x)=\frac{16}{9x-x^2}-1;\quad f'(x)=\frac{-16(9-2x)}{\left(9x-x^2\right)^2}=\frac{-144+32x}{\left(9x-x^2\right)^2}.$$

Teacher: For exercise (b), set $f'(x)=0$, and solve for x.

Student:
$$f'(x)=\frac{-144+32x}{\left(9x-x^2\right)^2}=\frac{16}{\left(9x-x^2\right)^2}(2x-9)=0\Rightarrow x=\frac{9}{2}.$$

Teacher: We get extreme candidates at $x=\frac{9}{2}$. Fill in the table that summarizes the two intervals determined by $x=\frac{9}{2}$ within the interval $(1,8)$.

Student:

Interval	$\left(1,\frac{9}{2}\right)$	$\frac{9}{2}$	$\left(\frac{9}{2},8\right)$
Sign of $f'(x)$	$-$	0	$+$
Conclusion	Decreasing		Increasing

Teacher: Well done. So, f is decreasing on the intervals $\left(1,\frac{9}{2}\right)$ and increasing on the interval $\left(\frac{9}{2},8\right)$. Applying the derivative test, we conclude that $f(x)$ has an minimum at $x=\frac{9}{2}$, which the maximum angle between vectors $\overrightarrow{a}$ 、$\overrightarrow{b}$ occur.

Teacher: For exercise (c), follow the local linear approximation formula

$f(x)\approx f(a)+f'(a)(x-a)$ at $x=5$. What do you get?

Student: $f(x)\approx f(5)+f'(5)(x-5)$.

Teacher: Simplify it.

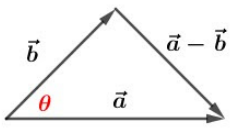
Student: $f(x)\approx\frac{16}{20}-1+\left(\frac{16}{400}\right)(x-5)=\frac{1}{25}x-\frac{2}{5}$

Teacher: Applying $f(x)\approx\frac{1}{25}x-\frac{2}{5}$ with $x=4.96$ gives

Student: $f(4.96)\approx\frac{1}{25}\times4.96-\frac{2}{5}=-\frac{126}{625}$

Teacher: Wonderful! So, when $x=4.96$, the value of $\cos\theta$ is rounded to $-\frac{126}{625}$ by the local linear approximation.

老師： 第(a)題，我們知道三角形應如下圖。請先寫出餘弦定理。



學生： $\cos\theta=\frac{\left|\overrightarrow{a}\right|^2+\left|\overrightarrow{b}\right|^2-\left|\overrightarrow{a}-\overrightarrow{b}\right|^2}{2\left|\overrightarrow{a}\right|\left|\overrightarrow{b}\right|}$

老師： 很好！將已知條件放入公式。

學生： $\cos\theta=\frac{\left|\overrightarrow{a}\right|^2+\left|\overrightarrow{b}\right|^2-\left|\overrightarrow{a}-\overrightarrow{b}\right|^2}{2\left|\overrightarrow{a}\right|\left|\overrightarrow{b}\right|}=\frac{x^2+(9-x)^2-7^2}{2x(9-x)}=\frac{x^2-9x+16}{9x-x^2}=\frac{16}{9x-x^2}-1$

老師：很好，所以我們得到 $f(x)=\dfrac{16}{9x-x^2}-1$，再來請找出其導函數。

學生：$f(x)=\dfrac{16}{9x-x^2}-1$. $f'(x)=\dfrac{-16(9-2x)}{\left(9x-x^2\right)^2}=\dfrac{-144+32x}{\left(9x-x^2\right)^2}$.

老師：第(b)題，令 $f'(x)=0$，找其解。

學生：$f'(x)=\dfrac{-144+32x}{\left(9x-x^2\right)^2}=\dfrac{16}{\left(9x-x^2\right)^2}(2x-9)=0\Rightarrow x=\dfrac{9}{2}$.

老師：我們得到極值的候選點為 $x=\dfrac{9}{2}$，在 $(1,8)$ 範圍內的實根將數線分成二個區間，請完成表格。

學生：

區間	$\left(1,\dfrac{9}{2}\right)$	$\dfrac{9}{2}$	$\left(\dfrac{9}{2},8\right)$
$f'(x)$ 的正負	−	0	+
結論	遞減		遞增

老師：很棒。所以 f 在區間 $\left(1,\dfrac{9}{2}\right)$ 上遞減，在區間 $\left(\dfrac{9}{2},8\right)$ 上遞增。利用極值檢定法知，$f(x)$ 在 $x=\dfrac{9}{2}$ 有最小值，使得 $\overrightarrow{a}$ 、 $\overrightarrow{b}$ 的夾角 θ 最大。

老師：第(c)題，利用一次估計公式 $f(x)\approx f(a)+f'(a)(x-a)$ 代入 $x=5$，得到？

學生：$f(x)\approx f(5)+f'(5)(x-5)$

老師：請化簡。

學生：$f(x)\approx\dfrac{16}{20}-1+\left(\dfrac{16}{400}\right)(x-5)=\dfrac{1}{25}x-\dfrac{2}{5}$

老師：接著使用一次估計法，將 $x=4.96$ 代入 $f(x)\approx\dfrac{1}{25}x-\dfrac{2}{5}$，得到？

學生：$f(4.96)\approx\dfrac{1}{25}\times4.96-\dfrac{2}{5}=-\dfrac{126}{625}$

老師：非常好，因此當 $x=4.96$ 時，利用一次估計法得 $\cos\theta$ 約為 $-\dfrac{126}{625}$。

單元七　積分的意義
The Meaning of Integration

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■ 前言 Introduction

積分是微積分學的重要概念，在這個單元我們將以曲線下面積的計算為起點，透過將曲線分割成微小的部分再進行加總、近似，以求得曲線下區域的總面積。除了可以求得曲線下面積的定積分外，我們也將介紹連結微分、積分的微積分基本定理與不定積分的運算方式。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
integral / integration	積分	inscribed rectangle	內接矩形
upper sum	上和	circumscribed rectangle	外接矩形
lower sum	下和	**fundamental theorem of calculus**	微積分基本定理
partition	分割		
Riemann sum	黎曼和	antiderivative	反導函數
definite integral	定積分	**indefinite integral**	不定積分

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ To ________ we can start with ________.

例句：**To** understand integration, **we can start with** an area problem.

為了要了解積分的運算，我們可以從一個面積問題開始談起。

❷ We can approximate ________.

例句：**We can approximate** the area of region S by using the areas of these rectangles.

我們可以由這些矩形的面積逼近（近似）區域 S 的面積。

❸ We define ________.

例句：**We define** an inscribed rectangle lying inside the *i*th subregion.

我們定義一個「內接矩形」落在第 *i* 個子區域內。

❹ Check ________ with ________.

例句：**Check** the definition of the area of a region in the plane carefully **with** Riemann sums.

請小心地用黎曼和驗證（確認）平面上所圍區域的面積的定義。

❺ The ________ is called ________.

例句：**The** limit **is called** the definite integral of $f(x)$ from a to b.

這個極限稱為函數 $f(x)$ 從 a 到 b 的積分。

❻ This theorem states that ________.

例句：**This theorem states that** differentiation and integration are inverse operations

這個定理描述了微分與積分互為反運算。

■ 問題講解 Explanation of Problems

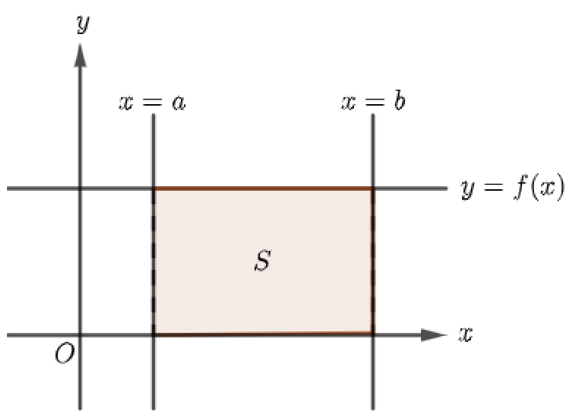
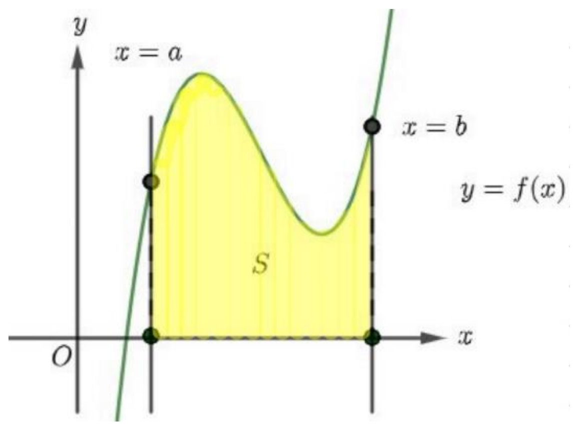
☙ 說明 ❧

I. The area problem

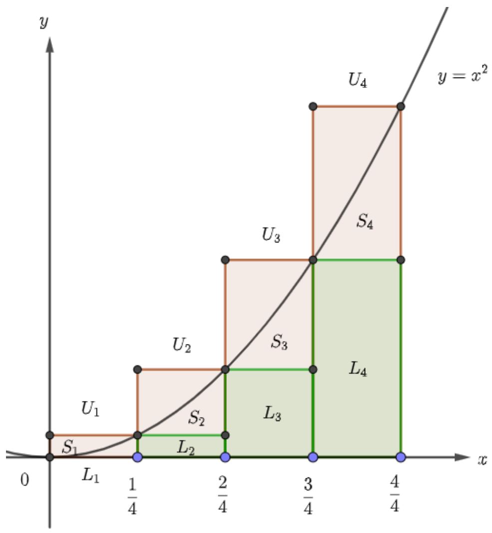
To understand integration, we can start with an area problem:

Find the area of the region *S* that lies under the curve $y = f(x)$ from $x = a$ to $x = b$.

