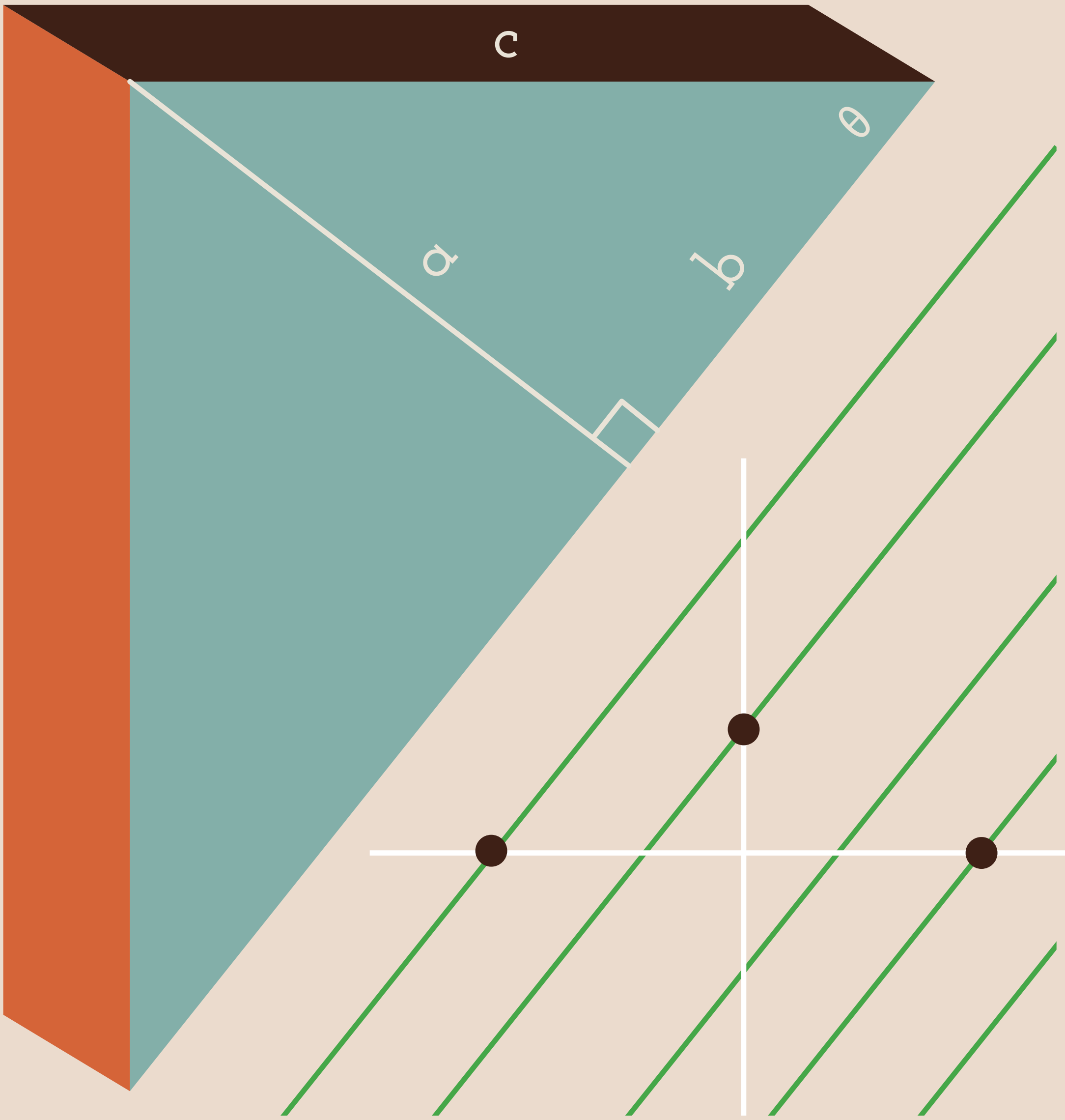
高中數學領域

雙語教學資源手冊 英語授課用語

A Reference Handbook for Senior High School Bilingual Teachers in the Domain of Mathematics: Instructional Language in English

〔高二下學期〕



臺師大雙語教學研究中心 發行

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單元一　空間概念
The Three-Dimensional Coordinate System

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■ 前言 Introduction

本單元介紹空間中的點線面，空間中直線與直線的關係，平面與平面的關係。兩平面相交時會產生兩面角，藉著帶學生利用公式推出兩面角的同時，複習計算機的使用。最後介紹三垂線定理及其應用。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
three-dimensional coordinate system	三維坐標系	octant	卦限
ordered triple	有序三元組	skew lines	歪斜線
distinct	不同的	coincident	重合的
dihedral angle	兩面角	the foot of the perpendicular	垂足
angle between two planes	兩平面夾角	Theorem of three perpendiculars	三垂線定理

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ _____ is perpendicular to _____.

例句：The z-axis **is perpendicular to** both the x-axis and the y-axis.

z 軸同時垂直 x 軸與 y 軸

❷ _____ intersect ______.

例句：These two planes **intersect** in a line.

這兩個平面相交於一直線。

❸ How might _____?

例句：**How might** these two lines be related when they are not coplanar?

當這兩條線不在同一平面，這兩條線可以有哪些關連性？

■ 問題講解 Explanation of Problems

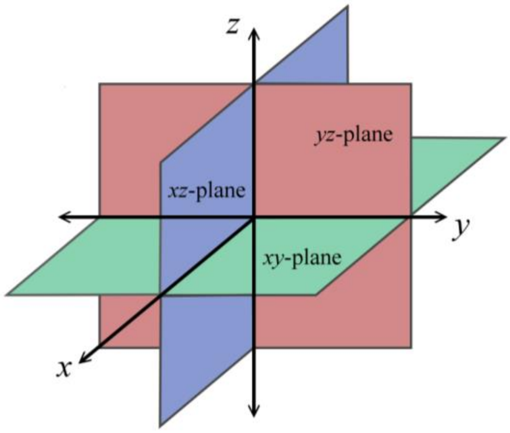
說明

[The three-dimensional coordinate system]

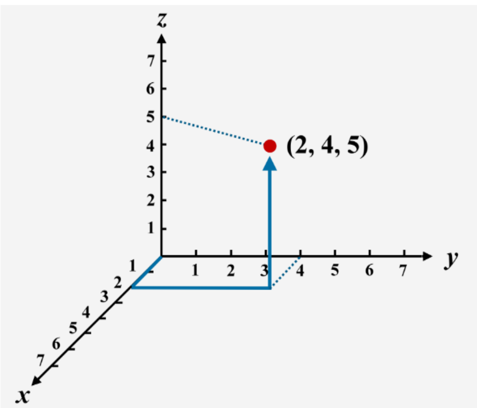
The Cartesian coordinate plane is formed by two perpendicular lines: the x-axis and the y-axis. A point on a plane can be identified by an ordered pair (x, y). Today, we will extend from a plane to a space, from 2D to 3D. A three-dimensional coordinate system can be constructed by three perpendicular lines: x-axis, y-axis and z-axis, while z-axis is perpendicular to both x-axis and y-axis at the origin.

Now open your right hand and make your four fingers point to the right, just as the positive *x*-axis. Next, hold your right hand with fingers curving from the positive *x*-axis toward the positive *y*-axis. Imagine that you are giving people a thumbs-up. The direction that your thumb is pointing is the positive *z*-axis. We say that these three axes form a right-handed coordinate frame.

These three axes determine three planes. The *x*-axis and *y*-axis determine the *xy*-plane, the *x*-axis and *z*-axis determine the *xz*-plane, *y*-axis and *z*-axis determine the *yz*-plane. These three planes separate the space into eight octants; while the *x*-axis and the *y*-axis separate the plane into four quadrants. Here's the sketch.



In the three-dimensional coordinate system, a point can be identified by an ordered triple (x, y, z), while x, y and z are all real numbers. The coordinate of the origin is $(0, 0, 0)$.



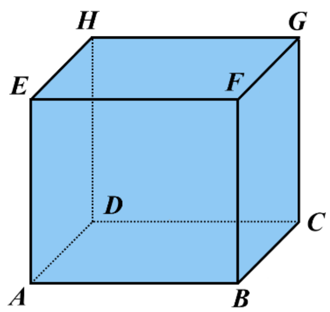
In the above graph, the coordinates of the Point P are $(2, 4, 5)$. The x coordinate, 2, represents the distance from the *yz*-plane to P. The y coordinate, 4, represents the distance from the *xz*-plane to P. The z coordinate, 5, represents the distance from the *xy*-plane to P.

In the two-dimensional coordinate plane, the points on the x-axis have one thing in common: their y-coordinates are all 0. The equation of the x-axis is $y = 0$. Can you guess the equation of the xy-plane with the same logical thinking? The xy-plane is determined by the x-axis and the y-axis. The points on the xy-plane have the form $(x, y, 0)$. Their z-coordinates are all 0. Therefore, the equation of the xy-plane is $z = 0$. The points on the yz-plane have the form $(0, y, z)$. Their x-coordinates are all 0, so the equation of the yz-plane is $x = 0$. The points on the xz-plane have the form $(x, 0, z)$, and the equation of the xz-plane is $y = 0$.

[Lines in space]

Hold up two pens and put them on the desk, pretending that they are coplanar lines. How many possible intersections are there? If two lines are parallel, there are no points of intersection. If two lines intersect, there is one intersection. If two lines are coincident, it means that they appear as a single line, and there are infinitely many intersections.

Now we are going to extend from a plane to a space. Please hold up two pens in front of your face, pretending that they are two lines in a 3D space. How might these two lines be related when they are not coplanar? We may find that there might be one intersection for intersecting lines. There might be infinitely many intersections for two coincident lines. There might be no points of intersection when these lines don't touch each other. If these two lines are coplanar and don't intersect, then we call these two lines parallel lines. If two lines are not coplanar and don't intersect, then we call these two lines skew lines. (The teacher can show skew lines in a 3D space with fingers, presenting the visual image.)



Take the cube as an example, $\overleftrightarrow{EH}$ and $\overleftrightarrow{FG}$ are coplanar and they don't intersect. We say that they are parallel and denote that $\overleftrightarrow{EH} \parallel \overleftrightarrow{FG}$.

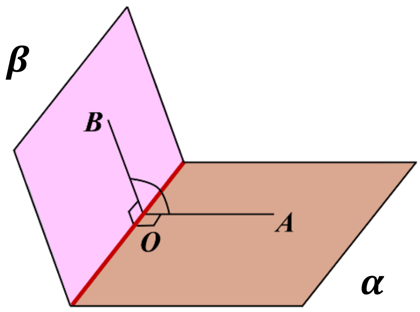
$\overleftrightarrow{EH}$ and $\overleftrightarrow{CD}$ are NOT coplanar and they don't intersect neither, therefore $\overleftrightarrow{EH}$ and $\overleftrightarrow{CD}$ are skew lines.

[Planes in space]

How might two planes be related in terms of possible intersections?

Two distinct planes in three-dimensional space might be parallel, coincident, or intersect in a line.

If two planes E_1 and E_2 are parallel to each other, we denote that $E_1 \| E_2$. If these two planes intersect in a line, we would be interested in the angle between these two planes. How do we measure the angle? Let's sketch the angle first. Mark a point O on the intersecting line. Construct a segment $\overline{OA}$ in plane α perpendicular to the intersecting line, and a segment $\overline{OB}$ in plane β perpendicular to the intersecting line. As shown in the graph below, the measure of $\angle AOB$ is the angle between these two planes, and we call it a *dihedral angle*. The graph looks like a laptop. When opening a laptop, the keyboard is the plane α, and the screen is the plane β. The angle between the keyboard and the screen is $\angle AOB$.

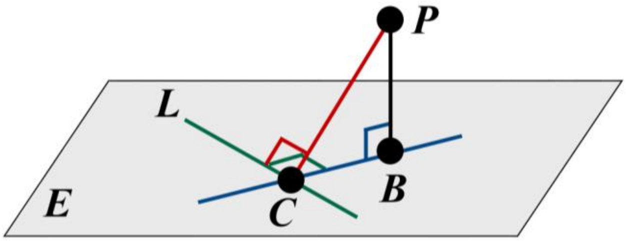


Take a tissue box as another example. The lateral faces are perpendicular to the base. Therefore, the angle between the base and one of the lateral faces is 90°. How do we find the measure of the angle? Connect $\overline{AB}$. Now can you see a triangle $\triangle ABO$? Let's find the length of three sides, and then apply the law of cosine: $cos\angle AOB = \frac{\overline{OA}^2+\overline{OB}^2-\overline{AB}^2}{2\cdot\overline{OA}\cdot\overline{OB}}$. Then we can use the inverse of the cosine function to find out the angle, by using a calculator.

[Theorem of three perpendiculars 三垂線定理]

I want you to pair up with one of your classmates. You two have to work together to place the pens. First, hold a pen vertically, and let it be perpendicular to your desk. Put the second pen on the desk, and make it perpendicular to the first pen. Next, put the third pan on the table, and make it perpendicular to the second pen. Last, take the fourth pen to connect the top of the first pen and the intersecting point of the 2nd and the 3rd pens. Do you think the fourth pen is perpendicular to the third pen? Maybe you need one more person to measure the angle with the protractor!

Yes, they are perpendicular. This is the theorem of three perpendiculars.



As shown in the graph above. $\overline{PB}$ is perpendicular to the plane E. B is the foot of the perpendicular. If $\overleftrightarrow{BC}$ is perpendicular to any line L in the plane E, then $\overline{PC}$ is also perpendicular to line L. Please remember this theorem, and we will solve some related questions later.

運算問題的講解

例題一

說明：

（英文）Which line(s) or plane(s) fit the description?

(a) line(s) parallel to $\overleftrightarrow{EH}$ and containing point F

(b) line(s) skew to $\overleftrightarrow{EH}$ and containing point F

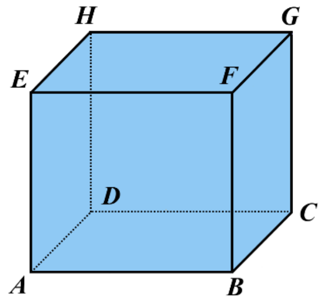
(c) planes(s) parallel to plane $ABCD$ and containing point F.

（中文）哪些線或平面符合描述？

(a) 與 $\overleftrightarrow{EH}$ 平行，且包含點 F 的線

(b) 與 $\overleftrightarrow{EH}$ 歪斜，且包含點 F 的線

(c) 與平面 $ABCD$ 平行，且包含點 F 的面



Teacher: Which lines are parallel to $\overleftrightarrow{EH}$?

Student: I only see segments, not lines.

Teacher: Please think of each segment in the figure as part of a line. When you answer the questions, please use “line”. Now which lines are parallel to $\overleftrightarrow{EH}$, and containing point F?

Student: $\overleftrightarrow{FG}$, $\overleftrightarrow{AD}$, $\overleftrightarrow{BC}$ are parallel to $\overleftrightarrow{EH}$, but only $\overleftrightarrow{FG}$ contains F.

Teacher: Correct. Which lines skew to $\overleftrightarrow{EH}$ and contains point F?

Student: I forgot the definition of “skew lines”.

Teacher: If two lines are not coplanar and don’t intersect, then these two lines are skew lines. Now do you see any lines skew to $\overleftrightarrow{EH}$?

Student: Not coplanar, not intersecting.… The lines in plane $EFGH$ should be excluded.

Student: Is $\overleftrightarrow{AD}$ skew to $\overleftrightarrow{EH}$?

Teacher: $\overleftrightarrow{AD}$ and $\overleftrightarrow{EH}$ are both in the plane $ADHE$. They are parallel lines, but they are not skew lines.

Student: $\overleftrightarrow{FB}$, $\overleftrightarrow{GC}$ $\overleftrightarrow{CD}$, $\overleftrightarrow{AB}$ are skew to $\overleftrightarrow{EH}$. They are not coplanar, and do not intersect with each other.

Teacher: Which one contains point *F*?

Student: $\overleftrightarrow{FB}$.

Teacher: Yes. Which plane is parallel to plane *ABCD* and contains point *F*?

Student: Easy. It's *EFGH*.

Teacher: Correct.

老師： 哪些線與 $\overleftrightarrow{EH}$ 平行？

學生： 我只看到線段，沒有看到線。

老師： 請把圖中每條線段都視為一條直線的一部分。回答問題時，請使用「直線」。現在哪些直線與 $\overleftrightarrow{EH}$ 平行，並且包含點 *F*？

學生： $\overleftrightarrow{FG}$、$\overleftrightarrow{AD}$、$\overleftrightarrow{BC}$ 與 $\overleftrightarrow{EH}$ 平行，但只有 $\overleftrightarrow{FG}$ 包含 *F*。

老師： 正確。

哪些線與 $\overleftrightarrow{EH}$ 歪斜且包含點 *F*？

學生： 我忘了什麼是歪斜。

老師： 如果兩條線不在同一平面上且不相交，則這兩條線是歪斜的線。現在你有看到任何與 $\overleftrightarrow{EH}$ 歪斜的線了嗎？

學生： 不在同一平面、不相交…… 那應該排除在平面 *EFGH* 上的線。

學生： $\overleftrightarrow{AD}$ 與 $\overleftrightarrow{EH}$ 歪斜嗎？

老師： $\overleftrightarrow{AD}$ 和 $\overleftrightarrow{EH}$ 都在平面 *ADHE* 內，是平行線，並不是歪斜線。

學生： $\overleftrightarrow{FB}$、$\overleftrightarrow{GC}$、$\overleftrightarrow{CD}$、$\overleftrightarrow{AB}$ 與 $\overleftrightarrow{EH}$ 歪斜。它們和 $\overleftrightarrow{EH}$ 不在同一平面上，且和 $\overleftrightarrow{EH}$ 不相交。

老師： 哪一條包含點 *F*？

學生： $\overleftrightarrow{FB}$。

老師： 對。哪個平面與平面 *ABCD* 平行且包含點 *F*？

學生： 簡單，就是 *EFGH*。

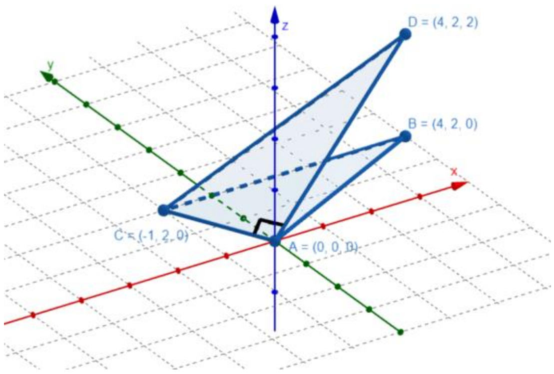
老師： 答對了。

例題二

說明： 利用餘弦定理推得二面角，並教學生使用計算機算出反餘弦，及討論角度之弳度量與度量之值。

（英文）Two right tringles $\triangle ADC$ and $\triangle ABC$ intersects in $\overleftrightarrow{AC}$. It is known that $\overline{DA} \perp \overline{AC}$, and $\overline{BA} \perp \overline{AC}$. Find the dihedral angle between two planes ABC and ADC.

（中文）兩直角三角形 ADC 與 ABC 相交於 $\overleftrightarrow{AC}$。已知$\overline{DA} \perp \overline{AC}$及$\overline{BA} \perp \overline{AC}$。找平面 ABC 與平面 ADC 所夾的兩面角。



Teacher: The right tringles $\triangle ADC$ and $\triangle ABC$ intersect in $\overleftrightarrow{AC}$. $\overline{DA} \perp \overline{AC}$, and $\overline{BA} \perp \overline{AC}$. The angle $\angle DAB$ formed by these two planes is the dihedral angle between two planes. How do we find out the measure of the angle?

Student: You can connect $\overline{DB}$. Draw a triangle first.

Teacher: Good. What's the next step?

Student 1: Measure three sides, and then apply the formula.

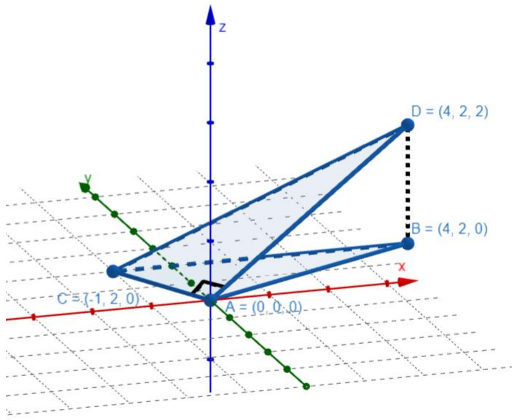
Student 2: Which formula? I forgot.

Teacher: Do you remember the law of cosine? We can find the third side of a triangle when we know the two sides and the angle between them.

The formula is $a^2 + b^2 - 2ab\cos\theta = c^2$, while θ is the angle between sides a and b. We can also write the formula as $\cos\theta = \frac{a^2+b^2-c^2}{2 \cdot ab}$. This formula helps us find the angles of a triangle when we know all three sides. We are looking for $\cos\angle DAB$ in this question.

Student: We can find out the length of three sides $\overline{AB}, \overline{AD}, \overline{BD}$, and apply the formula. But $\overline{AD}$ is in the 3D space, how do we find its length?

Teacher: It is a good question. I would like to bring your attention to $\triangle DBA$. What kind of triangle is it?



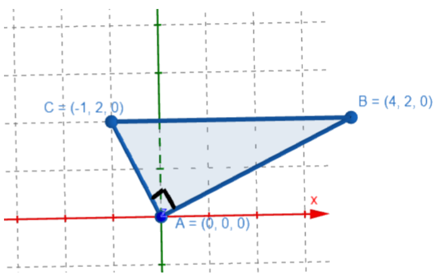
Student: It looks like a right triangle! Point D is 2 units above the point B.

Teacher: Correct. $\overleftrightarrow{DB}$ is perpendicular to the xy-plane. $\triangle DBA$ is a right triangle. We have an easy way to find the ratio of cosine function $\cos\angle DAB$, in a right triangle.

Student: The adjacent side to the hypotenuse.

Teacher: Yes. $\cos\angle DAB = \frac{\overline{AB}}{\overline{AD}}$. We just need to find out $\overline{AB}, \overline{AD}$.

Let's focus on ΔABC. Since the z coordinates of three points are 0, $\triangle ABC$ is in the xy-plane. If we look at the triangle from the top, here is the sketch. If we only consider the coordinates in the xy-plane, the coordinates of B will be (4,2) and the coordinates of A will be (0, 0). What is the length of $\overline{AB}$?



Student: I can use the distance formula, $\overline{AB} = \sqrt{4^2 + 2^2} = \sqrt{20}$

Teacher: Good. $\overline{AD}$ is the hypotenuse of the right triangle $\triangle DBA$. Point D is 2 units above the point B, implying that $\overline{DB} = 2$.

Therefore, the hypotenuse $\overline{AD} = \sqrt{20 + 2^2} = \sqrt{24}$.

You have everything you need now, please find out the value of $\cos\angle DAB$.

Student: Oh my god. I need a calculator.

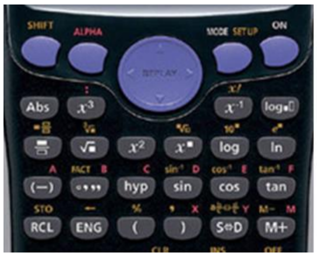
Teacher: Indeed. Please take out your calculator and calculate $cos\angle DAB = \frac{\overline{AB}}{\overline{AD}} = \frac{\sqrt{20}}{\sqrt{24}}$.

Student: I got 0.912871.

Teacher: $cos\angle DAB = 0.912871$. Therefore, $\angle DAB = cos^{-1}(0.912871)$. Find the inverse cosine button "cos^{-1}" in your calculator.

Student: Where is it? I only see "cos".

Teacher: Perhaps your calculators are different, and I use the Casio calculator as an example. Do you see "cos"? You may see "cos^{-1}" above the button "cos", which is written in yellow. Do you see "shift" in the upper left corner? Press "shift" and then press "cos", you can see the inverse cosine "cos^{-1}" on the screen.

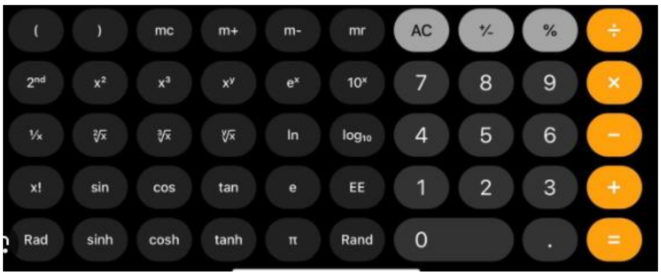


Student: I don't have a calculator. May I use the calculator on my iPhone?

Teacher: Ok, but only for the calculator. No games.

Student: I only see "cos" on my iPhone. I don't see any "cos^{-1}".

Teacher: Do you see the "2nd" button on the first column? Hit the "2nd" button, and you will see the "cos^{-1}" button. Now, do you find "$cos^{-1}(0.912871)$"?



Student 1: I got 24.0948, about 24.1.

Student 2: But I got 0.420534, about 0.42.

Teacher: Well, which answer is more reasonable?

Student: The angle in the graph looks like 24.1 degrees. It is not 0.42 degrees.

Teacher: Your estimation is right. It is 24.1 degrees. That's because the setting in your calculator is "radian", it gives you 0.42 radian as the answer.
Change your calculator's setting from "Rad" to "Deg", and then you will have 24.1 degrees. Besides this method, you can simply convert 0.42 radian to its degree.

Student: $0.42 \times \frac{180}{\pi} = 24.06$. I think I got it!

Teacher: Great!

老師： $\triangle ADC$ 和 $\triangle ABC$ 相交於$\overleftrightarrow{AC}$。已知$\overline{DA} \perp \overline{AC}$, $\overline{BA} \perp \overline{AC}$. $\angle DAB$ 是由這兩個平面形成的角度，也是二面角。。我們如何找出這個角度？

學生： 你可以連接 $\overline{DB}$。先畫一個三角形。

老師： 好。下一步是什麼？

學生 1： 測量三個邊，然後應用公式。

學生 2： 哪個公式？我忘了。

老師： 你還記得餘弦定理嗎？當我們知道兩邊和它們之間的角度時，我們可以找到三角形的第三邊。
公式是 $a^2 + b^2 - 2ab\cos\theta = c^2$，其中 θ 是兩邊 a 和 b 之間的角度。我們也可以將公式寫成 $\cos\theta = \frac{a^2+b^2-c^2}{2 \cdot ab}$。這樣有助於我們在知道三邊的情況下找出三角形的角度。這題，我們要找的是 $\cos\angle DAB$.

學生： 我們找到$\overline{AB}, \overline{AD}, \overline{BD}$的長，就可以帶入公式。但是$\overline{AD}$ 在 3D 空間中，要怎麼找到它的長度？

老師： 這是一個好問題。我想帶你們看一下ΔDBA，這是甚麼樣的三角形？
看起來是個直角三角形。D 在 B 的上方距離 2 單位長。
對，$\overleftrightarrow{DB}$垂直於 XY 平面，ΔDBA是直角三角形，我們有個簡單的方法可以找出直角三角形裡面的$\cos\angle DAB$

學生： 臨邊對斜邊。

老師： 對，$\cos\angle DAB = \frac{\overline{AB}}{\overline{AD}}$，我們只需要找 $\overline{AB}, \overline{AD}$. 先看$\triangle ABC$，三點坐標中的 Z 坐標都是 0，$\triangle ABC$ 在 xy 平面上。如果我們從頂端往下看，圖形如下。如果我們只考慮 xy 平面上的坐標，B 就是(4,2) ，A 是 (0,0)，那麼 $\overline{AB}$ 為何？

學生：我可以用距離公式 $\overline{AB}=\sqrt{4^2+2^2}=\sqrt{20}$

老師：好。$\overline{AD}$ 是直角三角形ΔDBA的斜邊，D 在 B 的上方距離 2 單位長，所以$\overline{DB}=$ 2.，因此斜邊$\overline{AD}=\sqrt{20+2^2}=\sqrt{24}$。你現在有所有需要的東西了，請算出 $\cos\angle DAB$ 的值。

學生：天啊。我需要用計算機。

老師：確實。請拿出你的計算機並計算 $cos\angle DAB=\frac{\overline{AB}}{\overline{AD}}=\frac{\sqrt{20}}{\sqrt{24}}$。

學生：我得到了 0.912871。

老師：$cos\angle DAB=0.912871$，因此 $\angle DAB=cos^{-1}(0.912871)$。
在你的計算機中找到反餘弦按鈕 "cos^{-1}"。

學生：它在哪裡？我只看到 "cos"。

老師：或許你的計算機不同，我以卡西歐計算機作為例子。你看到 "cos"嗎？
你可能會看到 "cos" 按鈕上方黃色的 "cos^{-1}" 字樣。有看到左上角有 "shift" 嗎？按 "shift"，然後再按 "cos"，你就會在螢幕上看到反餘弦 "cos^{-1}"。

學生：我沒有計算機。我可以用 iPhone 的計算機嗎？

老師：可以，但只能用來計算。不要玩遊戲。

學生：我在 iPhone 上只看到 "cos"，沒看到 "cos^{-1}"。

老師：你在第一列看到 "2nd" 按鈕了嗎？按 "2nd"，你就會看到 "cos^{-1}"。
現在，算出 "$cos^{-1}(0.912871)$"

學生 1：我得到了 24.0948, 大約是 24.1.。

學生 2：但我得到了 0.420534, 大約是 0.42.