In Figure1, the shaded region is a rectangle. We can easily find the area by multiplying the length ($f(x)$) and width ($b - a$). However, if we want to find the area in Figure2 it is not that easy. To find this area, we should slice the function $f(x)$ into small sections. We can approximate the region *S* by using the areas of these rectangles and then we take the limit of the areas of these rectangles as we increase the number of rectangles. The following example illustrates this procedure:

<table>
<tr><th>Straight Line</th><th>Curved Line</th></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>Figure1</td><td>Figure2</td></tr>
</table>

Example 1 Use four rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1.



<illustration>

To approximate the area of a region, we begin by subdividing the interval [0,1] into 4 subintervals. (We can also divide the interval into n subintervals to get a more precise value.) The four subintervals are:

$$\left[0,\frac{1}{4}\right],\left[\frac{1}{4},\frac{2}{4}\right],\left[\frac{2}{4},\frac{3}{4}\right],\left[\frac{3}{4},\frac{4}{4}\right]\text{ (Each subinterval has the width }\frac{1-0}{4}\text{)}$$

The function $y=x^2$ is continuous. We can find the minimum and maximum value of $f(x)$ in each subinterval. That is:

$$f(m_i)=\text{the minimum value of } f(x) \text{ in } i\text{th subinterval}$$

$$f(M_i)=\text{the maximum value of } f(x) \text{ in } i\text{th subinterval}$$

Then we define an **inscribed rectangle** (rectangle L_1, L_2, L_3, and L_4) lying inside the ith subregion and a **circumscribed rectangle** (rectangle U_1, U_2, U_3, and U_4) extending outside the ith subregion. The height of the ith inscribed rectangle is $f(m_i)$ and the height of ith circumscribed rectangle is $f(M_i)$. For each i, the area of the inscribed rectangle is less than or equal to the area of the circumscribed rectangle. We have the following relationship between the inscribed rectangle, the circumscribed rectangle, and the actual area (S_1, S_2, S_3, S_4) of each section:

$$\text{Area of inscribed rectangle}=f(m_i)\times\frac{1}{4}\le\text{ Actual area }\le f(M_i)\times\frac{1}{4}=\text{Area of circumscribed rectangle}$$

The sum of the areas of the inscribed rectangles is called the **lower sum**, and the sum of the areas of the circumscribed rectangles is called the **upper sum**.

Now we can find the lower sum and upper sum of this question:

Lower sum:

$$\frac{1}{4}f(0)+\frac{1}{4}f(\frac{1}{4})+\frac{1}{4}f(\frac{2}{4})+\frac{1}{4}f(\frac{3}{4})=\frac{1}{4}\left[f(0)+f(\frac{1}{4})+f(\frac{2}{4})+f(\frac{3}{4})\right]=0.21875$$

Upper sum:

$$\frac{1}{4}f(\frac{1}{4})+\frac{1}{4}f(\frac{2}{4})+\frac{1}{4}f(\frac{3}{4})+\frac{1}{4}f(\frac{4}{4})=\frac{1}{4}\left[f(\frac{1}{4})+f(\frac{2}{4})+f(\frac{3}{4})+f(\frac{4}{4})\right]=0.46875$$

<Extension>

Now we divide the interval [0,1] into n subintervals.

Each interval has width $\frac{1}{n}$.

The endpoints of the intervals are 0, $\frac{1}{n}$, $\frac{2}{n}$,… , $\frac{n}{n}$.

We can represent the lower sum and upper sum as n.

Lower sum:

$$L = L_1 + L_2 + \ldots + L_n = \sum_{i=1}^{n} f\left(\frac{i-1}{n}\right)\frac{1}{n} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{i-1}{n}\right)^2 = \frac{1}{n^3}[0^2 + 1^2 + \ldots + (n-1)^2] = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}$$

Upper sum:

$$U = U_1 + U_2 + \ldots + U_n = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2 = \frac{1}{n^3}(1^2 + 2^2 \ldots + n^2) = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$

Hence, we have:

$$L = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2} \leq \text{ Actual area } \leq \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} = U$$

Take the limit of n to infinity, by the squeeze theorem we can have the actual area equals $\frac{1}{3}$

From the illustration and extension above, we have the following conclusion:

A plane region is bounded above by the graph of nonnegative, continuous function $y = f(x)$. The region is bounded below by the x-axis, and the left and right boundaries of the region are the vertical lines $x = a$ and $x = b$. To approximate region's area, we divide the interval $[a,b]$ into n pieces with the width $\frac{b-a}{n}$.

We can find the minimum ($f(m_i)$) and maximum ($f(M_i)$) value of $f(x)$ in each subinterval to get the inscribed and circumscribed rectangles:

Area of inscribed rectangle	$= f(m_i)\times\frac{b-a}{n} \leq$	Actual area	$\leq f(M_i)\times\frac{b-a}{n} =$	Area of circumscribed rectangle

Finally, we take the lower sum = $s(n) = \sum_{i=1}^{n} f(m_i)\Delta x$, and the upper sum = $S(n) = \sum_{i=1}^{n} f(M_i)\Delta x$.

$$s(n) \leq \text{ Area of region } \leq S(n)$$

We can determine the actual area by taking the limit as n tends to infinity and applying the **squeeze theorem**. (Squeeze theorem: the limit of a function that is bounded by two other functions.)

Limit of the lower and upper sums

Let f be continuous and nonnegative on the interval [a, b]. The limits as n tend to be infinity of both the lower and upper sums, which exist and are equal. That is

$$\lim_{n\to\infty} s(n) = \lim_{n\to\infty}\sum_{i=1}^{n} f(m_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f(M_i)\Delta x = \lim_{n\to\infty} S(n)$$

Where $\Delta x = \dfrac{b-a}{n}$ and $f(m_i)$ and $f(M_i)$ are the minimum and maximum values of f on the subinterval.

Area of a region in the plane

Let f be continuous and nonnegative on the interval [a, b]. The area of the region bounded by the graph of f, the x-axis, and the vertical lines $x=a$ and $x=b$ is:

$$\text{Area} = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x \quad , x_{i-1} \le c_i \le x_i$$

Where $\Delta x = \dfrac{b-a}{n}$

II. **Riemann sums and definite integrals**

In the preceding section, we used the limit of a sum to define the area of a region in the plane. This is one of the many applications of finding the area. In this section we will introduce the relationship between the Riemann sums and the definite integrals. Let's see the following definition. (In this section, the function $f(x)$ has no restrictions, while the function $f(x)$ in the preceding section was assumed to be continuous and nonnegative.)

Riemann sums

Let f be defined on the closed interval $[a,b]$, and let Δ be a partition of $[a,b]$ given by

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$$

where Δx_i is the width of the ith subinterval. If c_i is any point in the ith subinterval, then the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i, \quad x_{i-1} \le c_i \le x_i$$

is called the Riemann sum of f for the partition Δ.

<Hint>

Please check the definition of "the area of a region in the plane" carefully with "Riemann sums".

To define the definite integral, consider the following limit:

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i \quad \text{(the limit of a Riemann sum)}$$

If the limit exists, we call this the definite integral of $f(x)$.

The definite integral

If f is defined on the closed interval $[a,b]$ and the limit $\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$ exists.

We say f is integrable on $[a,b]$ and the limit is denoted by:

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)dx$$

The limit is called the **definite integral of** f **from** a **to** b. The number a is the lower limit of integration, and the number b is the upper limit of integration.

The properties of the definite integral

Suppose $f(x)$, $g(x)$ are continuous functions defined on the interval $[a, b]$, then the following properties hold true:

(1) $\int_a^b kdx = k(b-a)$, k is a constant.

(2) $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$

(3) $\int_a^b kf(x)dx = k\int_a^b f(x)dx$, k is a constant.

(4) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, for $a < c < b$

The definite integral as the area of a region

If f is continuous and nonnegative on the closed interval $[a,b]$, then the area of the region bounded by the graph of f, the x-axis, and the vertical lines $x=a$ and $x=b$ is given by:

$$\text{area}=\int_a^b f(x)dx$$

III. The fundamental theorem of calculus

So far, we have learned two major branches of calculus: differential calculus and integral calculus. These two problems have a very close connection. The relationship between these two problems was discovered by Isaac Newton and Gottfried Leibniz and is stated in a theorem, known as the **fundamental theorem of calculus**.

This theorem states that differentiation and integration are inverse operations, just like the inverse relationship between division and multiplication. Let's look at the theorem below:

The fundamental theorem of calculus (I)

$f(x)$ is continuous on $[a,b]$.

Suppose $g(x)=\int_a^x f(t)dt,\ a\le x\le b,$ then $g(x)$ is differentiable on (a,b) and $g(x)$ has derivative $f(x)$. i.e. $g'(x)=f(x)$

<Hint> You can check the proof of this theorem in the following video:
https://youtu.be/pWtt0AvU0KA (Khan Academy)

Definition of an antiderivative

(1) If $F'(x)=f(x)$, then we say $F(x)$ is an antiderivative of $f(x)$

(2) Suppose $f(x)$ is a continuous function, then the **indefinite integral** of $f(x)$ is all the possible antiderivatives of it.

<Hint> If function $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$, then $F(x)=G(x)+C$, C is a constant.

The fundamental theorem of calculus (II)

$f(x)$ is continuous on $[a,b]$ and $F'(x)=f(x)$, i.e. $F(x)$ is an antiderivative of $f(x)$, then $\int_a^b f(x)dx=F(b)-F(a)$.