老師：那麼，哪個答案更合理？

學生：圖中的角度看起來像是 24.1 度。不是 0.42 度。

老師：你的推測是對的，是 24.1 度。因為你的計算機設置為"弧度"，它給出了 0.42 弳作為答案。
將你的計算機設置 "Rad" 改為 "Deg"，然後你就會得到 24.1 度。
除此之外，你還可以將 0.42 弳簡單地轉換為度數。

學生：$0.42\times\frac{180}{\pi}=24.1$。我想我明白了！

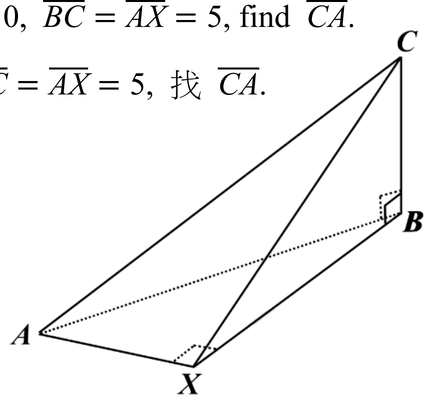
老師：太好了！

例題三

說明：利用三垂線定理找出未知線段長

（英文）It is given that $\overline{CB} \perp \overline{BX}, \overline{BX} \perp \overline{AX}$. If $\overline{BX} = 10,\ \overline{BC} = \overline{AX} = 5$, find $\overline{CA}$.

（中文）已知 $\overline{CB} \perp \overline{BX}$、$\overline{BX} \perp \overline{AX}$，若 $\overline{BX} = 10,\ \overline{BC} = \overline{AX} = 5$, 找 $\overline{CA}$.



Teacher: It is given that $\overline{CB} \perp \overline{BX}, \overline{BX} \perp \overline{AX}$, can you tell the relationship between $\overline{CX}$ and $\overline{AX}$?

Student: Perpendicular?

Teacher: Yes, they are perpendicular, according to the theorem of three perpendiculars. In order to find $\overline{CA}$, we need to know the length of $\overline{CX}$ and $\overline{AX}$ first. Then we can apply the Pythagorean theorem.

Student: $\overline{AX} = 5$ is given. But $\overline{CX}$…. I don't know.

Teacher: Do you see a standing triangle $\triangle CBX$? What kind of triangle is it?

Student: A right triangle.

Teacher: Yes, $\overline{CX}$ is the hypotenuse of this right triangle. Can you apply the Pythagorean theorem to find out its length?

Student: $\sqrt{5^2 + 10^2} = \sqrt{125}$.

Teacher: $\triangle AXC$ is also a right triangle. Now you have two legs, $\sqrt{125}$ and 5. What is the hypotenuse?

Student: $\sqrt{5^2 + 125} = \sqrt{150}$.

Teacher: Write the answer in the simplest form.

Student: $5\sqrt{6}$.

老師： 已知 $\overline{CB} \perp \overline{BX}$、$\overline{BX} \perp \overline{AX}$，你能告訴我 $\overline{CX}$ 和 $\overline{AX}$ 的關係嗎？

學生： 垂直嗎？

老師： 是的，根據三垂線定理，它們是垂直的。為了找出 $\underline{CA}$ 的長度，我們首先需要知道 $\overline{CX}$ 和 $\overline{AX}$ 的長度，然後我們可以應用畢氏定理。

學生： 給定 $\overline{AX} = 5$。但是 $\overline{CX}$... 我不知道。

老師： 你看到一個直立的三角形 CBX 了嗎？這是什麼樣的三角形？

學生： 是直角三角形。

老師： 是的，$\overline{CX}$ 是這個直角三角形的斜邊。你能應用畢氏定理找出它的長度嗎？

學生： $\sqrt{5^2 + 10^2} = \sqrt{125}$。

老師： 三角形 AXC 也是一個直角三角形。現在你有兩條邊，$\sqrt{125}$ 和 5。那麼斜邊是多少？

學生： $\sqrt{5^2 + 125} = \sqrt{150}$。

老師： 以最簡形式寫出答案。

學生： $5\sqrt{6}$。

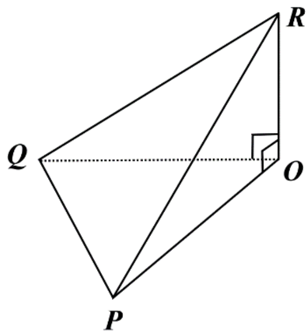
應用問題 / 學測指考題

例題一

說明：利用兩面角和三垂線定理證明兩三角形的面積比。

（英文）$\overline{RO}$ is perpendicular to the plane OPQ. Let θ be the angle between the plane RPQ and OPQ. Show that $\frac{area\ of\ \triangle OPQ}{area\ of\ \triangle RPQ} = cos\theta$

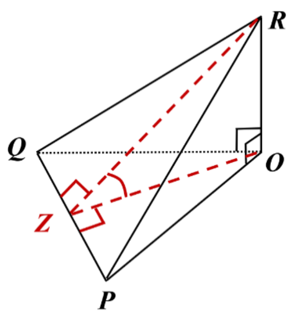
（中文）$\overline{RO}$ 垂直於平面 OPQ. 已知 θ 是兩平面 RPQ 和 OPQ 的夾角，證明兩三角形的面積比為 $\frac{\triangle OPQ}{\triangle RPQ} = cos\theta$.



（HKCEE A Math 2003 #18）

Teacher: $\overline{RO}$ is perpendicular to the plane OPQ, therefore $\overline{RO}$ is perpendicular to the segments in the planes, such as $\overline{PO}$ and $\overline{QO}$. You can see the right angle signs in the graph.

Let's construct $\overline{OZ}$ perpendicular to $\overline{PQ}$. Is $\overline{RO}$ perpendicular to $\overline{OZ}$? Why or why not?



Student: $\overline{RO}$ is perpendicular to the segments in the plane OPQ. Yes, they are perpendicular.

Teacher: Correct. Next, connect R and Z. Let me ask you a question. $\overline{OZ}$ perpendicular to $\overline{PQ}$, and $\overline{RO}$ is perpendicular to $\overline{OZ}$. According to the theorem of three perpendiculars, how are $\overline{RZ}$ and $\overline{PQ}$ related?

Student: They are perpendicular.

Teacher: Good. Do you see the right triangle $\triangle ROZ$?

Student: Yes.

Teacher: What is the measure of $\angle RZO$?

Student: It is not given. I don't know.

Teacher: $\angle RZO$ is the angle between the planes OPQ and RPQ. Let's assume $\angle RZO = \theta$. This question is asking for the ratio of two triangles. What is the area of $\triangle OPQ$? What is the area of $\triangle RPQ$? Can you answer them in terms of segments?

Student: $\triangle OPQ = \frac{1}{2} \cdot \overline{PQ} \cdot \overline{OZ}$; $\triangle RPQ = \frac{1}{2} \cdot \overline{PQ} \cdot \overline{RZ}$.

Teacher: Hence $\frac{\text{area of } \triangle OPQ}{\text{area of } \triangle RPQ} = \frac{\frac{1}{2} \cdot \overline{PQ} \cdot \overline{OZ}}{\frac{1}{2} \cdot \overline{PQ} \cdot \overline{RZ}} = \frac{\overline{OZ}}{\overline{RZ}}$. The ratio of the areas is $\frac{\overline{OZ}}{\overline{RZ}}$.

How is it related to $cos\theta$?

Student: Trigonometry! The adjacent side over the hypotenuse. It is $cos\vartheta$.

Teacher: Yes, $\cos\theta = \frac{\overline{OZ}}{\overline{RZ}}$. Therefore, $\frac{\text{area of } \triangle OPQ}{\text{area of } \triangle RPQ} = \cos\theta$.

老師：$\overline{RO}$ 垂直於平面 OPQ，因此 $\overline{RO}$ 垂直於平面內的線段，例如 $\overline{PO}$ 和 $\overline{QO}$。你可以在圖中看到直角標誌。
我們作線段 $\overline{OZ}$ 使其與 $\overline{PQ}$ 垂直。$\overline{RO}$ 垂直於 $\overline{OZ}$ 嗎？為什麼？

學生：因為 $\overline{RO}$ 垂直於平面 OPQ 內的線段，所以是的，$\overline{RO}$ 垂直於 $\overline{OZ}$。

老師：正確。接下來，連接 R 和 Z。我問你們一個問題。
$\overline{OZ}$ 垂直於 $\overline{PQ}$，而 $\overline{RO}$ 垂直於 $\overline{OZ}$。根據三垂線定理，$\overline{RZ}$ 和 $\overline{PQ}$ 有什麼關係？

學生：它們是垂直的。

老師：很好。你看到一個直角三角形 ROZ 嗎？

學生：是的。

老師：$\angle RZO$ 的度數是多少？

學生：　題目沒給，我不知道。

老師：　$\angle RZO$ 是平面 OPQ 和 RPQ 之間的角度。讓我們假設 $\angle RZO = \theta$。這個問題要求兩個三角形面積的比率。三角形 OPQ 的面積是多少？三角形 RPQ 的面積呢？你能用線段來回答嗎？

學生：　$\triangle OPQ = \frac{1}{2} \cdot \overline{PQ} \cdot \overline{OZ}$；$\triangle RPQ = \frac{1}{2} \cdot \overline{PQ} \cdot \overline{RZ}$。

老師：　現在代入 $\frac{\triangle OPQ\text{ 的面積}}{\triangle RPQ\text{ 的面積}} = \frac{\frac{1}{2}\overline{PQ}\cdot\overline{OZ}}{\frac{1}{2}\overline{PQ}\cdot\overline{RZ}} = \frac{\overline{OZ}}{\overline{RZ}}$。面積的比率是 $\frac{\overline{OZ}}{\overline{RZ}}$。

它跟 cosθ 有什麼關係？

學生：　三角函數！鄰邊除以斜邊，是 cosθ。

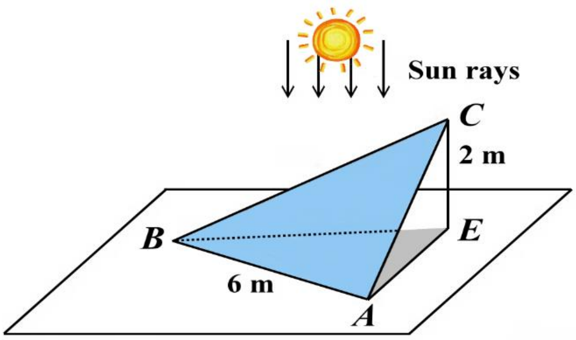
老師：　是的，$\cos\theta = \frac{\overline{OZ}}{\overline{RZ}}$。因此，$\frac{\triangle OPQ\text{的面積}}{\triangle RPQ\text{的面積}} = \cos\theta$。

例題二

說明：利用三垂線定理解決生活中的數學問題。

（英文）Ken set up a tent. He erected a 2-m-pole vertically at point E on the horizontal ground. He elevated a triangular tent ABC of area 12 m^2 by the pole such that side AB touches the ground and vertex C is fastened to the top of the pole. $\overline{AB} = 6\text{m}$. At noon, the sun's rays vertically cast a shadow of the tent on the ground.

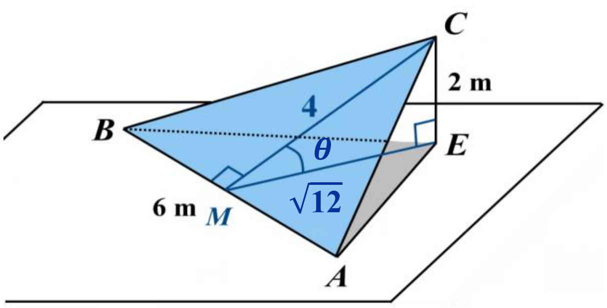
Find the area of the shadow.



（中文）Ken 在搭一個帳篷。他在 E 點垂直豎立一根 2 公尺高的桿子，架高了一個面積為 12 m^2 的三角形帳篷，讓 AB 邊碰到地面，點 C 繫在杆子的頂端。AB 邊長為 6 m，中午太陽光垂直照射影子在地面上，求影子面積。

（revised from HKCEE A Math 2003 #18）

Teacher: First, let's draw $\overline{CM}$, the altitude of $\triangle ABC$. $\overline{CM}$ is perpendicular to $\overline{AB}$. The area of $\triangle ABC$ is 12, and the base $\overline{AB}$ is 6. Can you find the length of the altitude, $\overline{CM}$?



Student: 4.

Teacher: Ok. Connect E and M. Is $\overline{EM}$ perpendicular to $\overline{AB}$? What do you think?

Student: You taught us the theorem of three perpendiculars. But it is a little bit different.

I know that if $\overline{CE}$ is perpendicular to $\overline{EM}$, and $\overline{EM}$ is perpendicular to $\overline{AB}$, then $\overline{CM}$ is perpendicular to $\overline{AB}$.

Teacher: You are right. Listen up. If $\overline{CE}$ is perpendicular to $\overline{EM}$, $\overline{CM}$ is perpendicular to $\overline{AB}$, then $\overline{EM}$ is perpendicular to $\overline{AB}$. This is also the theorem of three perpendiculars.

This means if we find out the length of $\overline{EM}$, then we can get the area of $\triangle ABE$, with $\overline{AB}$ as the base.

Student: Ok.

Teacher: Do you see the right triangle $\triangle CEM$? How do you get $\overline{EM}$?

Student: Pythagorean theorem. It is $\sqrt{12} = 2\sqrt{3}$.

Teacher: Correct. Find the area.

Student: $2\sqrt{3} \times 6 \times \frac{1}{2} = 6\sqrt{3}$.

Teacher: Correct.

老師： 首先，我們畫出 $\overline{CM}$，這是 $\triangle ABC$ 的高，$\overline{CM}$ 垂直於 $\overline{AB}$。$\triangle ABC$ 的面積是 12，底邊 $\overline{AB}$ 為 6。你能找出高 $\overline{CM}$ 的長度嗎？

學生： 4

老師： 好的。連接點 E 和 M。你覺得 $\overline{EM}$ 垂直於 $\overline{AB}$ 嗎？

學生： 你教過我們三垂線定理，但這裡好像有點不同。我知道如果 $\overline{CM}$ 垂直於 $\overline{EM}$，而 $\overline{EM}$ 垂直於 $\overline{AB}$，那麼 $\overline{CM}$ 就垂直於 $\overline{AB}$。

老師： 沒錯。如果 $\overline{CE}$ 垂直於 $\overline{EM}$，$\overline{CM}$ 垂直於 $\overline{AB}$，那麼 $\overline{EM}$ 就垂直於 $\overline{AB}$。這也是三垂線定理！

這表示如果我們找到 $\overline{EM}$ 的長度，我們就可以得到以 $\overline{AB}$ 為底的 $\triangle ABE$ 的面積。

學生： 了解。

老師： 你看到一個直角三角形 $\triangle CEM$ 嗎？如何求 $\overline{EM}$？

學生： 用畢氏定理。是 $\sqrt{12} = 2\sqrt{3}$。

老師： 正確，那麼接著算出面積。

學生： $2\sqrt{3} \times 6 \times \frac{1}{2} = 6\sqrt{3}$

老師： 答對了。

單元二　空間向量的坐標表示式
Vectors in Space-Representations and Operations

國立新竹科學園區實驗高級中等學校　吳珮蓁老師

■ 前言 Introduction

本單元介紹介紹空間坐標系中兩點的距離及中點的坐標表示法，空間向量的定義及表示法，空間向量的加法、減法、係數積等運算及其在坐標平面上的幾何意義，最後介紹空間向量線性組合，及分點公式。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
directed line segment	有向線段	initial point	起點
terminal point	終點	magnitude	量

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ ____ units in the _____ direction.

例句：Plot a point 2 **units in the** negative x **direction**.

在負 x 的方向上，$x=-2$ 的地方畫一個點。

❷ _____ as a linear combination of ______.

例句：Write the vector $\vec{a}$ **as a linear combination of** $\vec{b}$ and $\vec{c}$.

將向量 $\vec{a}$ 表示為 $\vec{b}$ 和 $\vec{c}$ 的線性組合。

❸ the relationship between ____ and ____.

例句：**The relationship between** $\overrightarrow{AC}$ **and** $\overrightarrow{AB}$ is $\overrightarrow{AC}=-2\overrightarrow{AB}$.

向量 $\overrightarrow{AC}$ 和 $\overrightarrow{AB}$ 的關係式為 $\overrightarrow{AC}=-2\overrightarrow{AB}$。

❹ ____ divides ____ into ______, and the ratio of _____ is _____.

例句：The point P **divides** $\overline{AB}$ **into** two line segments, **and the ratio of** two segments **is** $2:3$.

點 P 把 $\overline{AB}$ 分為兩線段，線段長為 2:3。

■ 問題講解 Explanation of Problems

☙ 說明 ❧

[Point in space]

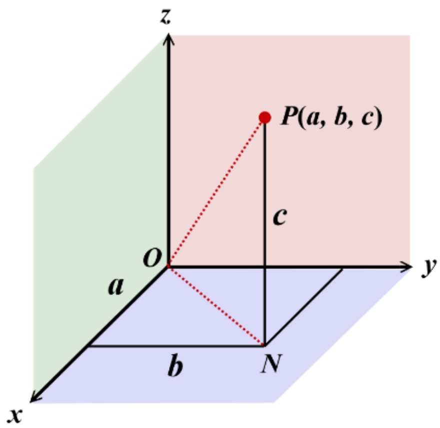
In the last section, we introduced the three-dimensional coordinate system. A point P in space can be determined by an ordered triple (a, b, c), while a, b and c are all real numbers.

The x coordinate, a, represents the directed distance from the yz-plane to P.

The y coordinate, b, represents the directed distance from the xz-plane to P.

The z coordinate, c, represents the directed distance from the xy-plane to P.

The origin is (0, 0, 0). How do we find out the distance between two points in space?



Pretend that a light is right above point P, its shadow is located on point $N(a, b, 0)$. $\overline{ON}$ is the hypotenuse of a right triangle, with two legs a and b. We can use the Pythagorean theorem to get the distance of $\overline{ON}$. $\overline{ON} = \sqrt{a^2 + b^2}$.

$\overline{PN}$ is perpendicular to the xy-plane, so $\overline{PN}$ is perpendicular to $\overline{ON}$. The point P is c unit vertically above the point N, so $\overline{PN} = c$.

$\triangle OPN$ is a right triangle, so $\overline{OP} = \sqrt{\overline{ON}^2 + \overline{NP}^2} = \sqrt{a^2 + b^2 + c^2}$.

The distance formula established for the two-dimensional coordinate system can be extended to three dimensions. In a plane, the distance between two points (x_1, y_1), (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. In space, the distance the distance between two points (x_1, y_1, z_1), (x_2, y_2, z_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$. The midpoint of (x_1, y_1, z_1), (x_2, y_2, z_2) is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$.

[Definitions of vectors]

In the past, we learned about the vectors in a plane. Let's review what we've learned.

"Vector" is a quantity with magnitude and direction, and we usually use a directed line segment to represent a vector. If the vector $\vec{u}$ has the initial point at the origin $O(0,0)$, and the terminal point is $P(x,y)$, then the vector is $\vec{u}=\overrightarrow{OP}$, representing the direction from O to P. The component form of the vector $\vec{u}$ is represented by $\vec{u}=(x-0,y-0)=(x,y)$.

The first number is the *x* component of $\vec{u}$, which is the change in *x*. The second number is the y component of $\vec{u}$, which is the change in *y*. The "magnitude" of the vector $|\vec{u}|=\sqrt{x^2+y^2}$ is to represent the length of the vector.

Now we extend this concept to represent a vector in 3D space using *x*, *y*, and *z* axes.

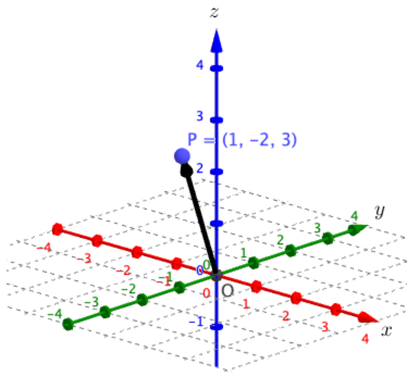
The vector $\vec{u}$ has the initial point at the origin $O(0,0,0)$, and the terminal point is $P(1,-2,3)$, then the vector $\vec{u}=(1-0,\ -2-0,3-0)=(1,-2,3)$.

It is the component form of a vector $\vec{u}$.

The first number, 1, is the *x* component of $\vec{u}$, which is the change in *x*.

The second number, -2, is the *y* component of $\vec{u}$, which is the change in *y*.

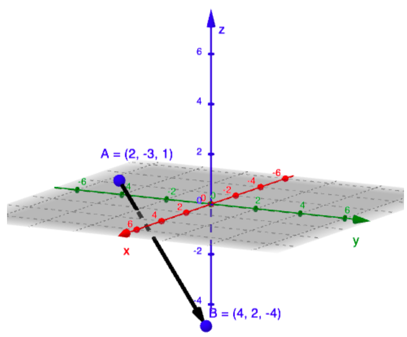
The last number, 3, is the *z* component of $\vec{u}$, which is the change in *z*.



The "magnitude" of the vector, meaning "length", is $|\vec{u}|=\sqrt{1^2+(-2)^2+3^2}=\sqrt{14}$.

What if the initial point is not the origin? In the following graph, the initial point is $A(2,\ -3,\ 1)$ and the terminal point is $B(4,\ 2,\ -4)$. These two points are separated by the *xy*-plane. Point *A* is above the plane, and *B* is beneath the plane. I draw the vector from *A* to *B*, and the vector is $(4-2,\ 2-(-3),\ -4-1)=(2,\ 5,\ -5)$.

2 is the change in *x*, 5 is the change in *y*, and -5 is the change in *z*. The negative number means the negative direction.

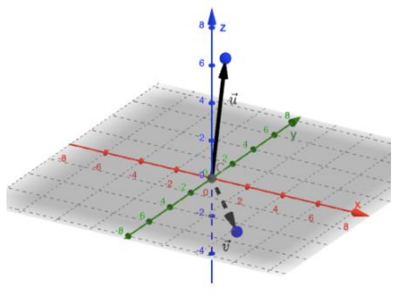
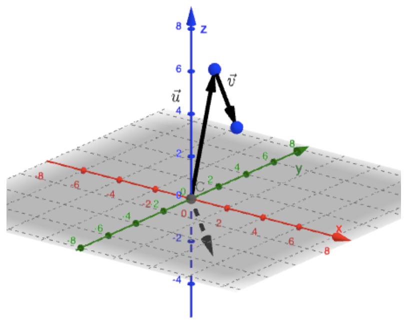
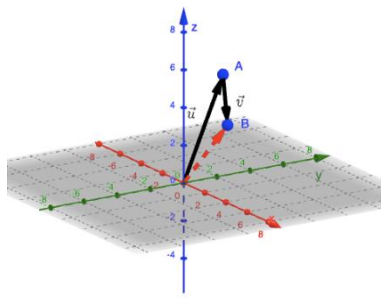


In space, the initial point is $P(p_1, p_2, p_3)$ and the terminal point is $Q(q_1, q_2, q_3)$.
The component form of the vector is given by $\overrightarrow{PQ} = (q_1 - p_1, q_2 - p_2, q_3 - p_3)$.
The magnitude of the vector, which is the length of the vector, can be figured out by the distance formula. The magnitude is $|\overrightarrow{PQ}| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2 + (q_3 - p_3)^2}$.

[Operations of vectors in space: addition and subtraction]

Some properties of vectors can be extended to three dimensions. Let's start with addition and subtraction.
Assuming that $\vec{u} = (-1, 3, 5)$ and $\vec{v} = (2, -1, -2)$, what would $\vec{u} + \vec{v}$ be?
To take the sum of two vectors, we add up their *x* components to get the new *x* component: $-1 + 2 = 1$. We add up their *y* components to get the new *y* component: $3 + (-1) = 2$. Then add up the *z* components to get a new *z* component: $5 + (-2) = 3$. The resulting vector is $(1, 2, 3)$. I don't think that simple math calculations would bother you here. Let's check the geometric meaning of adding vectors in space.

Let's draw $\vec{u} = (-1, 3, 5)$ and $\vec{v} = (2, -1, -2)$ in standard position, as shown in graph (1).
We are going to add $\vec{v}$ to $\vec{u}$, and we have to shift vector $\vec{v}$ over so that its initial point starts at the terminal point of vector $\vec{u}$'s terminal point, as shown in graph (2).
In graph (3), the resulting vector $(1, 2, 3)$, in dashed line, describes the movement from the initial point of $\vec{u}$ to the terminal point of $\vec{v}$: move 1 unit toward the positive *x* direction, move 2 units toward the positive *y* direction, and move 3 units up toward the positive *z* direction.
Following the direction of arrowheads, the addition can be expressed as $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$.
Point *A* is the connecting point.
Do you remember the parallelogram when we learned the addition of vectors in planes? We also have a parallelogram here in three-dimensional space. $\vec{u} + \vec{v}$ is regarded as the diagonal of a parallelogram formed by $\vec{u}$ and $\vec{v}$.

(1)	(2)	(3)
		

Assuming that $\vec{u}=(-1,3,5)$ and $\vec{v}=(2,-1,-2)$, what would $\vec{u}-\vec{v}$ be? The expression $\vec{u}-\vec{v}$ can written as $\vec{u}+(-\vec{v})$. The vector $-\vec{v}$ has the same magnitude as $\vec{v}$, but in the opposite direction. $-\vec{v}=(-2,1,2)$. To add $-\vec{v}$ to $\vec{u}$, we shift vector $-\vec{v}$ over so that its initial point starts at the terminal point of vector $\vec{u}$'s terminal point.

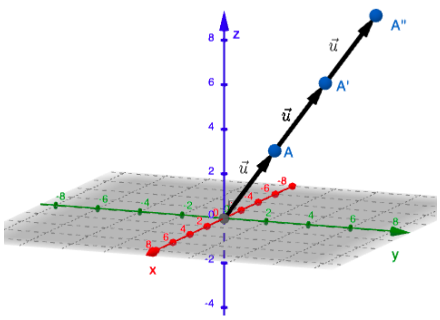
We add the x components, y components, and z components correspondingly, and we will have $\vec{u}+(-\vec{v})=(-1,3,5)+(-2,1,2)=(-3,4,7)$.

[Scalar multiplication: Scale a vector]

The algebraic property, the scalar multiplication, can also be extended to three dimensions.

The scale multiplication can change its magnitude, or change its direction.

Let's draw a vector $\vec{u}=(-1,2,3)$ in the standard position. Put the initial point at the origin, and the terminal point at $(-1,2,3)$. Multiply the vector by 3, and 3 is the scalar. We extend the line, and translate the point A to A' and A'', while $\overline{OA}=\overline{AA'}=\overline{A'A''}$.



We multiply each component by 3, and we will have

$$3\vec{u}=(-3,6,9)=\overrightarrow{OA''}$$

We can tell that the direction doesn't change, but the magnitude changes. It has been tripled in size.

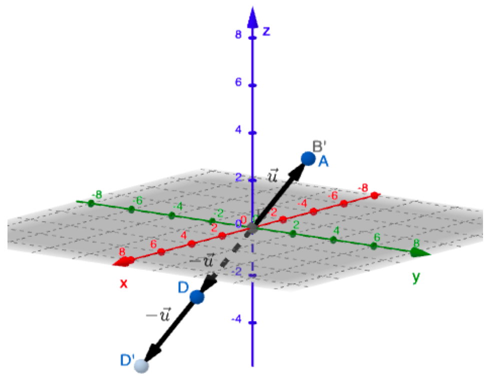
Let's try another scalar multiplication. Multiply the vector by -2.

We multiply each component by -2, and we will have $-2\vec{u} = (2, -4, -6)$.

Please draw the vector in the standard position and observe the relative position between $\vec{u}$ and $-2\vec{u}$

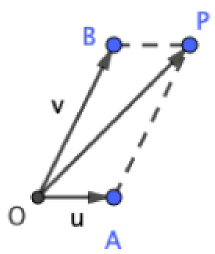
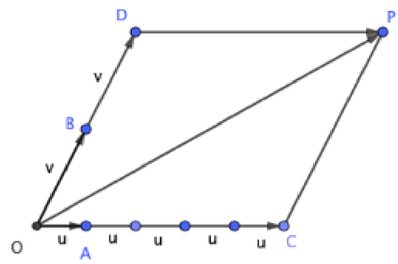
The direction has been flipped by 180 degrees, and it is in the opposite direction. They still sit on the same line.

The magnitude of the new vector is increased by a factor of 2. This is twice the magnitude of the original vector.



[Linear combination of vectors and section formula]

Three non-colinear points, O, A, and B, can determine a plane, so any linear combination of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ will be located on the plane. For example, (see two graphs below)

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB}$	$\overrightarrow{OP} = 5\overrightarrow{OA} + 2\overrightarrow{OB}$
	

$\overrightarrow{OP}$ is expressed in a linear function of $\overrightarrow{OA}$ and $\overrightarrow{OB}$, and it is called the linear combination.

$\overrightarrow{OP} = x\overrightarrow{OA} + y\overrightarrow{OB}$, while $\overrightarrow{OA}$, $\overrightarrow{OB}$ are non-zero vectors, unparalleled, coplanar vectors, and x, y are scalars.

This concept can be extended in a three-dimensional system. Let's work on an example.

A point P $(10, -12, -22)$ is located in the plane determined by three points $O(0, 0, 0)$, $A(1, 0, -3)$, and $B(-2, 3, 4)$ and $\overrightarrow{OP} = x\overrightarrow{OA} + y\overrightarrow{OB}$. How do we find the value of x and y?