<Hint> You can check the proof of this theorem in the following video:
https://youtu.be/Cz_GWNdf_68 (Khan Academy)

運算問題的講解

例題一

說明：利用黎曼和求面積。

（英文）Let $f(x)=x^3$. Use the partition and approximation methods to find the area bounded by $f(x)$, $x=0$, $x=1$ and $y=0$. (Hint: Use Riemann sums)

（中文）請用「分割」與「逼近」的方式，計算由函數 $f(x)=x^3$、$x=0$、$x=1$、$y=0$ 所圍成的面積。（提示：使用黎曼和）

Teacher: Now we have finished the first example, let's move on to the practice. In this question, you are asked to find out the area bounded by several equations. You must slice the area into small pieces of rectangles and use the Riemann sums we mentioned. Let's read this question aloud together.

Student: Let … (Read the question.)

Teacher: Very good. Where should we start?

Student: Find a suitable partition.

Teacher: Correct, and then?

Student: Find the area of each small rectangle and take the limit to approach the actual area.

Teacher: Great. In the first part, we should slice the interval into smaller pieces. Can anyone tell us how to do it?

Student: We can divide the interval $[0, 1]$ into n subintervals to get:

$0=x_0<x_1<x_2<...<x_n=1$.

Suppose $\Delta x=\dfrac{1-0}{n}=\dfrac{1}{n}$, in the interval $[x_i, x_{i+1}]$ take

$t_i=x_i=0+\dfrac{1}{n}i,\ i=1,\ 2,\ 3,\ ...,\ n.$

Then we can have the Riemann sum…

Uh… , sorry teacher, I don't know how to find the correct Riemann sum.

Teacher: That's okay. Please give him a big hand. He just helped us find the proper partition. I'll help you finish the rest of the question.

The Riemann sum of this question is:

$$\sum_{i=1}^{n} f(t_i)\Delta x = \sum_{i=1}^{n} (\frac{i}{n})^3 \frac{1}{n} = \frac{1}{n^4}\sum_{i=1}^{n} i^3 = \frac{1}{n^4}\cdot\frac{n^2(n+1)^2}{4} = \frac{(n+1)^2}{4n^2}$$

Now we have the Riemann sum of this question. What should we do next?

Student: Take the limit.

Teacher: Correct! For $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\left[\frac{(n+1)^2}{4n^2}\right] = \frac{1}{4}$

Then we have the area bounded by these functions equals $\frac{1}{4}$.

老師：在看過前面的例子後，接下來請同學嘗試這個練習，在這題當中，我們要用「分割」與「逼近」的方式計算由數個函數所圍成的面積。你需要把大區域的面積切割成小塊的矩形，並用黎曼和的方法求出面積。請大家一起把題目讀出來。

學生：（讀題目。）

老師：非常好，我們要從哪裡開始呢？

學生：找到合適的分割。

老師：沒錯，然後呢？

學生：找出每一個小塊矩形的面積，並取極限去逼近面積。

老師：很棒，首先，我們要把區間「切」成小段，有誰可以告訴我們怎麼做嗎？

學生：首先，我們把區間 [0, 1] 切成 n 個子區間，可以得到：

$$0 = x_0 < x_1 < x_2 < \ldots < x_n = 1$$

假設 $\Delta x = \frac{1-0}{n} = \frac{1}{n}$，並在區間 $[x_i,\ x_{i+1}]$ 中取 $t_i = x_i = 0 + \frac{1}{n}i,\ i = 1,\ 2,\ 3,\ \ldots,\ n.$

然後我們可以得到黎曼和為…

嗯…老師，我好像卡住了，我不太知道怎麼找到後面合適的黎曼和。

老師：沒關係，請給剛剛的同學掌聲鼓勵，他幫我們找到了合適的區間分割。接下來我會幫大家完成剩下的問題。這題的黎曼和可以表示為：

$$\sum_{i=1}^{n} f(t_i)\Delta x = \sum_{i=1}^{n} (\frac{i}{n})^3 \frac{1}{n} = \frac{1}{n^4}\sum_{i=1}^{n} i^3 = \frac{1}{n^4}\cdot\frac{n^2(n+1)^2}{4} = \frac{(n+1)^2}{4n^2}$$

現在我們得到這題的黎曼和了，接下來我們該做甚麼呢？

學生： 取極限。

老師： 正確！$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\left[\frac{(n+1)^2}{4n^2}\right] = \frac{1}{4}$，我們可以得出所圍面積是$\frac{1}{4}$。

例題二

說明：利用定積分的性質求值。

（英文）Suppose $g(x)$ is a continuous function, and $\int_0^3 g(x)dx = 15$, $\int_0^7 g(x)dx = 7$.

Find the value of $\int_7^3 g(x)dx$.

（中文）設$g(x)$是一個連續函數，已知$\int_0^3 g(x)dx = 15$、$\int_0^7 g(x)dx = 7$。求積分$\int_7^3 g(x)dx$的值。

Teacher: Now we will use the properties of the definite integral to find the value of the definite integral. We have: first, $g(x)$ is a continuous function. Second, some definite integral values of $g(x)$. How can we do it?

Student: Combine these values with the definite integral.

Teacher: Yes, but be careful, this question asks for the value of $\int_7^3 g(x)dx$.

Student: We can either calculate directly using the previous method, or do we need to swap the order of 3 and 7?

Teacher: How?

Student: Add a negative in front of the integral sign.

Teacher: Great, the answer will be $\int_7^3 g(x)dx = G(3) - G(7) = -[G(7) - G(3)] = -\int_3^7 g(x)dx$.

(The fundamental theorem of calculus (II))

But, how can we get the value of $\int_3^7 g(x)dx$?

Student: For $\int_0^7 g(x)dx = \int_0^3 g(x)dx + \int_3^7 g(x)dx$, so we have

$\int_3^7 g(x)dx = \int_0^7 g(x)dx - \int_0^3 g(x)dx = 7 - 15 = -8$. Hence we have

$\int_7^3 g(x)dx = -\int_3^7 g(x)dx = -(-8) = 8$.

Teacher: Well done.

Student: How come the definite integral $\int_3^7 g(x)dx$ has a negative value?

Teacher: Good observation! That is because the function $g(x)$ is not always positive. We will talk about that in the next section.

老師： 接下來我們要利用定積分的性質求值。首先，我們知道$g(x)$是一個連續函數。再來，我們有一些函數$g(x)$定積分的值。我們可以怎麼開始？

學生： 利用定積分把這些值和在一起。

老師： 是的，但請注意，這一題問的定積分是$\int_7^3 g(x)dx$的值。

學生： 我們可以使用先前的方法，或是我們要把 3 跟 7 的位置對調計算。

老師： 怎麼做呢？

學生： 在積分前面加上負號。

老師： 很好，所以我們的所求$\int_7^3 g(x)dx = -\int_3^7 g(x)dx$。但我們又要怎樣得到定積分$\int_3^7 g(x)dx$的值呢？

學生： 因為$\int_0^7 g(x)dx = \int_0^3 g(x)dx + \int_3^7 g(x)dx$，所以我們可以得到：

$\int_3^7 g(x)dx = \int_0^7 g(x)dx - \int_0^3 g(x)dx = 7 - 15 = -8$。因此，我們可以得到最後的結果是 $\int_7^3 g(x)dx = -\int_3^7 g(x)dx = -(-8) = 8$.

老師： 很好。

學生： 等等，怎麼定積分$\int_3^7 g(x)dx$的值會是負的呢？

老師： 你的觀察很敏銳喔，那是因為函數$g(x)$在這個區間中不是全部都為正。我們會在下一個單元談到這個問題。

應用問題 / 學測指考題

例題一

說明：利用黎曼和求極限的問題。

（英文）Find the limit $\lim_{n\to\infty}\frac{10^{10}}{n^{10}}\left[1^9+2^9+3^9+\ldots+(2n)^9\right]$.

（中文）試求極限 $\lim_{n\to\infty}\frac{10^{10}}{n^{10}}\left[1^9+2^9+3^9+\ldots+(2n)^9\right]$ 的值。

（110 學年指定科目考試－數學甲）

Teacher: Now let's take a look at this college entrance exam question. You are asked to find the limit of $\frac{10^{10}}{n^{10}}\left[1^9+2^9+3^9+\ldots+(2n)^9\right]$ as n tends to infinity. Do you have any ideas?

Student: The denominator is n to the power of ten, and the numerator's maximum power of n is nine. When n tends to infinity, the denominator will be larger than the numerator. The answer will be zero.

Teacher: Good, you compared the powers of the denominator and numerator, but be careful, the numerator has so many turns. You cannot just take the last turn $(2n)^9$ to make the comparison.

Student: Is this a question about Riemann sums?

Teacher: What do you think?

Student: We should put the denominator n to the power of nine into the square bracket to get:

$$\frac{10^{10}}{n^{10}}\left[1^9+2^9+3^9+\ldots+(2n)^9\right]=10^{10}\cdot\frac{1}{n}\left[(\frac{1}{n})^9+(\frac{2}{n})^9+(\frac{3}{n})^9+\ldots+(\frac{2n}{n})^9\right]$$

Teacher: Well done. You are very close to the answer. Keep going.

Student: On the right hand side, $\frac{1}{n}$ is the width of the rectangles and $(\frac{1}{n})^9,(\frac{2}{n})^9,\ldots(\frac{2n}{n})^9$ are the lengths of the rectangles. Now we can convert this limit question into a definite integral question. That is:

$$\lim_{n\to\infty}\frac{10^{10}}{n^{10}}\left[1^9+2^9+3^9+\ldots+(2n)^9\right]=10^{10}\cdot\int_0^2 x^9dx=10^{10}\cdot\frac{x^{10}}{10}\Big|_0^2=10^9\cdot(2^{10}-0)=10^9\cdot2^{10}$$

The answer to this question is $10^9\cdot2^{10}$.