First, find the vectors. $\overrightarrow{OA} = (1, 0, -3)$ and $\overrightarrow{OB} = (-2, 3, 4)$.

$\overrightarrow{OP}$ can be expressed $x\overrightarrow{OA} + y\overrightarrow{OB}$.

We get $\overrightarrow{OP} = x(1, 0, -3) + y(-2, 3, 4) = (x - 2y, 3y, -3x + 4y)$.

$\overrightarrow{OP} = (10, -12, -22)$ so we can have the following equations by comparing the components:

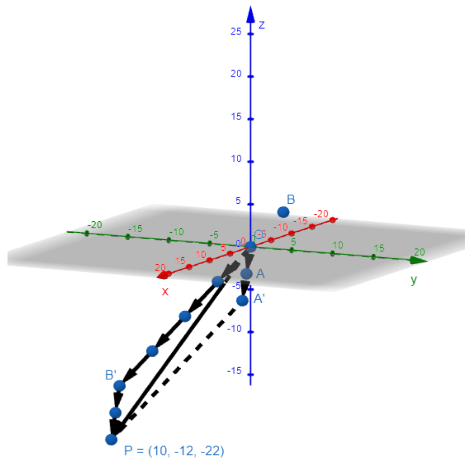
$$\begin{cases} x - 2y = 10 \\ 3y = -12 \\ -3x + 4y = -22 \end{cases}.$$

Solve the equations, and we will get $x = 2,\ y = -4$.

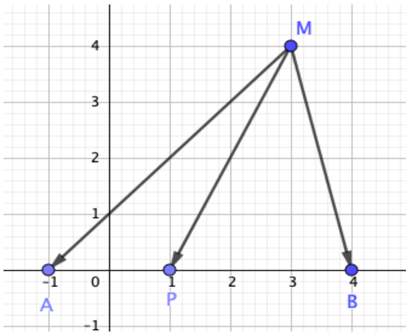
You can see the graph.

The plane is formed by $2\overrightarrow{OA}$ and $-4\overrightarrow{OB}$,

and $\overrightarrow{OP} = (10, -12, -22)$.



In the graph, the point P divides $\overline{AB}$ internally into two segments, and the ratio of two segments is $2:3$. Point M can be generalized to any point on the plane. The position vector of P can be expressed by $\overrightarrow{MP} = \frac{3}{5}\overrightarrow{MA} + \frac{2}{5}\overrightarrow{MB}$. If the point P divides internally the segment joining A and B in the ratio of $x:y$, then the position vector of P is $\overrightarrow{MP} = \frac{y}{x+y}\overrightarrow{MA} + \frac{x}{x+y}\overrightarrow{MB}$.

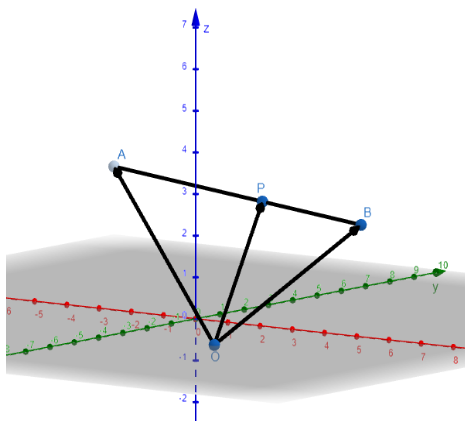


This section formula can be extended to three dimensions.

If the point P divides internally the segment joining A and B in the ratio of $m{:}n$, then the position vector of P is $\overrightarrow{OP} = \overrightarrow{OA} + \frac{m}{m+n}\overrightarrow{OB}$.

Assuming that $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$,

the coordinates of P are $\left(\frac{nx_1+mx_2}{m+n}, \frac{ny_1+my_2}{m+n}, \frac{nz_1+mz_2}{m+n}\right)$



運算問題的講解

例題一

說明：在空間坐標中描繪點

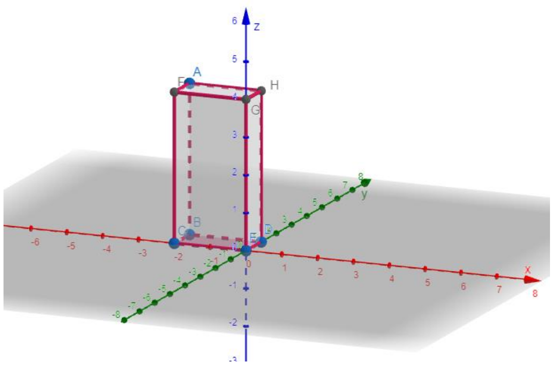
（英文）Sketch a point $(-2, 1, 4)$ in three-dimensional space.

（中文）在 3D 空間中描繪點 $(-2, 1, 4)$.

Teacher: First sketch a three-dimensional coordinate system. Make sure that the z axis is perpendicular to both x axis and y axis. Then construct a prism.

Student: I don't know how.

Teacher: Sketch three sides of a rectangular prism along the coordinate axes. 2 units in the negative x direction, 1 unit in the positive y direction, and 4 units in the positive z direction. The sketch is as shown below.

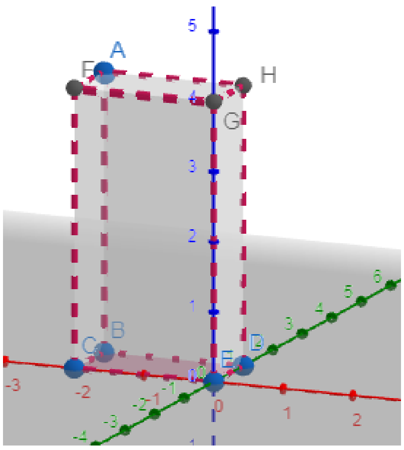


Student: But I just need a point, not a prism.

Teacher: A prism can help you identify the position of the point in the three-dimensional system.

When you construct the prism, the vertex $A\ (-2, 1, 4)$ is the point!

You can change the sides of the prism into dashed lines.

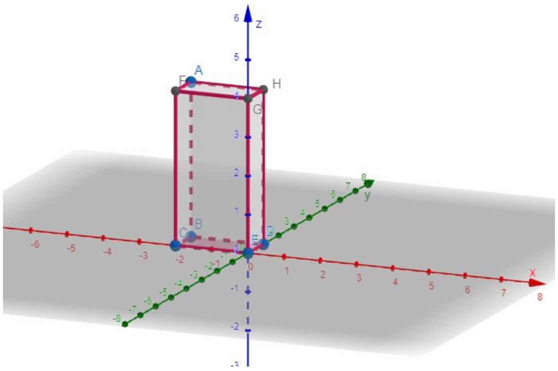


Student: Ok. I get it.

老師： 首先畫出一個三維坐標系統，確認 z 軸垂直於 x 軸和 y 軸。然後畫出一個角柱。

學生： 我不知道該怎麼做。

老師： 沿著坐標軸畫出一個長方體的三面。往負 x 方向移動 2 單位，往正 y 方向移動 1 單位，往正 z 方向移動 4 單位。如下：



學生： 但我只需要一個點，不需要一個長方體耶。

老師： 角柱可以幫助你在三維坐標中確定點的位置。當你畫出角柱時，頂點 $A(-2, 1, 4)$ 就是題目要求的點！你可以把角柱的邊改成虛線。

學生： 好的，我明白了。

例題二

說明：以坐標表示法表示向量減法及其長度，及練習係數積。

（英文）Assuming that $\vec{u}=(4,-1,2)$ and $\vec{v}=(0,-1,3)$, find the following vectors.

(1) $\vec{u}-2\vec{v}$　　(2) $|\vec{u}-2\vec{v}|$　　(3) $|2\vec{u}-4\vec{v}|$

（中文）已知 $\vec{u}=(4,-1,2)$，$\vec{v}=(0,-1,3)$，找出下列向量：

(1) $\vec{u}-2\vec{v}$　　(2) $|\vec{u}-2\vec{v}|$　　(3) $|2\vec{u}-4\vec{v}|$

Teacher: $\vec{u}-2\vec{v}$ can be regarded as $\vec{u}+(-2\vec{v})$. You can use the scalar multiplication: multiply each component of $\vec{v}$ by -2.

Student: $-2\vec{v}=(0,2,-6)$

Teacher: Good. You remember to change the sign when multiplying a negative number. Next, add $\vec{u}$ and $-2\vec{v}$.

Student: $\vec{u}+(-2\vec{v})=(4,-1,2)+(0,2,-6)=(4,1,-4)$

Teacher: Good. Next, how do you find $|\vec{u}-2\vec{v}|$, the magnitude?

Student: Square root!

Teacher: Can you complete it?

Student: $\sqrt{4^2+1+(-4)^2}=\sqrt{33}$

Teacher: Next, what is the magnitude of $|2\vec{u}-4\vec{v}|$?

Student: I know. Use the scalar multiplication. Multiply the $\vec{u}$ by 2, and multiply $\vec{v}$ by -4. Use addition and the magnitude formula.

Teacher: Very good. Please do so and share your answer.

Student: $2\vec{u}+(-4\vec{v})=(8,-2,4)+(0,4,-12)=(8,2,-8)$.

The magnitude is $\sqrt{64+4+64}=\sqrt{132}$

Teacher: Please simplify the square root.

Student: $2\sqrt{33}$.

Teacher: Good. Have you noticed any relationship between $2\sqrt{33}$ and the previous question?

Student: $2\sqrt{33}$. equals $\sqrt{33}$ times 2.

Teacher: You can take out the common factor 2. $|2\vec{u}-4\vec{v}|=2|\vec{u}-2\vec{v}|$. When the vector is doubled, the magnitude is also doubled.

Student: I see. I can just multiply $\sqrt{33}$ by 2.

Teacher: Correct.

老師： $\vec{u}-2\vec{v}$ 可以視為 $\vec{u}+(-2\vec{v})$。你可以使用純量乘法：將 $\vec{v}$ 的每個分量乘上 -2。

學生： $-2\vec{v}=(0,2,-6)$

老師： 很好。乘以負數時要記得變號。接下來，將 $\vec{u}$ 和 $-2\vec{v}$ 相加。

學生： $\vec{u}+(-2\vec{v})=(4,-1,2)+(0,2,-6)=(4,1,-4)$

老師： 很好。接下來，如何求 $|\vec{u}-2\vec{v}|$，算出它的大小？

學生： 開根號！

老師： 怎麼做呢？

學生： $\sqrt{4^2+1+(-4)^2}=\sqrt{33}$

老師： 接下來，$|2\vec{u}-4\vec{v}|$ 的向量大小是？

學生： 我知道了。使用純量乘法，將 $\vec{u}$ 乘以 2，將 $\vec{v}$ 乘以 -4。然後相加。

老師： 非常好，做完分享你的答案。

學生： $2\vec{u}+(-4\vec{v})=(8,-2,4)+(0,4,-12)=(8,2,-8)$。

向量是 $\sqrt{64+4+64}=\sqrt{132}$

老師： 請化簡。

學生： $2\sqrt{33}$。

老師： 很好。你有沒有注意到 $2\sqrt{33}$ 和前一個問題之間的關係？

學生： $2\sqrt{33}$ 等於 $\sqrt{33}$ 乘以 2。

老師： 你可以提出公因數 2，$|2\vec{u}-4\vec{v}|=2|\vec{u}-2\vec{v}|$。當向量加倍時，大小也加倍。

學生： 我明白了，其實只需要將 $\sqrt{33}$ 乘以 2。

老師： 沒錯。

例題三

說明：利用向量性質，判斷空間中的點是否共線。

（英文）$A(5,-2,3)$, $B(0,4,4)$, $C(-5,10,5)$, $D(0,6,7)$ are four points in the space.

(1) Find the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$.

(2) Determine whether A, B and C are colinear or not, and explain your answer.

(3) Determine whether A, B and D are colinear or not, and explain your answer.

（中文）$A(5,-2,3)$、$B(0,4,4)$、$C(-5,10,5)$、$D(0,6,7)$ 為空間中四點。

(1) 求向量 $\overrightarrow{AB}$、$\overrightarrow{AC}$、$\overrightarrow{AD}$

(2) 判斷 A, B, C 三點是否共線。

(3) 判斷 A, B, D 三點是否共線。

Teacher: How do you find the vector $\overrightarrow{AB}$?

Student: The terminal point minus the initial point.

Teacher: Which one is the initial point? Which one is the terminal point?

Student: $\overrightarrow{AB}$ The vector starts from A and goes to B. A is the initial point and B is the terminal point.

Teacher: Correct. $\overrightarrow{AB}=(0-5, 4+2, 4-3)=(-5, 6, 1)$. Please find the other two vectors.

Student: $\overrightarrow{AC}=(-5-5, 10+2, 5\text{-}3)=(-10, 12, 2)$.

Student: $\overrightarrow{AD}=(-0-5, 6+2, 7-3)=(-5, 8, 4)$.

Teacher: To check whether *A*, *B*, and *C* are colinear or not, we can check whether the vector $\overrightarrow{AB}$ equals $t\cdot\overrightarrow{AC}$ or not, while *t* is a constant. If $\overrightarrow{AB}=t\cdot\overrightarrow{AC}$, then *A*, *B*, and *C* are colinear. Do you see this relationship between $\overrightarrow{AB}$ and $\overrightarrow{AC}$?

Student: $\overrightarrow{AB}=\frac{1}{2}\overrightarrow{AC}$. Can *t* be a fraction?

Teacher: Yes. $\frac{1}{2}$ is a constant. Do you think *A*, *B*, and *C* are colinear?

Student: Yes.

Teacher: I would like to ask one more question. *A* is the initial point, where are *B* and *C*? Can you sketch a draft?

Student: $\overrightarrow{AB}$ is half of $\overrightarrow{AC}$. *B* is in the middle of *A* and *C*.

Teacher: Great. *B* and *C* are on the same side of *A*.

If *t* is a negative number, such as $\overrightarrow{AB}=-\frac{1}{2}\overrightarrow{AC}$, where are *B* and *C*?

Student: Opposite side?

Teacher: Correct. $\overrightarrow{AB}$ and $\overrightarrow{AC}$ go in different directions. *A* is between *B* and *C*, and they are still colinear, as long as $\overrightarrow{AB}=t\cdot\overrightarrow{AC}$.

Student: I see.

Teacher: Next, are *A*, *B*, and *D* colinear? Please work on your own.

Student: $\overrightarrow{AB}=(-5, 6, 1)$ and $\overrightarrow{AD}=(-5, 8, 4)$. I can't find the constant.

Teacher: There is no such *t* which makes $\overrightarrow{AB}=t\cdot\overrightarrow{AD}$. Therefore, *A*, *B*, and *D* are not colinear.