Teacher: Wonderful. You found that this question is actually about Riemann sums and got the right answer. Please give her a big hand. Next time when you have a question that can extract $\frac{1}{n}$ (or $\frac{k}{n}$), you should be aware of this. It might be a question about Riemann sums.

老師： 現在我們來看看這一題指考的考題。在這一題當中，你要去找當 n 趨近於無限大時 $\frac{10^{10}}{n^{10}}\left[1^9+2^9+3^9+\ldots+(2n)^9\right]$ 的極限值。大家有甚麼想法嗎？

學生： 在這題當中，這個極限的分母是 n 的十次方，分子 n 的最大次方是九次方。當 n 趨近於無限大的時候，因為分母的次方比較大，分母會大過分子，這題的答案就會是零。

老師： 很好，你比較了分子與分母的次方，但是請注意，分子的項數很多，你不能只取最大的 $(2n)^9$ 去進行比較。

學生： 這是一題跟黎曼和有關的問題嗎？

老師： 你怎麼想呢？

學生： 我們應該要把分母中 n 的九次方乘到中括號中，得到以下的式子：

$$\frac{10^{10}}{n^{10}}\left[1^9+2^9+3^9+\ldots+(2n)^9\right]=10^{10}\cdot\frac{1}{n}\left[(\frac{1}{n})^9+(\frac{2}{n})^9+(\frac{3}{n})^9+\ldots+(\frac{2n}{n})^9\right]$$

老師： 非常好，很接近答案了，繼續往下說。

學生： 在右側的式子中，$\frac{1}{n}$ 是分割後矩形的寬，而 $(\frac{1}{n})^9$、$(\frac{2}{n})^9$、...$(\frac{2n}{n})^9$ 是分割後矩形的長。透過這樣的表示方式，我們可以把原本的極限問題轉換為一個定積分問題如下：

$$\lim_{n\to\infty}\frac{10^{10}}{n^{10}}\left[1^9+2^9+3^9+\ldots+(2n)^9\right]=10^{10}\cdot\int_0^2 x^9dx=10^{10}\cdot\frac{x^{10}}{10}\bigg|_0^2=10^9\cdot(2^{10}-0)=10^9\cdot 2^{10}$$

所以這題的答案就是 $10^9\cdot 2^{10}$。

老師： 太棒了，你看出這題其實是一個計算黎曼和的問題，並找到了正確的答案。請大家給她掌聲鼓勵。下一次，當你遇到一個問題可以提出 $\frac{1}{n}$ (或 $\frac{k}{n}$)時，請特別注意，這可能是一個需要用黎曼和才有辦法處理的問題。

例題二

說明：利用微積分基本定理解函數。

（英文）Suppose $f(x)$ is a polynomial equation with real coefficients and $xf(x) = 3x^4 - 2x^3 + x^2 + \int_1^x f(t)\,dt$, for all $x \geq 1$ holds true. Answer the following questions.

(1) Find the value of $f(1)$.

(2) Find the equation $f'(x)$.

(3) Find the equation $f(x)$.

(4) Prove there is only one real number $a > 1$ such that $\int_0^a f(x)dx = 1$.

（中文）設 $f(x)$ 為實係數多項式函數，且 $xf(x) = 3x^4 - 2x^3 + x^2 + \int_1^x f(t)\,dt$ 對 $x \geq 1$ 恆成立。試回答下列問題。

(1) 試求 $f(1)$。

(2) 試求 $f'(x)$。

(3) 試求 $f(x)$。

(4) 試證明恰有一個大於 1 的正實數 a 滿足 $\int_0^a f(x)dx = 1$。

（108 學年指定科目考試－數學甲）

Teacher: This question is not a multiple choice or gap-fill question. It's a calculation and proof question. You need to write all the details carefully. In question one, we're asked to find the value of $f(1)$. Now try to solve it.

Student: We can put $x=1$ into the equation $xf(x) = 3x^4 - 2x^3 + x^2 + \int_1^x f(t)dt$, then we can have the answer. That is:

$$1 \cdot f(1) = 3 \cdot 1^4 - 2 \cdot 1^3 + 1^2 + \int_1^1 f(t)dt = 3 - 2 + 1 + 0 = 2$$

Teacher: Correct. We only need to put $x=1$ into the equation then we can get the right answer. How about question two?

Student: We can differentiate the given equation $xf(x) = 3x^4 - 2x^3 + x^2 + \int_1^x f(t)dt$. We use the fundamental theorem of calculus to differentiate $\int_1^x f(t)dt$. We get the following result: $f(x) + xf'(x) = 12x^3 - 6x^2 + 2x + f(x)$. We can eliminate $f(x)$ to get $f'(x) = 12x^2 - 6x + 2$.

Teacher: Great. In this question, we'll need the fundamental theorem of calculus to differentiate $\int_1^x f(t)dt$. This is the hardest part in this question. Let's move on to question three. Find the equation $f(x)$.

Student: We got the equation $f'(x)=12x^2-6x+2$ and the initial value $f(1)=2$. We can find the indefinite integral: $f(x)=\int f'(x)dx=4x^3-3x^2+2x+c$, and we know that $f(1)=4-3+2+c=2$ and $c=-1$. We get the answer: $f(x)=4x^3-3x^2+2x-1$.

Teacher: Very good. With the indefinite integral and initial value, we can find the function $f(x)$. Let's finish the last question. We need to prove that there is only one real number $a>1$ such that $\int_0^a f(x)dx=1$. This question is more challenging than the others. Let's do it together.

For $\int_0^a f(x)dx=1$, we have $\int_0^a (4x^3-3x^2+2x-1)dx=1$. Calculate the definite integral: $a^4-a^3+a^2-a=1$ and we have $a(a-1)(a^2+1)-1=0$.

Let $g(a)=a(a-1)(a^2+1)-1$. For $g(1)=-1<0$ and $g(2)=9>0$, from the intermediate value theorem, we know that $g(a)$ has real root(s) in the interval $(1,\ 2)$. Finally, consider the derivative of $g(a)$, $g'(a)=4a^3-3a^2+2a-1$, $g'(a)>0$ for all $a>1$. $g(a)=a(a-1)(a^2+1)-1$ is increasing for all $a>1$.

Hence, we find that there is only one real number $a>1$ such that $\int_0^a f(x)dx=1$.

That's all for today's practice. Please make sure that you can do it!

老師： 接下來的這個問題不是多選題或是填充題，而是一題計算證明題，你需要詳細地把你的計算過程寫下來。在第一題當中，我們需要找出 $f(1)$ 的值，請各位同學試著解解看。

學生： 我們把 $x=1$ 帶入方程式 $xf(x)=3x^4-2x^3+x^2+\int_1^x f(t)dt$ 中，如此一來我們可以得到 $1\cdot f(1)=3\cdot 1^4-2\cdot 1^3+1^2+\int_1^1 f(t)dt=3-2+1+0=2$。所以 $f(1)$ 的值為 2。

老師： 正確，我們只需要把 $x=1$ 帶入方程式就可以輕鬆地找到正確的答案。那麼第二題呢？

學生：　我們可以把給定的函數 $xf(x)=3x^4-2x^3+x^2+\int_1^x f(t)dt$ 進行微分，這題我們將會用到微積分基本定理去確認 $\int_1^x f(t)dt$ 的值。微分過後，我們可以得到下列的函數：$f(x)+xf'(x)=12x^3-6x^2+2x+f(x)$，接著我們把等號兩側的 $f(x)$ 消去即可得到所求 $f'(x)=12x^2-6x+2$。

老師：　很棒，在這題當中，我們需要用到微積分基本定理去計算積分 $\int_1^x f(t)dt$ 的值，這是這題最困難的部分。接著我們來看看第三題，找出函數 $f(x)$。

學生：　由第二題我們知道 $f'(x)=12x^2-6x+2$，以及函數的初始值 $f(1)=2$。我們可以計算函數 $f'(x)$ 的不定積分：$f(x)=\int f'(x)dx=4x^3-3x^2+2x+c$。

此外，我們知道 $f(1)=4-3+2+c=2$，$c=-1$。我們可以得到這題的答案是：$f(x)=4x^3-3x^2+2x-1$。

老師：　非常好，透過不定積分與初始值我們可以找出函數 $f(x)$。最後，我們一起完成剩下的一題吧。我們需要證明恰有一個大於 1 的正實數 a 滿足 $\int_0^a f(x)dx=1$。這題相較於前面的題目更有挑戰性，我們一起完成它吧！

因為 $\int_0^a f(x)dx=1$，我們有 $\int_0^a (4x^3-3x^2+2x-1)dx=1$。

計算定積分得到：$a^4-a^3+a^2-a=1$ 透過分解可知：$a(a-1)(a^2+1)-1=0$。

令 $g(a)=a(a-1)(a^2+1)-1$，已知 $g(1)=-1<0$ 且 $g(2)=9>0$，由中間值定理（勘根定理）知函數 $g(a)$ 在區間 $(1,\ 2)$ 間必有實根。

最後，考慮函數 $g(a)$ 的微分，$g'(a)=4a^3-3a^2+2a-1$，且 $g'(a)>0$ 對於所有 $a>1$。故 $g(a)=a(a-1)(a^2+1)-1$ 在 $a>1$ 恆為遞增。

綜合上述可知，恰有一個大於 1 的正實數 a 滿足 $\int_0^a f(x)dx=1$。

以上就是今天的練習，請確認你都能掌握住喔！

單元八　積分的應用
Application of Integration

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■ 前言 Introduction

在前一個單元中，我們了解了積分的運算方式與基本概念，除了計算數學中的問題，積分也可以應用在許多不同的領域上，包含物理、化學、生物、經濟學等。在這個單元中，我們將透過總量變化、平均以及不規則區域的面積、體積（旋轉體體積）等問題，展現積分的應用方式。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
displacement	位移	**cross-section**	橫截面
distance	路徑長	slab	板 / 片
velocity	速度	revolve	旋轉
speed	速率	**disk/slice method**	圓盤法
net change	淨變化	**shell method**	球殼法
cylinder	圓柱	pyramid	金字塔

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ If ________ then ________.

例句：**If** a function $f(x)$ is continuous on the interval $[a, b]$, **then** $\int_a^b f(x)dx = F(b) - F(a)$.

若函數 $f(x)$ 在區間$[a, b]$連續，則$\int_a^b f(x)dx = F(b) - F(a)$ 。

❷ Take ________ as an example.

例句：**Take** the displacement of a particle **as an example.**

以質點的位移為例。

❸ Find the ________.

例句：**Find the** area of the region enclosed by the parabola and the straight line.

找出被拋物線與直線所圍成的區域面積。

❹ _______ represents ________ with respect to ________.

例句： $f'(x)$ **represents** the rate of change of $y = f(x)$ **with respect to** x.

$f'(x)$表示對於 x 的變化，函數 $y = f(x)$ 的變化率。

❺ Take _______ tends to infinity to approximate ________.

例句：**Take** n **tends to infinity to approximate** the real volume.

取 n 趨近於無限大去逼近實際的體積。

■ 問題講解 Explanation of Problems

說明

I. **Application – Net Change and Average**

With the fundamental theorem of calculus (II), if a function $f(x)$ is continuous on the interval $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

$F(x)$ is any antiderivative of $f(x)$. That is, $F'(x) = f(x)$. We can rewrite the equation as

$$\int_a^b F'(x)dx = F(b) - F(a)$$

$F'(x)$ represents the rate of change of $y = F(x)$ with respect to *x*, and $F(b) - F(a)$ is the change in *y* when *x* changes from *a* to *b*. We can use this principle for all the problems about the rate of change (or net change) in sciences and social sciences as in the following examples.

1. **Displacement of a particle**

   The velocity of a particle is $V(t) = S'(t)$ ($S(t)$ stands for the distance.), so

   $$\int_a^b V(t)dt = S(b) - S(a)$$

   is the change in position of the particle from time *a* to time *b*.

2. **Net change of a population**

   The rate of growth of a population is $n'(t)$, then

   $$\int_{t_1}^{t_2} n'(t)dt = n(t_2) - n(t_1)$$

   is the net change of a population from time t_1 to time t_2.

3. **Mass of a rod**

   The mass of a rod measured from the left end to point *x* is: $m(x)$, for the rod has density $\rho(x) = m'(x)$, so

   $$\int_a^b \rho(x)dx = m(b) - m(a)$$

   is the mass of the segment of the rod that lies between $x = a$ and $x = b$.

With the examples above, you can see how to use integrals to measure the net change of real-world problems. Also, if we divide the net change by the differences of variables *x* or *t*, we can get the average of net change. Take **displacement of a particle** as an example.

The **velocity** of a particle is: $V(t)$

The **change in position** from time *a* to time *b* is: $\int_a^b V(t)dt = S(b) - S(a)$.

The **difference of *t*** from time *a* to time *b* is: $b-a$.

The **"average" of change in position** is: $\frac{1}{b-a}\int_a^b V(t)dt = \frac{S(b)-S(a)}{b-a}$, which is the average speed of the particle from time *a* to time *b*.

Net change

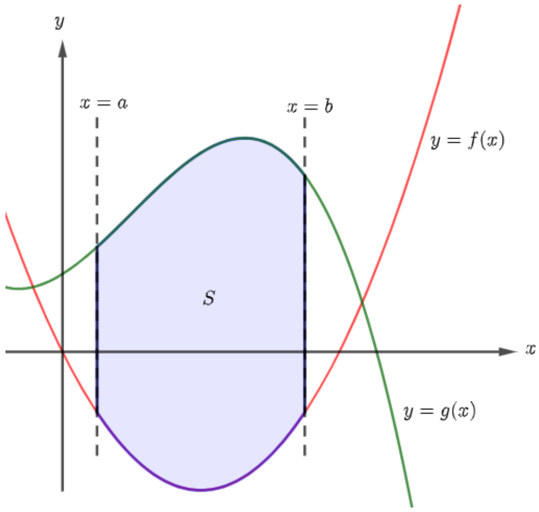
The integral of a rate of change is the net change. $$\int_a^b F'(t)dt = F(b) - F(a)$$

The average value of a continuous function

The average value of a continuous function $f(x)$ on the interval $[a,b]$ is, $$\frac{1}{b-a}\int_a^b f(x)dx$$ ($f(x)$ can be any continuous function mentioned above.)

II. **Application – Areas Between Curves**

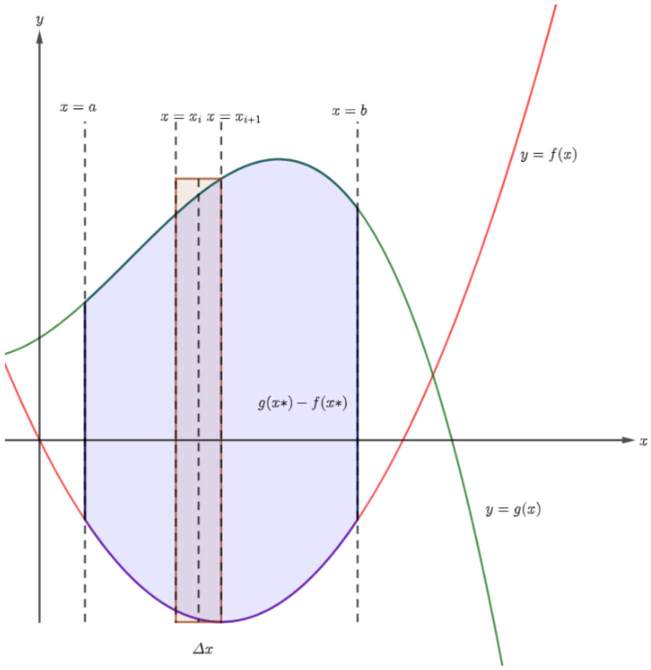
We have talked about the areas of regions that lie under the graphs of functions in the Meaning of Integration unit. Here, we use integrals to find the area of a region between the graphs of two functions. Consider the region *S* that lies between two curves $y = f(x)$, $y = g(x)$ and two vertical lines $x = a$, $x = b$ as in the figure shown below. The functions *f* and *g* are both continuous functions and $g(x) \ge f(x)$ for all *x* in $[a, b]$.



To get the area, we divide the region into *n* strips of equal width, and then we can approximate the *i*th strip by a rectangle with width

$\Delta x = x_{i+1} - x_i = \dfrac{b-a}{n}$ and length $g(x^*) - f(x^*)$. (x^* is an arbitrary point in the interval $[x_i, x_{i+1}]$.) Then the Riemann sum of the region *S* equals:

$$\sum_{i=1}^{n}[g(x^*) - f(x^*)]\Delta x$$



To approximate the actual area of region *S*, take *n* tends to be infinity. Then we can define the area *A* of the region *S* as the limiting value of the sum of the areas of the rectangles.

$$\text{Area of the region} = A = \lim_{n\to\infty}\sum_{i=1}^{n}[g(x^*) - f(x^*)]\Delta x$$

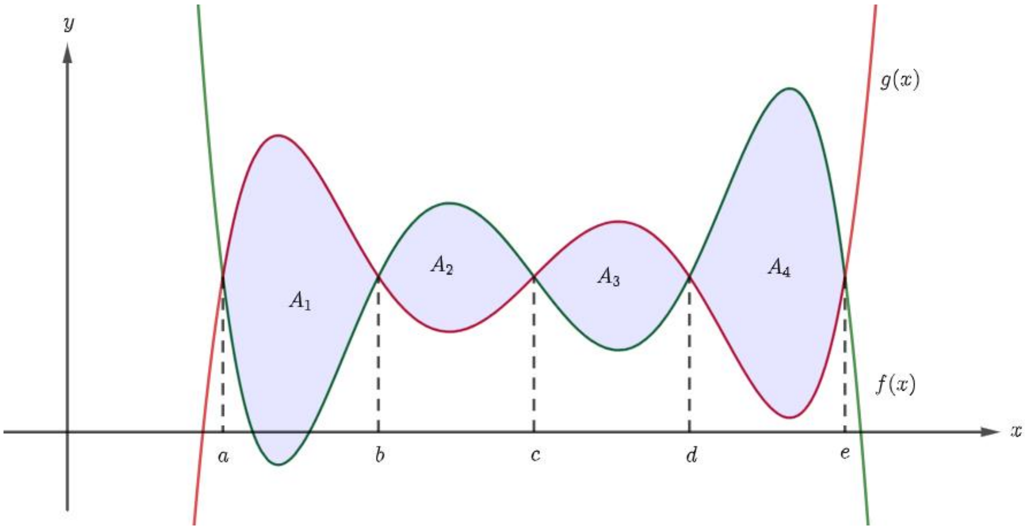
We can find that the limit above is the definite integral of $g(x) - f(x)$ from $x = a$ to $x = b$. Therefore, we get the following conclusion:

The area between continuous functions

$f(x), g(x)$ are continuous functions and $g(x) \ge f(x)$ for all $a \le x \le b$. Then the area of the region bounded by the curves $f(x), g(x)$, and the lines $x = a, x = b$ is:

$$A = \int_a^b [g(x) - f(x)]dx$$

Alternate regions



$f(x), g(x)$ are continuous functions, in the interval $[a,\ e]$. If we are asked to find the area between the curves $f(x),\ g(x)$, where $f(x) \geq g(x)$ for some values of *x* and $g(x) \geq f(x)$ for some values of *x,* we should split the given region *S* into subregions S_1, S_2, S_3,… with areas A_1, A_2, A_3,… as shown in the figure above. We then define the area of the region *S* as the sum of the areas of the smaller regions. Hence, the area above is:

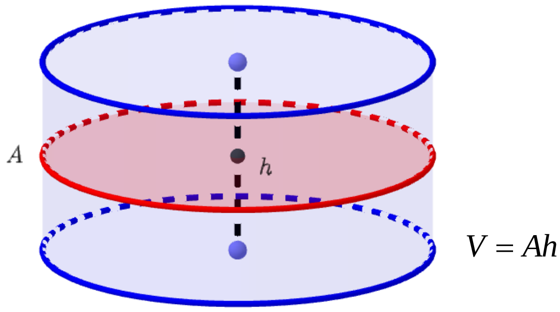
$$A_1 + A_2 + A_3 + A_4 = \int_a^b g(x) - f(x)dx + \int_b^c f(x) - g(x)dx + \int_c^d g(x) - f(x)dx + \int_d^e f(x) - g(x)dx$$

$$= \int_a^e |f(x) - g(x)|dx$$

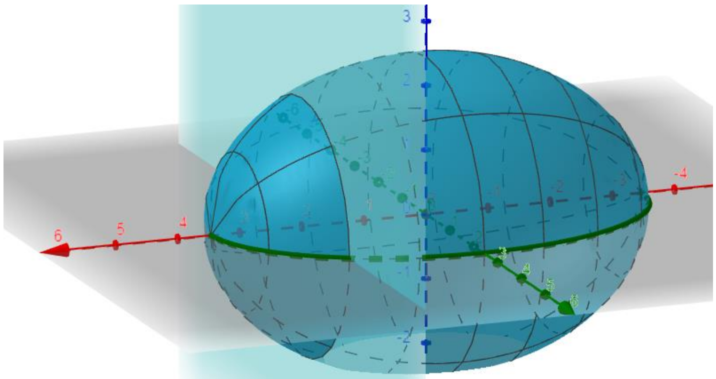
<**Hint**> To find the area, the integral function should be all positive.

III. **Application – Volumes**

In the previous part, we learned one of the applications of the definite integral. Another important application of the definite integral is its use in finding the volume of a three-dimensional solid. We start with a simple type of solid – a cylinder (as shown in the figure below). The cylinder is bounded below by a plane region P_1 (base), and bounded above by another congruent region P_2. If the area of the base is *A* and the height of the cylinder is *h*, then the volume *V* of the cylinder is defined as:



Now we can find the volume of a cylinder with the formula above easily, but how can we get the solid S that isn't a cylinder? First, we slice the solid into pieces. Second, we estimate the area of each piece. Finally, we add all the areas of each piece to approximate the volume of the solid. (See the figure below)



We slice the volume by planes.

We slice the solid region S into n "slabs" of equal width Δx with planes and obtain cross-sections of S. Let the function $A(x)$ be the area of the cross-section of S in a plane P_x perpendicular to the x-axis and passing through the point x, where $a \le x \le b$. The cross-sectional area $A(x)$ will vary as x increases from a to b.

Hence, we have the following:

(1) The height of each slab:

$$\Delta x = \frac{b-a}{n}$$

(2) The cross-section area of each slab:

$$A(x_i^*), x_i^* \in [x_{i-1}, x_i]$$

Then, we can find the volume of each slab to be:

$$V(S_i) = A(x_i^*)\Delta x_i$$

Adding the volumes of these slabs, we get the approximate volume to be:

$$\sum_{i=1}^{n} A(x_i^*)\Delta x_i$$

Finally, we take n tends to infinity to approximate the real volume.

Volumes of solids with cross-sections

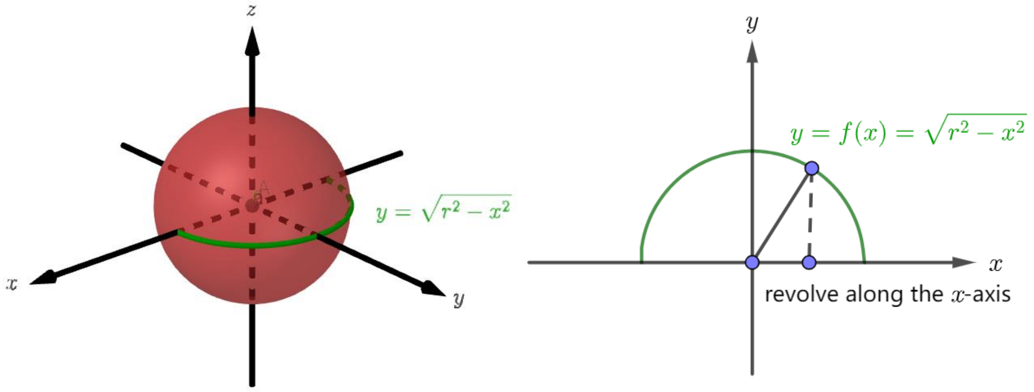
Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where $A(x)$ is a continuous function, then the volume of S can be represented as:

$$V = \lim_{n\to\infty}\sum_{i=1}^{n} A(x_i^*)\Delta x_i = \int_a^b A(x)dx$$

We can also take the cross-section that perpendicular to the y-axis between $y = c$ and $y = d$ with area $A(y)$, where $A(y)$ is a continuous function, then the volume of S can be represented as:

$$V = \lim_{n\to\infty}\sum_{i=1}^{n} A(y_i^*)\Delta y_i = \int_c^d A(y)dy$$

If we revolve a region about a line, we obtain a solid of revolution. For example, a sphere of radius r can be considered as function $y = f(x) = \sqrt{r^2 - x^2}$ revolve along the x-axis, as shown in the figure below.



Hence, the volume of a sphere can be considered in another way: $\int_{-r}^{r} \pi[f(x)]^2 dx$

(We can slice the sphere into small pieces of cylindrical with radius $y = f(x) = \sqrt{r^2 - x^2}$ and altitude dx. The volume of cylindrical element is: $dV = \pi y^2 dx$. The sum of the cylindrical elements from 0 to r is a hemisphere. Hence, $V = \int_{-r}^{r} \pi[f(x)]^2 dx = 2\int_0^r \pi[f(x)]^2 dx = \frac{4}{3}\pi r^3$.)

We can extend this result to any polynomial function to get the following result:

Volumes of solids with revolution

The volume of a solid generated by revolving the region bounded by a function $y = f(x)$ and x-axis on the interval $[a,\ b]$ about the x-axis is:

$$\int_a^b \pi[f(x)]^2\,dx \quad , f(x) \geq 0 \ \text{ for all } \ x \in [a,\ b]$$

We can also find the volume of a solid generated by revolving the region about the y-axis.

<**Hint**> This method to find the volume is also known as "the disk method".

<**Hint**> Another method to find the volume is "the shell method", which you will learn in college.

運算問題的講解

例題一

說明：利用積分求連續函數的平均。

（英文）Find the average value of the function $f(x) = \sqrt{9-x^2}$ from $x = -3$ to $x = 3$.

（中文）求函數 $f(x) = \sqrt{9-x^2}$ 在 $x = -3$ 到 $x = 3$ 之間的平均。（區間 $[-3,\ 3]$）

Teacher: Now, we are going to find the "average value" of the function. In this example, we are going to find the average value of $f(x) = \sqrt{9-x^2}$ from $x = -3$ to $x = 3$. What should we do first?

Student: Find the integral value of $f(x)$ from $x = -3$ to $x = 3$.

Teacher: Good, and then?

Student: Divide the integral value with the length of the interval $[-3,\ 3]$.

Teacher: Correct. Hence, we have the following:

The average value of $f(x)$ is: $\frac{1}{3-(-3)}\int_{-3}^{3}\sqrt{9-x^2}\,dx = \frac{1}{6}\int_{-3}^{3}\sqrt{9-x^2}\,dx$, but how can we get the value of $\int_{-3}^{3}\sqrt{9-x^2}\,dx$?

Student: It's the area under the curve $y = \sqrt{9-x^2}$. It is a semicircle with radius 3.

Teacher: Very good. We only need to find the semicircle's area and divide it by 6.

The answer to this question is: $\frac{1}{6}\int_{-3}^{3}\sqrt{9-x^2}\,dx=\frac{1}{6}\cdot\frac{1}{2}\pi(3)^2=\frac{3}{4}\pi$. Please write it down.

老師： 現在我們要來計算函數的「平均值」。在這個例子中要計算 $f(x)=\sqrt{9-x^2}$ 在 $x=-3$ 到 $x=3$ 間的平均值，一開始該怎麼做呢？

學生： 找出函數 $f(x)$ 在 $x=-3$ 到 $x=3$ 間的積分值。

老師： 很好，然後呢？

學生： 把所得到的積分值除以區間[−3, 3]的長度。

老師： 沒錯，因此我們可以得到下列的結果：

函數 $f(x)$ 在[−3, 3]間的平均值為：$\frac{1}{3-(-3)}\int_{-3}^{3}\sqrt{9-x^2}\,dx=\frac{1}{6}\int_{-3}^{3}\sqrt{9-x^2}\,dx$ 。

但我們要如何計算積分 $\int_{-3}^{3}\sqrt{9-x^2}\,dx$ 的值呢？

學生： 這是曲線 $y=\sqrt{9-x^2}$ 下的區域面積（$y=\sqrt{9-x^2}$ 與 x 軸所圍成的區域面積）。也就是一個半徑為三的半圓面積。

老師： 非常好，所以我們只要找出這個半圓形的面積再除以六後就可以得到這題的結果。這題的答案為：$\frac{1}{6}\int_{-3}^{3}\sqrt{9-x^2}\,dx=\frac{1}{6}\cdot\frac{1}{2}\pi(3)^2=\frac{3}{4}\pi$ ，請把它紀錄下來。

例題二

說明：利用定積分求面積。

（英文）Find the area of the region bounded by the equations $y=2x^2+10$ and $y=3x+9$.