老師：如何求向量 $\overrightarrow{AB}$？

學生：終點減起點。

老師：哪個是起點？哪個是終點呢？

學生：$\overrightarrow{AB}$ 向量是從 *A* 出發到 *B*。*A* 是起點，*B* 是終點。

老師：正確，$\overrightarrow{AB}=(0-5, 4+2, 4-3)=(-5, 6, 1)$。請算出另外兩個向量。

學生：$\overrightarrow{AC}=(-5-5, 10+2, 5\text{-}3)=(-10, 12, 2)$.

學生：$\overrightarrow{AD}=(-0-5, 6+2, 7-3)=(-5, 8, 4)$.

老師：要檢查 *A*、*B* 和 *C* 是否共線，我們可以檢查向量 $\overrightarrow{AB}$ 是否等於 $t\cdot\overrightarrow{AC}$，其中 *t* 是一個常數。如果 $\overrightarrow{AB}=t\cdot\overrightarrow{AC}$，那麼 *A*、*B* 和 *C* 就是共線的。有觀察到 $\overrightarrow{AB}$ 和 $\overrightarrow{AC}$ 之間有這種關係嗎？

學生：　$\overrightarrow{AB} = \frac{1}{2}\overrightarrow{AC}$。$t$ 可以是分數嗎？

老師：　可以。$\frac{1}{2}$ 是一個常數。你認為 A、B 和 C 是否共線呢？

學生：　是的。

老師：　我想再問一個問題。A 是起點，B 和 C 會在哪裡？能畫出大概在哪裡嗎？

學生：　$\overrightarrow{AB}$ 是 $\overrightarrow{AC}$ 的一半，所以 B 會在 A 和 C 的中間。

老師：　很好。B 和 C 都跟 A 在同一側。

如果 t 是一個負數，比如 $\overrightarrow{AB} = -\frac{1}{2}\overrightarrow{AC}$，那麼 B 和 C 在哪裡？

學生：　不同側嗎？

老師：　沒錯。$\overrightarrow{AB}$ 和 $\overrightarrow{AC}$ 為不同的方向。只要 $\overrightarrow{AB} = t\cdot\overrightarrow{AC}$，$A$ 在 B 和 C 之間，它們仍然三點共線。

學生：　我明白了。

老師：　接下來，A、B 和 D 三點是否共線？請自己想想看。

學生：　$\overrightarrow{AB} = (-5, 6, 1)$ 和 $\overrightarrow{AD} = (-5, 8, 4)$。我找不到常數。

老師：　沒有 t 使 $\overrightarrow{AB} = t\cdot\overrightarrow{AD}$。因此，$A$、$B$ 和 D 三點不共線。

應用問題 / 學測指考題

例題一

說明：練習空間中兩點的距離公式，並利用向量解決問題。

（英文）Find the radius of the sphere with the center $O(3, 0, -2)$ containing the point $P(5, 2, -1)$. Find the other endpoint of the diameter which starts at $P(5, 2, -1)$.

（中文）一圓圓心 $O(3, 0, -2)$ 通過點 $P(5, 2, -1)$，求此圓之半徑，及以 $P(5, 2, -1)$ 為直徑起始點的另一端點的坐標。

Teacher: The radius of the sphere is the distance between the center and the point on the sphere. Find the distance between $O(3, 0, -2)$ and $P(5, 2, -1)$.

Student: $\sqrt{(3-5)^2 + (0-2)^2 + (-2+1)^2} = \sqrt{9} = 3$

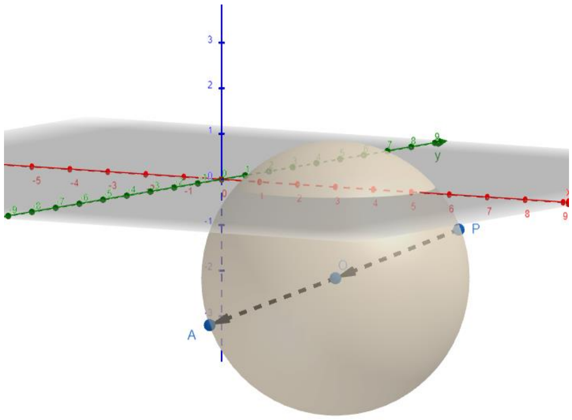
Teacher: Correct. The radius is 3. Next, where is the other endpoint of the diameter?

Student: I know that the diameter is 6. Does this information help?

Teacher: How about using the “vector”? Let me sketch the sphere and the points.

Connect the points P and O, extend the radius and meet the other endpoint of the diameter A.

What is the vector $\overrightarrow{PO}$?



Student: $\overrightarrow{PO} = (3-5, 0-2, -2+1) = (-2, -2, -1)$

Teacher: Good. What is $\overrightarrow{OA}$?

Student: Looks the same as $\overrightarrow{PO}$.

Teacher: P, O, and A are colinear, and $\overline{PO} = \overline{OA}$. Hence, the vectors are the same.

$\overrightarrow{OA} = \overrightarrow{PO} = (-2, -2, -1)$. Can you find the coordinate of A? You can assume the coordinate of A is (x, y, z).

Student: $\overrightarrow{OA} = (x-3, y-0, z+2) = (-2, -2, -1)$, so $x = 1, y = -2, z = -3$.

Teacher: Correct. $A\ (1, -2, -3)$

老師： 球的半徑是從中心到球上一點的距離。求 $O(3, 0, -2)$ 和 $P(5, 2, -1)$ 之間的距離。

學生： $\sqrt{(3-5)^2 + (0-2)^2 + (-2+1)^2} = \sqrt{9} = 3$

老師： 正確。半徑是 3。接下來，直徑的另一端點在哪裡？

學生： 我知道直徑是 6。這個資訊有幫助嗎？

老師： 那試試使用「向量」吧。我先畫出球和點，連接點 P 和 O，延伸半徑並找到直徑的另一端點 A。

$PO^{\rightharpoonup}$ 向量是多少？

學生： $\overrightarrow{PO} = (3-5, 0-2, -2+1) = (-2, -2, -1)$

老師： 很好。$\overrightarrow{OA}$ 是什麼？

學生： 看起來和 $\overrightarrow{PO}$ 一樣。

老師： P、O、A 三點共線，且 $\overline{PO}=\overline{OA}$。

因此，向量一樣。$\overrightarrow{OA}=\overrightarrow{PO}=(-2,-2,-1)$。你能找到點 A 的坐標嗎？

可以假設 A 的坐標是 (x,y,z)。

學生： $\overrightarrow{OA}=(x-3,y-0,z+2)=(-2,-2,-1)$，所以 $x=1$、$y=-2$、2, $z=-3$。

老師： 正確。$A\,(1,-2,-3)$

例題二

說明：利用向量的線性組合，討論向量的最小值及相對應的 k 值。

（英文）$\vec{a}=(2,-3,1)$, $\vec{b}=(1,2,0)$. Let $\vec{v}=\vec{a}+k\vec{b}$, find the minimum of $|\vec{v}|$.

（中文）$\vec{a}=(2,-3,1)$、$\vec{b}=(1,2,0)$，令 $\vec{v}=\vec{a}+k\vec{b}$，求 $|\vec{v}|$ 的最小值。

Teacher: $\vec{v}=\vec{a}+k\vec{b}$, $\vec{v}$ is expressed as a linear combination of $\vec{a}$ and $\vec{b}$. Can you write $\vec{v}$ in terms of k?

Student: $\vec{v}=\vec{a}+k\vec{b}=(2,-3,1)+k(1,2,0)=(2+k,-3+2k,1)$

Teacher: What does $|\vec{v}|$ mean?

Student: The magnitude of the vector. $|\vec{v}|=\sqrt{(2+k)^2+(-3+2k)^2+1^2}$

Teacher: Correct. Please expand the expression and simplify it. I will give you some time.

Student: $|\vec{v}|=\sqrt{5k^2-8k+14}$

Teacher: What is the minimum of $|\vec{v}|$?

Student: Very small.

Teacher: Can $|\vec{v}|$ be a negative number?

Student: Maybe.

Teacher: A square root of something should be a positive number or 0, if we are mentioning real numbers.

Student: I see. The minimum is 0.

Teacher: Don't make the conclusion so quickly. I need to complete the square. Do you remember the steps? Take out the common factor for the first two terms….

$$5k^2 - 8k + 14 = 5\left(k^2 - \frac{8}{5}k\right) + 14.$$

Which number do you add in the parenthesis to make it a square?

Student: Is it the half of $\frac{8}{5}$?

Teacher: It should be $\left(\frac{4}{5}\right)^2$. You add $\frac{16}{25}$ and you should take away $\frac{16}{25}$ at the same time, in order to keep the original expression the same. You will get:

$$5k^2 - 8k + 14 = 5\left(k^2 - \frac{8}{5}k + \left(\frac{4}{5}\right)^2\right) - 5\left(\frac{4}{5}\right)^2 + 14$$

Student: The process looks familiar.

Teacher: Yes, because we learned it before. The expression can be simplified as

$$5k^2 - 8k + 14 = 5\left(k - \frac{4}{5}\right)^2 - 5\left(\frac{4}{5}\right)^2 + 14 = 5\left(k - \frac{4}{5}\right)^2 + \frac{54}{5}.$$

$5\left(k - \frac{4}{5}\right)^2 + \frac{54}{5}$ is greater than or equal to $\frac{54}{5}$.

Student: Why? I forgot the reason.

Teacher: $\left(k - \frac{4}{5}\right)^2$ has to be a positive number or 0. After you multiply it by 5, it will be a bigger positive number or 0. We say $5\left(k - \frac{4}{5}\right)^2 \geq 0$.

You add $\frac{54}{5}$ to the both sides of the inequality, and you will get

$$5\left(k - \frac{4}{5}\right)^2 + \frac{54}{5} \geq 0 + \frac{54}{5}.$$

Student: The minimum is $\frac{54}{5}$.

Teacher: You are almost there. We were discussing the minimum of $5k^2 - 8k + 14$. However, the question is asking for the minimum of $\sqrt{5k^2 - 8k + 14}$. What should we do?

Student: Add a square root! $\sqrt{\frac{54}{5}}$ is the minimum.

Teacher: Good. When $|\vec{v}|$ reaches the minimum, what is the value of k?

Student: When $5\left(k-\frac{4}{5}\right)^2=0$, $|\vec{v}|$ has the minimum.

Teacher: Correct. When $k=\frac{4}{5}$, $|\vec{v}|$ has the minimum.

老師： $\vec{v}=\vec{a}+k\vec{b}$，$\vec{v}$ 可以表示為 $\vec{a}$ 和 $\vec{b}$ 的線性組合。你能用 k 來表示 $\vec{v}$ 嗎？

學生： $\vec{v}=\vec{a}+k\vec{b}=(2,-3,1)+k(1,2,0)=(2+k,-3+2k,1)$

老師： $|\vec{v}|$ 代表什麼意思？

學生： 向量的大小。$|\vec{v}|=\sqrt{(2+k)^2+(-3+2k)^2+1^2}$。

老師： 正確，請展開並簡化。我會給你一些時間。

學生： $|\vec{v}|=\sqrt{5k^2-8k+14}$

老師： $|\vec{v}|$ 的最小值是多少？

學生： 非常小。

老師： $|\vec{v}|$ 可以是負數嗎？

學生： 也許。

老師： 對於實數來說，平方根應該是一個正數，或者是 0。

學生： 我懂了。最小值是 0。

老師： 別急著下結論。我們需要用配方法。你還記得步驟嗎？將前兩項提出共同因子......

$5k^2-8k+14=5\left(k^2-\frac{8}{5}k\right)+14$。要加入什麼數字在括號內使其成為一個完全平方？

學生： 是 $\frac{8}{5}$ 的一半嗎？

老師： 應該是 $\left(\frac{4}{5}\right)^2$。加上 $\frac{16}{25}$，同時減去 $\frac{16}{25}$ 這樣才能保持原來的表達式。你會得到：

$5k^2-8k+14=5\left(k^2-\frac{8}{5}k+\left(\frac{4}{5}\right)^2\right)-5\left(\frac{4}{5}\right)^2+14$。

學生： 這個過程看起來很熟悉。

老師： 是的，因為我們之前學過。這個表達式可以簡化為

$5k^2-8k+14=5\left(k-\frac{4}{5}\right)^2-5\left(\frac{4}{5}\right)^2+14=5\left(k-\frac{4}{5}\right)^2+\frac{54}{5}$。

$5\left(k-\frac{4}{5}\right)^2+\frac{54}{5}$ 大於或等於 $\frac{54}{5}$。

學生：為什麼？我忘了原因。

老師：$\left(k-\frac{4}{5}\right)^2$ 必須是一個正數或 0。當你將它乘以 5 時，它將成為一個更大的正數或 0。我們說 $5\left(k-\frac{4}{5}\right)^2 \geq 0$。將 $\frac{54}{5}$ 加到不等式的兩邊，會得到

$$5\left(k-\frac{4}{5}\right)^2+\frac{54}{5} \geq 0+\frac{54}{5}$$

學生：最小值是 $\frac{54}{5}$。

老師：差不多要算完成了。我們討論的是 $5k^2-8k+14$ 的最小值，不過題目問的是 $\sqrt{5k^2-8k+14}$ 的最小值。我們該怎麼做？

學生：加上平方根！最小值 $\sqrt{\frac{54}{5}}$。

老師：很好！ $|\vec{v}|$ 為最小值時，k 是多少？

學生：當 $5\left(k-\frac{4}{5}\right)^2=0$ 時，$|\vec{v}|$ 有最小值。

老師：正確。當 $k=\frac{4}{5}$ 時，$|\vec{v}|$ 有最小值。

單元三　空間向量的內積
Inner Product of Vectors in Space

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■　前言 Introduction

本單元的內容首先討論空間向量的內積及其運算性質，接下來運用內積求兩向量的夾角及計算向量的正射影，最後討論柯西不等式及其應用等相關問題。因學生已經學過平面向量的內積及運算性質等相關內容，建議老師可以先複習平面向量的內積、正射影、柯西不等式等相關英文詞彙及內容，再討論本單元空間向量的相關內容。最後以應用問題或學測題來加深學生們對本單元的了解。

■　詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
inner product (dot product)	**內積**	**projection**	**正射影**
space	空間	**orthogonal**	**正交的**
Cauchy inequality	**柯西不等式**	tetrahedron	四面體（三角錐）

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ If_____ and _____ are orthogonal, ________.

例句：**If** two vectors $\vec{u}$ **and** $\vec{v}$ **are orthogonal**, then their inner product is 0.

當兩向量 $\vec{u}$ 和 $\vec{v}$ 為正交時，其內積為零。

❷ A/an ________ is known as _______.

例句：**A** tetrahedron **is also known as** a triangular pyramid which is formed by four triangular faces.

四面體也可以稱為三角錐，它的四個面均為三角形。

■ 問題講解 Explanation of Problems

ღ 說明 ჲ

When discussing plane vectors in grade 10, we learned the contents of the inner product of vectors, the angle between vectors, the vector projection, and the Cauchy-Schwarz inequality. These contents still apply in space.

Teachers can guide students to review the contents of plane vectors, and then extend these concepts/formulas to space vectors. For example, the formula of the inner product in plane vectors is $\vec{x}\cdot\vec{y}=x_1y_1+x_2y_2$, where $\vec{x}=(x_1,x_2)$ and $\vec{y}=(y_1,y_2)$. In space, the inner product is $\vec{x}\cdot\vec{y}=x_1y_1+x_2y_2+x_3y_3$, where $\vec{x}=(x_1,x_2,x_3)$ and $\vec{y}=(y_1,y_2,y_3)$.

Similarly, we can find the angle between vectors is

$cos\theta=\frac{\vec{x}\cdot\vec{y}}{|\vec{x}||\vec{y}|}=\frac{x_1y_1+x_2y_2+x_3y_3}{\sqrt{x_1^2+x_2^2+x_3^2}\sqrt{y_1^2+y_2^2+y_3^2}}$ and the vector projection is $proj_v u=\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}$ in space. Finally, the Cauchy-Schwarz inequality space is

$|x_1y_1+x_2y_2+x_3y_3|\leq\sqrt{x_1^2+x_2^2+x_3^2}\cdot\sqrt{y_1^2+y_2^2+y_3^2}$ $(|\vec{x}\cdot\vec{y}|\leq|\vec{x}||\vec{y}|)$.

運算問題的講解

例題一

說明： 本題為求空間向量的內積及運用內積求兩空間向量的夾角。

（英文）In space, there are two different vectors $\overrightarrow{PQ}=(1,-2,-1)$ and $\overrightarrow{PR}=(1,1,2)$. Find

(1) $\overrightarrow{PQ}\cdot\overrightarrow{PR}$　　(2) $\angle QPR$

（中文）已知空間中兩向量 $\overrightarrow{PQ}=(1,-2,-1)$、$\overrightarrow{PR}=(1,1,2)$，試求：

(1) $\overrightarrow{PQ}\cdot\overrightarrow{PR}$　　(2) $\angle QPR$

Teacher: The formula for the inner product in space is similar to the inner product in the plane. So, we have
$\vec{x}\cdot\vec{y}=x_1y_1+x_2y_2+x_3y_3$, where $\vec{x}=(x_1,x_2,x_3)$ and $\vec{y}=(y_1,y_2,y_3)$.
What is the value of $\overrightarrow{PQ}\cdot\overrightarrow{PR}$?

Student: -3.

Teacher: Yes, the inner product (or dot product) is -3.
Also, we can use the inner product to find the angle between two nonzero vectors.
$\vec{x}\cdot\vec{y}=|\vec{x}||\vec{y}|cos\theta$, where θ is the angle between $\vec{x}$ and $\vec{y}$.
So, $cos\theta=\frac{\vec{x}\cdot\vec{y}}{|\vec{x}||\vec{y}|}$.
Find cos $\angle QPR$.

Student: $\cos\angle QPR=-\frac{1}{2}$.

Teacher: Very good.
Then, what is the measure of $\angle QPR$?

Student: $\angle QPR=120°$.

Teacher: Well done.

老師： 在空間中內積的公式與平面內積相似。因此，我們得到 $\vec{x}\cdot\vec{y}=x_1y_1+x_2y_2+x_3y_3$，其中 $\vec{x}=(x_1,x_2,x_3)$ 且 $\vec{y}=(y_1,y_2,y_3)$。
請問 $\overrightarrow{PQ}\cdot\overrightarrow{PR}$ 的值是多少？。

學生： -3

老師： 沒錯，內積是-3。我們還可以運用內積找到兩個非零向量之間的角度。

$\vec{x}\cdot\vec{y}=|\vec{x}||\vec{y}|cos\theta$，其中 θ 是 $\vec{x}$ 和 $\vec{y}$ 之間的角度。所以，$cos\theta=\frac{\vec{x}\cdot\vec{y}}{|\vec{x}||\vec{y}|}$。

找出 cos $\angle QPR$。

學生： cos $\angle QPR=-\frac{1}{2}$。

老師： 很好。那麼，$\angle QPR$ 的度數是多少？

學生： $\angle QPR=120°$

老師： 做得好。

例題二

說明：本題是討論空間中的平行向量及垂直向量的性質。

（英文）In space, there are three nonzero vectors $\vec{u}=(2,3,a)$, $\vec{v}=(-4,b,2)$ and $\vec{w}=(1,-3,c)$, where a,b,c are real numbers. If $\vec{u}\parallel\vec{v}$ and $\vec{v}\perp\vec{w}$, then what is the value of $a+b+$c?

（中文）已知空間中三個非零向量 $\vec{u}=(2,3,a)$、$\vec{v}=(-4,b,2)$ 及 $\vec{w}=(1,-3,c)$，其中 a,b,c 均為實數。若$\vec{u}\parallel\vec{v}$且$\vec{v}\perp\vec{w}$，則$a+b+$c 之值為何？

Teacher: We have $\vec{v}=t\cdot\vec{u}$ or $(-4,b,2)=t(2,3,a)$ if $\vec{u}\parallel\vec{v}$.
So, $-4=2t$, $b=3t$, and $2=t\cdot a$.
Then, $-4=2t$, and $t=-2$.
What are the values of a and b?

Student: $b=-6$, and $a=-1$.

Teacher: Very good.
So, $\vec{u}=(2,3,-1)$ and $\vec{v}=(-4,-6,2)$.
$\vec{v}\cdot\vec{w}=0$ if $\vec{v}\perp\vec{w}$.
$\vec{v}\cdot\vec{w}=(-4,-6,2)\cdot(1,-3,c)=0$
Find the value of c and $a+b+$c.

Student: c $=-7$and $a+b+$c $=-14$.

Teacher: Excellent.

老師：　當 $\vec{u} \parallel \vec{v}$，$\vec{v} = t \cdot \vec{u}$ 也就是 $(-4, b, 2) = t(2, 3, a)$。所以，$-4 = 2t$、$b = 3t$、$2 = t \cdot a$。又 $-4 = 2t$，得出 $t = -2$。

a 和 b 的值是多少？

學生：　$b = -6$、$a = -1$。

老師：　非常好。因此，$\vec{u} = (2, 3, -1)$，$\vec{v} = (-4, -6, 2)$。

$\vec{v} \perp \vec{w}$，則 $\vec{v} \cdot \vec{w} = 0$。

$\vec{v} \cdot \vec{w} = (-4, -6, 2) \cdot (1, -3, c) = 0$。

算出 c 以及 $a + b + c$。

學生：　$c = -7$，$a + b + c = -14$。

老師：　太棒了！

例題三

說明：本題是求空間向量的正射影。

（英文）In space, there are two vectors $\vec{u} = (4, 4, -3)$ and $\vec{v} = (1, -1, 2)$.

(a) Find the vector projection of $\vec{u}$ onto $\vec{v}$ ($proj_v u$).

(b) Use $proj_v u$ to write $\vec{u}$ as the sum of two orthogonal vectors.

（中文）(a) 試求向量 $\vec{u} = (4, 4, -3)$ 在向量 $\vec{v} = (1, -1, 2)$上的正射影（$proj_v u$）。

(b) 將 $\vec{u}$ 表示成其正射影($proj_v u$)及垂直於正射影($proj_v u$)的兩個向量之和。

Teacher:　The vector projection of $\vec{u}$ onto $\vec{v}$ is $proj_v u = \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}\right)\vec{v}$.

So, $\vec{u} \cdot \vec{v} = (4, 4, -3) \cdot (1, -1, 2) = -6$ and $|\vec{v}| = \sqrt{6}$.

Find $proj_v u$ now.

Student:　$proj_v u = (-1, 1, -2)$

Teacher:　Excellent. We can write $\vec{u} = proj_v u + \vec{w}$, where $\vec{w} = \vec{u} - proj_v u$.

Can you find $\vec{w}$?

Student:　Yes, $\vec{w} = (4, 4, -3) - (-1, 1, -2) = (5, 3, -1)$.

Teacher:　It is correct. Then $\vec{u}$ can be written as the sum of two orthogonal vectors.

$\vec{u} = (4, 4, -3) = (-1, 1, -2) + (5, 3, -1)$

We can check that $proj_v u$ and $\vec{w}$ are orthogonal because the inner product of the two vectors is zero.

$proj_v u \cdot \vec{w} = (-1, 1, -2) \cdot (5, 3, -1) = 0$

老師： 向量 $\vec{u}$ 在 $\vec{v}$ 上的正射影是 $proj_v u = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}$。

$\vec{u}\cdot\vec{v} = (4, 4, -3)\cdot(1, -1, 2) = -6$，$|\vec{v}| = \sqrt{6}.$。

現在找出 $proj_v u$。

學生： $proj_v u = (-1, 1, -2)$

老師： 很棒。我們可以將$\vec{u}$ 寫成$proj_v u + \vec{w}$，其中$\vec{w} = \vec{u} - proj_v u$。

$\vec{w}$ 是多少呢？

學生： $\vec{w} = (4, 4, -3) - (-1, 1, -2) = (5, 3, -1)$

老師： 答對了。 那麼 $\vec{u}$ 可以寫成兩個正交向量的和，也就是

$\vec{u} = (4, 4, -3) = (-1, 1, -2) + (5, 3, -1)$。

最後，透過確認兩個向量的內積是否為零，我們可以知道 $proj_v u$ 和 $\vec{w}$ 是否正交。

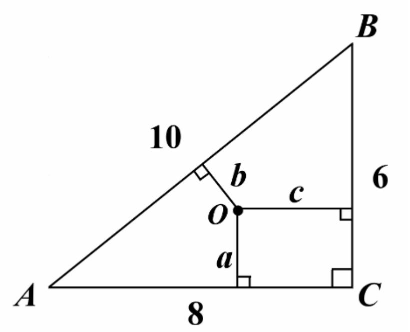
$proj_v u \cdot \vec{w} = (-1, 1, -2)\cdot(5, 3, -1) = 0$

例題四

說明：本題是運用柯西不等式求解。

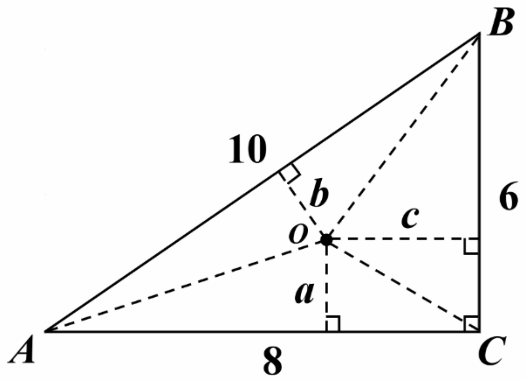
（英文）Refer to the diagram below, point O is inside right $\triangle ABC$. The distances from O to the three sides of $\triangle ABC$ are a, b, and c respectively. Find the smallest value of $a^2 + 4b^2 + 9c^2$.

（中文）如下圖所示，點 O 是直角 $\triangle ABC$ 內一點且此點到 $\triangle ABC$ 各邊之距離分別為 a、b、c。試求 $a^2 + 4b^2 + 9c^2$ 的最小值。



Teacher: Connect the three segments $\overline{OA}$, $\overline{OB}$, and $\overline{OC}$.

Assume that the distances from O to the sides $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ are a, b, and c respectively.



Area of $\triangle ABC$

=Area of $\triangle AOB$ + Area of $\triangle BOC$ + Area of $\triangle COA$

$$\frac{6\times 8}{2}=\frac{8a}{2}+\frac{10b}{2}+\frac{6c}{2}$$

We have $4a+5b+3c=24$.

By Cauchy inequality $|\vec{x}\cdot\vec{y}|\leq|\vec{x}||\vec{y}|$

$$|x_1y_1+x_2y_2+x_3y_3|\leq\sqrt{{x_1}^2+{x_2}^2+{x_3}^2}\cdot\sqrt{{y_1}^2+{y_2}^2+{y_3}^2}$$

Because $a^2+4b^2+9c^2=(a)^2+(2b)^2+(3c)^2$.

We have $|4a+5b+3c|\leq\sqrt{a^2+4b^2+9c^2}\cdot\sqrt{{y_1}^2+{y_2}^2+{y_3}^2}$.

Therefore, $|4a+5b+3c|\leq\sqrt{(a)^2+(2b)^2+(3c)^2}\cdot\sqrt{{y_1}^2+{y_2}^2+{y_3}^2}$

Find the values of y_1, y_2, and y_3.

Student: $y_1=4$, $y_2=\frac{5}{2}$, and $y_3=1$

Teacher: Yes, you are correct.

Because $y_1a=4a$, $y_2(2b)=5b$, and $y_3(3c)=3c$. We have $y_1=4$, $y_2=\frac{5}{2}$, and $y_3=1$. If we get the square of each side of the Cauchy inequality, then

$$(4a+5b+3c)^2\leq(a^2+4b^2+9c^2)\cdot(4^2+(\frac{5}{2})^2+1^2)$$

What is the smallest value of $(a^2+4b^2+9c^2)$?

Student: The smallest value of $(a^2+4b^2+9c^2)$ is $\frac{768}{31}$.

Teacher: Yes, your answer is correct.

$$a^2+4b^2+9c^2 \geq (4a+5b+3c)^2 \div \left(4^2+\left(\frac{5}{2}\right)^2+1^2\right)$$

and $(24)^2 \div \left(\frac{93}{4}\right)= \frac{768}{31}$.

So, $a^2+4b^2+9c^2 \geq \frac{768}{31}$.