(Use the definite integral, don't use a Riemann sum.)

（中文）試求由 $y=2x^2+10$ 與 $y=3x+9$ 所圍成的區域面積。

（請使用定積分的方法計算，不要使用黎曼和。）

Teacher: In this question, we'll find the area of a region bounded by two different equations. How can we do it?

Student: First, we should find the intersection of two equations. That is: $\begin{cases} y=2x^2+10 \\ y=3x+9 \end{cases}$

$\Rightarrow 2x^2-3x+1=(2x-1)(x-1)=0$. We have these two equations intersect at points $(\frac{1}{2}, \frac{21}{2})$ and $(1, 12)$.

Second, for $2x^2+10\le 3x+9$ on the interval $x\in[\frac{1}{2}, 1]$ the area of the region bounded by the equations $y=2x^2+10$ and $y=3x+9$ equals:

$$\int_{\frac{1}{2}}^{1}(3x+9)-(2x^2+10)dx=\int_{\frac{1}{2}}^{1}(-2x^2+3x-1)dx=\left[-\frac{2}{3}x^3+\frac{3}{2}x^2-x\right]_{\frac{1}{2}}^{1}=\frac{1}{24}.$$

Teacher: Wonderful, everyone please give her a big hand. In this question, first, we need to make sure of the intersections of the given equations. After that, we need to compare which equation is larger on the interval, and finally, we can calculate the area with a definite integral. Your classmate just described all the details. Well done!

老師：在這題中，我們要求由兩個不同方程式圍成的區域面積，我們該怎麼做呢？

學生：首先，我們要先找出這兩個方程式的交點，我們可以透過聯立方程式找出一個一元二次方程式，即：$\begin{cases} y=2x^2+10 \\ y=3x+9 \end{cases}\Rightarrow 2x^2-3x+1=(2x-1)(x-1)=0$。

我們可以知道這兩個方程式相交於 $(\frac{1}{2}, \frac{21}{2})$、$(1, 12)$ 兩點。

再來，在區間$[\frac{1}{2}, 1]$間$2x^2+10 \le 3x+9$恆成立。

綜合以上，我們可以用積分計算由函數$y=2x^2+10$與$y=3x+9$所圍成的區域面積如下：

$$\int_{\frac{1}{2}}^{1}(3x+9)-(2x^2+10)dx=\int_{\frac{1}{2}}^{1}(-2x^2+3x-1)dx=\left[-\frac{2}{3}x^3+\frac{3}{2}x^2-x\right]_{\frac{1}{2}}^{1}=\frac{1}{24}$$

故這題所求的面積為$\frac{1}{24}$。

老師：太棒了，請大家給她掌聲鼓勵。在這題當中，首先我們要確認所給定函數的交點；接著，我們要比較給定函數在相交的區間中的大小（我們才有辦法確認怎樣把面積夾成正的）；最後，我們再用定積分計算出面積。她剛剛的回答中點到了所有的細節，做得好！

例題三

說明：利用定積分求體積。

（英文）Find the volume of the solid obtained by rotating the region bounded by $y=\frac{r}{h}x$, $x=0$, $x=h$, and the x-axis about the x-axis. (The volume of a cone.)

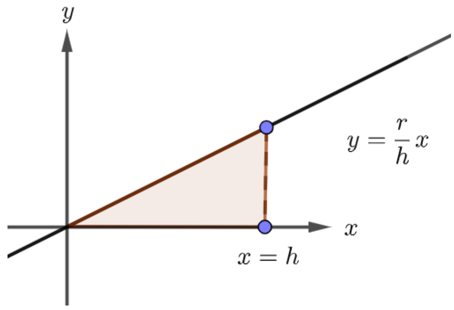
（中文）試求由$y=\frac{r}{h}x$的圖形與$x=0$、$x=h$及x軸所圍成的區域繞x軸旋轉所形成的旋轉體體積。（錐體體積）

Teacher: After finding the area bounded by two equations, we move on to the volume problem. We find the volume of the solid obtained by rotating the region bounded by $y=\frac{r}{h}x$, $x=0$, $x=h$, and the x-axis about the x-axis. Do you have any ideas?

Student: Is that a cone? I think this should be the volume of a cone.

Teacher: Great idea! What steps should we follow?

Student: First, we can start by drawing a diagram to represent the relationship of the given function.



Teacher: Good, what's the next step?

Student: Next, we can rotate the shaded region and use the method of integration (disk method) for rotational solids to calculate the volume.

Teacher: How should we choose the starting and ending x-values for the integration?

Student: The starting point is 0, and the ending point as shown in the figure, is $x=h$. So, the process for calculating this question is:

$$\int_0^h \pi\left[f(x)\right]^2 dx=\int_0^h \pi(\frac{r}{h}x)^2 dx=\frac{\pi r^2}{h^2}\int_0^h x^2 dx=\frac{\pi r^2}{h^2}\times\left[\frac{1}{3}x^3\right]\Bigg|_0^h=\frac{1}{3}\pi r^2 h\,.$$

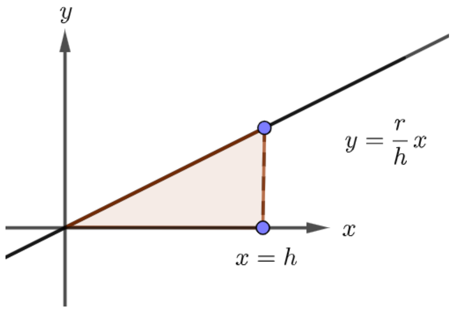
Teacher: Well done, this is the formula for the volume of a cone with a base radius of r and height h.

老師： 在計算過曲線間的面積後，接著我們要進行體積計算的問題。這題我們要求由 $y=\frac{r}{h}x$ 的圖形與 $x=0$ 、 $x=h$ 及 x 軸所圍成的區域繞 x 軸旋轉所形成的旋轉體體積。大家有甚麼想法嗎？

學生： 這是圓錐嗎？我覺得這應該是在求圓錐體的體積。

老師： 很好，那具體來說我們應該怎麼做呢？

學生：　首先，我們可以畫出圖形表示給定函數的關係，如下所示：



老師：　很好，下一步呢？

學生：　接下來，我們要將陰影區域旋轉，並用定積分求旋轉體體積的方法（圓盤法）去計算它的體積。

老師：　那我們要怎麼挑選積分起點與終點的 x 值呢？

學生：　起點為 $x=0$，終點如圖所示為 $x=h$，因此這題的計算過程如下：

$$\int_0^h \pi\left[f(x)\right]^2 dx=\int_0^h \pi(\frac{r}{h}x)^2 dx=\frac{\pi r^2}{h^2}\int_0^h x^2 dx=\frac{\pi r^2}{h^2}\times\left[\frac{1}{3}x^3\right]\Bigg|_0^h=\frac{1}{3}\pi r^2 h\,.$$

老師：　做得好！這就是底圓半徑為 r、高度為 h 的圓錐體積公式。

應用問題 / 學測指考題

例題一

說明：利用積分求物理量。(位移、路徑長、速度、速率)

(英文) A particle moves along a line so that its velocity at time t is $v(t)=t^2-2t-3$ (measured in meters per second).

(1) Find the displacement of the particle during the time period $2\le t\le 5$.

(2) Find the distance traveled during this time period.
(Hint: You should find out at which period the velocity will be negative.)

(3) Find the average velocity during this time period.

(4) Find the average speed during this time period.
(Hint: Velocity has both magnitude and direction, but speed has magnitude only.)

(中文) 一個質點沿直線運動，其速度函數為$v(t)=t^2-2t-3$ (公尺 / 秒)。

(1) 求在時段$2\le t\le 5$上，質點運動的總位移。

(2) 求在時段$2\le t\le 5$上，質點運動所經過的總路徑長。

(3) 求在時段$2\le t\le 5$上，質點平均速度。

(4) 求在時段$2\le t\le 5$上，質點平均速率。

Teacher: This is a physics-related applied problem that involves different physical quantities. Be careful because displacement and velocity are directional physical quantities, while path length (distance traveled) and speed are non-directional physical quantities. Let's complete these questions.

Student: For the displacement it is easier. We just need to integrate $v(t)=t^2-2t-3$ from $t=2$ to $t=5$, that is:

The displacement: $s(5)-s(2)=\int_2^5 t^2-2t-3dt=\left[\frac{1}{3}t^3-t^2-3t\right]_2^5=9 \quad (m)$

The average velocity: $\frac{s(5)-s(2)}{5-2}=\frac{9}{3}=3 \quad (\text{m/s})$

Teacher: Correct, how about the distance traveled and the speed?

Student: Note that $v(t)=t^2-2t-3=(t-3)(t+1)$ and so $v(t)\le 0$ on the interval $[2, 3]$

and $v(t) \geq 0$ on $[3, 5]$. Thus, the distance traveled is:

$$\int_2^5 |v(t)|dt = \int_2^3 [-v(t)]dt + \int_3^5 v(t)dt = -(-\frac{5}{3}) + \frac{32}{3} = \frac{37}{3} \quad (m)$$

The average speed during this time period is: $\frac{37}{3} \div (5-2) = \frac{37}{9}$ (m/s)

Teacher: Well done! When calculating path length and speed, it's crucial to pay special attention to the positive and negative values of the velocity function. Even when the velocity function is negative, we still need to calculate the distance traveled during that time period to obtain the correct result.

老師：這是一題與物理相關的應用問題，其中包含了許多物理量。小心，位移與速度是有方向性的物理量；路徑長與速率是沒有方向性的物理量。我們一起完成這題吧！

學生：位移是比較容易的，我們只要把$v(t) = t^2 - 2t - 3$從$t = 2$到$t = 5$進行積分。

位移是：$s(5) - s(2) = \int_2^5 t^2 - 2t - 3dt = \left[\frac{1}{3}t^3 - t^2 - 3t\right]_2^5 = 9 \quad (m)$

平均速度是：$\frac{s(5) - s(2)}{5-2} = \frac{9}{3} = 3$ (m/s)

老師：正確！那麼路徑長與平均速率呢？

學生：已知$v(t) = t^2 - 2t - 3 = (t-3)(t+1)$，在區間$[2, 3]$中$v(t) \leq 0$；在區間$[3, 5]$中$v(t) \geq 0$，因此此質點移動的路徑長是：

$$\int_2^5 |v(t)|dt = \int_2^3 [-v(t)]dt + \int_3^5 v(t)dt = -(-\frac{5}{3}) + \frac{32}{3} = \frac{37}{3} \quad (m)$$

此時點移動的平均速率是：

$$\frac{37}{3} \div (5-2) = \frac{37}{9} \quad (m/s)$$

老師：非常好，我們在找路徑長與速率的時候，要特別注意速度函數的正負值，當速度函數為負的時候，我們仍要計算在那段時間內所走的距離，我們才可以得到正確的結果。

例題二

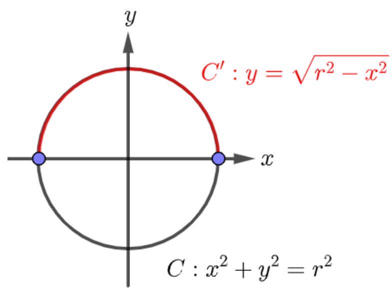
說明：利用圓盤法證明求體體積公式。

（英文）Prove that the volume of a sphere with radius r is $\frac{4}{3}\pi r^3$.

（中文）證明半徑為 r 的球體體積為 $\frac{4}{3}\pi r^3$ 。

Teacher: Now, I am going to take you through the proof of the formula for the volume of a sphere. Do you have any ideas?

Student: We can start by drawing a circle with a radius of r on the coordinate plane. This circle's equation can be $C: x^2+y^2=r^2$.



Teacher: Good, but how do we use the equation and disk method to find the volume in this case?

Student: We can extract the equation of the upper half of the circle, which is:

$C': y=\sqrt{r^2-x^2}$. Then we can rotate this curve along the x-axis to get the volume.

The integral will be:

$$\int_{-r}^{r}\pi\left[\sqrt{r^2-x^2}\right]^2dx=\pi\int_{-r}^{r}(r^2-x^2)dx=\pi\left[r^2x-\frac{1}{3}x^3\right]\Bigg|_{-r}^{r}=\frac{4}{3}\pi r^3$$

Teacher: Great, so the process for this question will be:

(1) Draw a circle centered at the origin with radius of r on the coordinate plane.

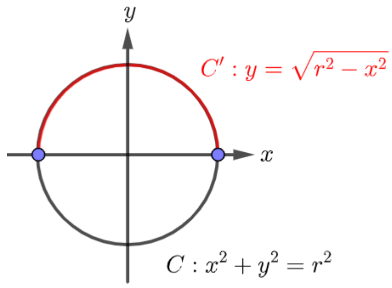
(2) Extract the equation of the upper half of the circle: ($y=\sqrt{r^2-x^2}$)

(3) Apply the disk method to find the volume by integrating this equation over the interval $[-r, r]$.

Then, we can have the volume of a sphere with radius r is: $\frac{4}{3}\pi r^3$

老師： 現在我要帶大家來證明球體的體積公式，大家有甚麼想法嗎？

學生： 首先，我們在坐標平面上畫一個半徑為 r 的圓（如下圖所示），這個圓的方程式為 $C: x^2+y^2=r^2$



老師： 很好，但我們怎麼運用旋轉體的體積(圓盤法)找到體積呢？

學生： 我們可以取出上半圓的方程式，也就是 $C': y=\sqrt{r^2-x^2}$ ，接著我們可以把這個曲線沿著 x 軸做旋轉得到體積。則我們的積分會是：

$$\int_{-r}^{r}\pi\left[\sqrt{r^2-x^2}\right]^2dx=\pi\int_{-r}^{r}(r^2-x^2)dx=\pi\left[r^2x-\frac{1}{3}x^3\right]\Bigg|_{-r}^{r}=\frac{4}{3}\pi r^3.$$

老師： 很棒，所以我們這一題的證明流程包含了：

(1) 在坐標平面上畫出一個以圓點為圓心、半徑為 r 的圓。

(2) 取出這個圓的上半圓方程式。($y=\sqrt{r^2-x^2}$)

(3) 運用旋轉體的體積（圓盤法）在區間 $[-r, r]$ 進行積分。

結合以上步驟，我們可以得證半徑為 r 的球體體積為 $\frac{4}{3}\pi r^3$ 。

國內外參考資源 More to Explore

國家教育研究院樂詞網	
查詢學科詞彙 https://terms.naer.edu.tw/search/	
教育雲：教育媒體影音	
為教育部委辦計畫雙語教學影片 https://video.cloud.edu.tw/video/co_search.php?s=%E9%9B%99%E8%AA%9E	
Oak Teacher Hub	
國外教學及影音資源，除了數學領域還有其他科目 https://teachers.thenational.academy/	
CK-12	
國外教學及影音資源，除了數學領域還有自然領域 https://www.ck12.org/student/	
Twinkl	
國外教學及影音資源，除了數學領域還有其他科目，多為小學及學齡前內容 https://www.twinkl.com.tw/	

Khan Academy
可汗學院，有分年級數學教學影片及問題的討論 https://www.khanacademy.org/
Open Textbook (Math)
國外數學開放式教學資源 http://content.nroc.org/DevelopmentalMath.HTML5/Common/toc/toc_en.html
MATH is FUN
國外教學資源，還有數學相關的小遊戲 https://www.mathsisfun.com/index.htm
PhET: Interactive Simulations
國外教學資源，互動式電腦模擬。除了數學領域，還有自然科 https://phet.colorado.edu/
Eddie Woo YouTube Channel
國外數學教學影音 https://www.youtube.com/c/misterwootube

國立臺灣師範大學數學系陳界山教授網站	
國高中數學雙語教學相關教材 https://math.ntnu.edu.tw/~jschen/index.php?menu=Teaching_Worksheets	
2024 年第五屆科學與科普專業英文(ESP)能力大賽	
科學專業英文相關教材，除了數學領域，還有其他領域 https://sites.google.com/view/ntseccompetition/%E5%B0%88%E6%A5%AD%E8%8B%B1%E6%96%87%E5%AD%B8%E7%BF%92%E8%B3%87%E6%BA%90/%E7%9B%B8%E9%97%9C%E6%95%99%E6%9D%90?authuser=0	
Desmos Classroom	
國外教學資源，也有免費繪圖功能 https://teacher.desmos.com/?lang=en	

高中數學領域雙語教學資源手冊：英語授課用語

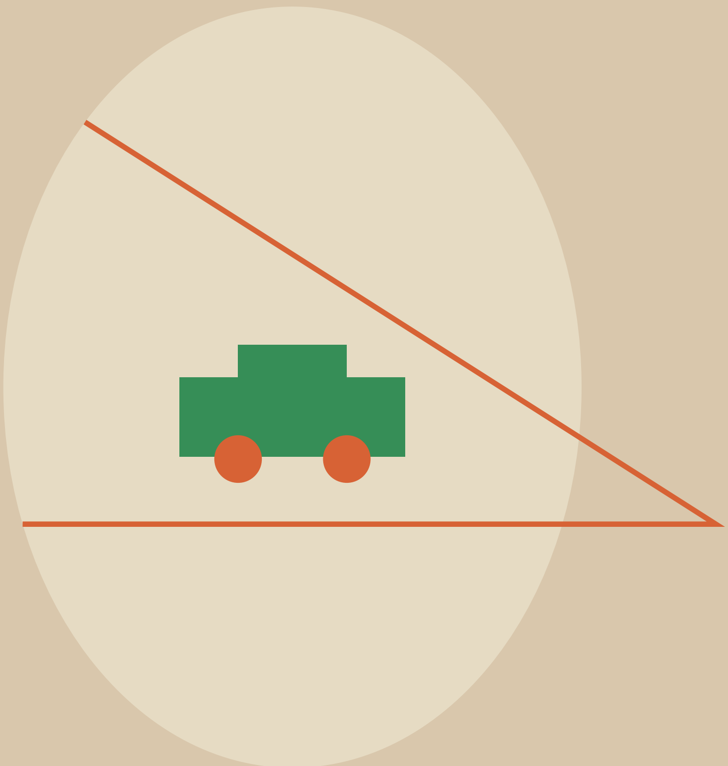
[十二年級上學期]

A Reference Handbook for Senior High School Bilingual Teachers in the Domain of Mathematics: Instructional Language in English

[12^{th} grade 1^{st} semester]

- 研編單位：國立臺灣師範大學雙語教學研究中心
- 指導單位：教育部師資培育及藝術教育司

- 撰稿：鄧宇凱、林佳葦、吳柏萱、蕭煜修
- 學科諮詢：鄭章華
- 語言諮詢：李壹明
- 綜合規劃：王宏均
- 排版：吳依靜
- 封面封底：JUPE Design



發行單位　臺師大雙語教學研究中心
NTNU BILINGUAL EDUCATION RESEARCH CENTER
指導單位　教育部師資培育及藝術教育司
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