老師：連接三條線段 $\overline{OA}$、$\overline{OB}$、$\overline{OC}$。假設從 O 到邊 $\overline{AB}$、$\overline{AC}$、$\overline{BC}$ 的距離分別為 a、b 和 c。

$\triangle ABC$ 的面積 $=\triangle AOB$ 的面積 $+\triangle BOC$ 的面積 $+\triangle COA$ 的面積

$\frac{6\times 8}{2}=\frac{8a}{2}+\frac{10b}{2}+\frac{6c}{2}$，得到 $4a+5b+3c=24$。

根據柯西不等式 $|\vec{x}\cdot\vec{y}| \leq |\vec{x}||\vec{y}|$

$$|x_1y_1+x_2y_2+x_3y_3| \leq \sqrt{{x_1}^2+{x_2}^2+{x_3}^2}\cdot\sqrt{{y_1}^2+{y_2}^2+{y_3}^2}$$

因為 $a^2+4b^2+9c^2=(a)^2+(2b)^2+(3c)^2$，帶入公式得出

$|4a+5b+3c| \leq \sqrt{a^2+4b^2+9c^2}\cdot\sqrt{{y_1}^2+{y_2}^2+{y_3}^2}$。

因此，$|4a+5b+3c| \leq \sqrt{(a)^2+(2b)^2+(3c)^2}\cdot\sqrt{{y_1}^2+{y_2}^2+{y_3}^2}$

現在求出 $y_1, y_2,$ 和 y_3。

學生：$y_1=4$、$y_2=\frac{5}{2}$、$y_3=1$

老師：答對了！因為 $y_1a=4a$、$y_2(2b)=5b$、$y_3(3c)=3c$，即可求出 $y_1=4$、$y_2=\frac{5}{2}$、$y_3=1$。如果我們把柯西不等式的兩邊取平方，

$(4a+5b+3c)^2 \leq (a^2+4b^2+9c^2)\cdot(4^2+(\frac{5}{2})^2+1^2)$，那 $a^2+4b^2+9c^2$ 的最小值是多少？

學生：$a^2+4b^2+9c^2$ 的最小值是 $\frac{768}{31}$。

老師：答對了。$a^2+4b^2+9c^2 \geq (4a+5b+3c)^2 \div \left(4^2+\left(\frac{5}{2}\right)^2+1^2\right)$，

$(24)^2 \div \left(\frac{93}{4}\right)=\frac{768}{31}$。因此，$a^2+4b^2+9c^2 \geq \frac{768}{31}$。

ꙮ 應用問題 / 學測指考題

例題一

說明：這題是運用內積求解。

（英文）In coordinate space, choose one vertex of a cube with side length 1 to be the point O. Then, randomly select two different vertices other than point O and name the two vertices P and Q respectively. Which of the following values is the expected value of the inner product $\overrightarrow{OP}\cdot\overrightarrow{OQ}$?

(1) $\frac{4}{7}$　(2) $\frac{5}{7}$　(3) $\frac{6}{7}$　(4) 1　(5) $\frac{8}{7}$

（中文）坐標空間中，考慮邊長為 1 的正立方體，固定一頂點 O。從 O 以外的七個頂點隨機選取相異兩點，設此兩點為 P、Q，試問所得的內積 $\overrightarrow{OP}\cdot\overrightarrow{OQ}$ 之期望值為下列哪一個選項？

(1) $\frac{4}{7}$　(2) $\frac{5}{7}$　(3) $\frac{6}{7}$　(4) 1　(5) $\frac{8}{7}$

（112 年學測數學 A 單選題第 6 題）

Teacher: We draw a cube in coordinate space, and label the vertices shown below.

Then P and Q can be any two different points from the vertices (1, 0, 0), (1, 1, 0), (0, 1, 0), (0, 0, 1), (1, 0, 1), (1, 1, 1) and (0, 1, 1).

Let P be the point (1, 0, 0), then $\overrightarrow{OP}\cdot\overrightarrow{OQ}$ could be:

(1, 0, 0)· (1, 1, 0) = 1

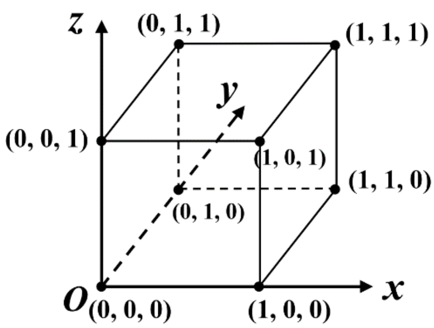
(1, 0, 0)· (0, 1, 0) = 0

(1, 0, 0)· (0, 0, 1) = 0

(1, 0, 0)· (1, 0, 1) = 1

(1, 0, 0)· (1, 1, 1) = 1

(1, 0, 0)· (0, 1, 1) = 0



Teacher: When $\overrightarrow{OP}$ is fixed, there are 6 different $\overrightarrow{OQ}$ vectors.
Besides, the inner product is 0 if two nonzero vectors are orthogonal.
So, the sum of the six different $\overrightarrow{OP}\cdot\overrightarrow{OQ}$ is 3.
Find the sum of the six different $\overrightarrow{OP}\cdot\overrightarrow{OQ}$ when P is the point (0, 1, 0).

Student: The sum of the six different $\overrightarrow{OP}\cdot\overrightarrow{OQ}$ is 3

Teacher: Very good. What is the sum when P is (0, 0, 1)?

Student: It is also 3.

Teacher: Yes, you are right. Then, find the sum when P is (1, 1, 0), (1, 0, 1) and (0, 1, 1)

Student: The sum of the six different inner products ($\overrightarrow{OP}\cdot\overrightarrow{OQ}$) is equal to 6.

Teacher: Great. Then find the final one when P is (1, 1, 1).

Student: The sum is 9.

Teacher: Excellent. So, the expected value of all the inner products $\overrightarrow{OP}\cdot\overrightarrow{OQ}$ is equal to

$(3\times 3+3\times 6+9)\div 42=\frac{6}{7}$.

(P and Q are two different vertices other than point O. There are 7 different $\overrightarrow{OP}$, and each $\overrightarrow{OP}$ has 6 different outcomes of $\overrightarrow{OP}\cdot\overrightarrow{OQ}$. So, the total number of $\overrightarrow{OP}\cdot\overrightarrow{OQ}$ is 42.)

The answer is (3).

老師：我們在坐標空間中畫一個立方體，並標示如下所示的頂點。那麼，P 和 Q 可以是來自頂點 (1, 0, 0)、(1, 1, 0)、(0, 1, 0)、(0, 0, 1)、(1, 0, 1)、(1, 1, 1) 和 (0, 1, 1) 中的任意兩個不同的點。
假設 P 是點 (1, 0, 0)，那麼 $\overrightarrow{OP}\cdot\overrightarrow{OQ}$ 可能為：
$(1, 0, 0)\cdot(1, 1, 0)=1$
$(1, 0, 0)\cdot(0, 1, 0)=0$
$(1, 0, 0)\cdot(0, 0, 1)=0$
$(1, 0, 0)\cdot(1, 0, 1)=1$
$(1, 0, 0)\cdot(1, 1, 1)=1$
$(1, 0, 0)\cdot(0, 1, 1)=0$
當 $\overrightarrow{OP}$ 固定時，會有 6 個不同的 $\overrightarrow{OQ}$ 向量。如果兩個非零向量垂直，則其內積為 0。因此，這六個不同的 $\overrightarrow{OP}\cdot\overrightarrow{OQ}$ 的總和為 3。找出當點 P 是 (0, 1, 0) 時這六個不同的 $\overrightarrow{OP}\cdot\overrightarrow{OQ}$ 的總和。

學生：這六個不同的 $\overrightarrow{OP}\cdot\overrightarrow{OQ}$ 的總和為 3。

老師：非常好。當 P 是 $(0,0,1)$ 時，總和是多少？

學生：也是 3。

老師：對，沒錯。那麼，當 P 分別是 $(1,1,0)$、$(1,0,1)$ 和 $(0,1,1)$ 時的個別總和為？

學生：這六個不同的內積 $(\overrightarrow{OP}\cdot\overrightarrow{OQ})$ 的個別總和都等於 6。

老師：太棒了。那麼最後一個，當 P 是 $(1,1,1)$ 時的總和是多少？

學生：總和是 9。

老師：你們超厲害！所以，所有內積 $\overrightarrow{OP}\cdot\overrightarrow{OQ}$ 的期望值就是 $(3\times3+3\times6+9)\div 42=\frac{6}{7}$。答案是 (3)。

(P 和 Q 是 O 以外的 7 個頂點中的相異兩點。所以有 7 種不同的$\overrightarrow{OP}$，每一 $\overrightarrow{OP}$ 有 6 種$\overrightarrow{OP}\cdot\overrightarrow{OQ}$。共有 42 個$\overrightarrow{OP}\cdot\overrightarrow{OQ}$。)

例題二

說明：本題是運用內積求兩非零向量的夾角。

（英文）In space, there is a tetrahedron $ABCD$. Please find the correct option(s) if $\overrightarrow{AD}$ is orthogonal to $\overrightarrow{AB}$ and $\overrightarrow{AC}$ respectively.

(1) $\overrightarrow{DB}\cdot\overrightarrow{DC}=\overline{DA}^2-\overrightarrow{AB}\cdot\overrightarrow{AC}$.

(2) If $\angle BAC$ is a right angle, then $\angle BDC$ is also a right angle.

(3) If $\angle BAC$ is an acute angle, then $\angle BDC$ is also an acute angle.

(4) If $\angle BAC$ is an obtuse angle, then $\angle BDC$ is also an obtuse angle.

(5) If $\overline{AB}<\overline{DA}$ and $\overline{AC}<\overline{DA}$, $\angle BDC$ is an acute angle.

（中文）空間中有一四面體 $ABCD$ 。假設 $\overrightarrow{AD}$ 分別與 $\overrightarrow{AB}$ 和 $\overrightarrow{AC}$ 垂直，請選出正確的選項。

(1) $\overrightarrow{DB}\cdot\overrightarrow{DC}=\overline{DA}^2-\overrightarrow{AB}\cdot\overrightarrow{AC}$

(2) 若 $\angle BAC$ 是直角，則 $\angle BDC$ 是直角

(3) 若 $\angle BAC$ 是銳角，則 $\angle BDC$ 是銳角

(4) 若 $\angle BAC$ 是鈍角，則 $\angle BDC$ 是鈍角

(5) 若 $\overline{AB}<\overline{DA}$ 且 $\overline{AC}<\overline{DA}$，則 $\angle BDC$ 是銳角

（106 年學測數學多選題第 13 題）

Teacher: By the given information, we have $\overrightarrow{AD}\cdot\overrightarrow{AB}=0$ and $\overrightarrow{AD}\cdot\overrightarrow{AC}=0$ because $\overrightarrow{AD}$

is orthogonal to $\overrightarrow{AB}$ and $\overrightarrow{AC}$ respectively.

$\overrightarrow{DB}\cdot\overrightarrow{DC}=(\overrightarrow{DA}+\overrightarrow{AB})\cdot(\overrightarrow{DA}+\overrightarrow{AC})$

$=\overrightarrow{DA}^2+\overrightarrow{DA}\cdot\overrightarrow{AC}+\overrightarrow{AB}\cdot\overrightarrow{DA}+\overrightarrow{AB}\cdot\overrightarrow{AC}$

$=\overrightarrow{DA}^2+\overrightarrow{AB}\cdot\overrightarrow{AC}$

Because $\overrightarrow{DA}\cdot\overrightarrow{AC}=(-\overrightarrow{AD})\cdot\overrightarrow{AC}=0$ and $\overrightarrow{AB}\cdot\overrightarrow{DA}=\overrightarrow{DA}\cdot\overrightarrow{AB}=0$.

So, the option (1) $\overrightarrow{DB}\cdot\overrightarrow{DC}=\overrightarrow{DA}^2-\overrightarrow{AB}\cdot\overrightarrow{AC}$ is false.

Let us move to the 2^{nd} option.

"If $\angle BAC$ is a right angle, then $\angle BDC$ is also a right angle."

To tell whether $\angle BDC$ is a right angle, we can find $\cos\angle BDC$ first.

What is $\cos\angle BDC$?

Student: $\cos\angle BDC=\frac{\overrightarrow{DB}\cdot\overrightarrow{DC}}{|\overrightarrow{DB}||\overrightarrow{DC}|}$

Teacher: Yes, you are right. From the former part, we know that

$\overrightarrow{DB}\cdot\overrightarrow{DC}=\overrightarrow{DA}^2+\overrightarrow{AB}\cdot\overrightarrow{AC}=\overrightarrow{DA}^2\ (\overrightarrow{AB}\cdot\overrightarrow{AC}=0)$

So, $\cos\angle BDC=\frac{\overrightarrow{DA}^2}{|\overrightarrow{DB}||\overrightarrow{DC}|}>0$.

Is $\angle BDC$ a right angle?

Student: No, $\angle BDC$ is an acute angle.

Teacher: Correct.

Now, we can use the same way to check the options (3) and (4).

If $\angle BAC$ is an acute angle, then $\overrightarrow{AB}\cdot\overrightarrow{AC}=|\overrightarrow{AB}||\overrightarrow{AC}|\cos\angle BAC>0$.

So, $\cos\angle BDC=\frac{\overrightarrow{DB}\cdot\overrightarrow{DC}}{|\overrightarrow{DB}||\overrightarrow{DC}|}=\frac{\overrightarrow{DA}^2+\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{DB}||\overrightarrow{DC}|}>0$.

$\angle BDC$ is also an acute angle when $\angle BAC$ is acute.

The option (3) is correct.

Please check the option (4) with your partner.

(After a few minutes)

Is the option (4) correct?

Student: No, because we do not know whether $\overrightarrow{DA}^2+\overrightarrow{AB}\cdot\overrightarrow{AC}$ is positive or negative.

Teacher: Excellent. We know that $\overrightarrow{AB}\cdot\overrightarrow{AC}<0$ when $\angle BAC$ is obtuse.

But $\overline{DA}^2+\overrightarrow{AB}\cdot\overrightarrow{AC}$ is undetermined. The option (4) is not correct.

Let us check the last option.

Can you tell whether $\overline{DA}^2+\overrightarrow{AB}\cdot\overrightarrow{AC}$ is positive, negative, or?

Student: Yes, $\overline{DA}^2+\overrightarrow{AB}\cdot\overrightarrow{AC}$ is positive.

Teacher: You are correct.

If $\overline{AB}<\overline{DA}$ and $\overline{AC}<\overline{DA}$, then $\left|\overrightarrow{AB}\cdot\overrightarrow{AC}\right|=\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right||\cos\angle BAC|\le\overline{DA}^2$.

So, $\cos\angle BDC>0$ because $\overline{DA}^2+\overrightarrow{AB}\cdot\overrightarrow{AC}$ is always positive.

$\angle BDC$ is an acute angle. The option (5) is correct.

The answer is (3) and (5).

老師：題目說，$\overrightarrow{AD}$ 分別與 $\overrightarrow{AB}$ 和 $\overrightarrow{AC}$ 垂直，因此我們可以知道 $\overrightarrow{AD}\cdot\overrightarrow{AB}=0$ 和 $\overrightarrow{AD}\cdot\overrightarrow{AC}=0$。

$\overrightarrow{DB}\cdot\overrightarrow{DC}=\left(\overrightarrow{DA}+\overrightarrow{AB}\right)\cdot\left(\overrightarrow{DA}+\overrightarrow{AC}\right)$

$=\overline{DA}^2+\overrightarrow{DA}\cdot\overrightarrow{AC}+\overrightarrow{AB}\cdot\overrightarrow{DA}+\overrightarrow{AB}\cdot\overrightarrow{AC}$

$=\overline{DA}^2+\overrightarrow{AB}\cdot\overrightarrow{AC}$

因為 $\overrightarrow{DA}\cdot\overrightarrow{AC}=(-\overrightarrow{AD})\cdot\overrightarrow{AC}=0$，且 $\overrightarrow{AB}\cdot\overrightarrow{DA}=\overrightarrow{DA}\cdot\overrightarrow{AB}=0$。所以，選項 (1) $\overrightarrow{DB}\cdot\overrightarrow{DC}=\overline{DA}^2-\overrightarrow{AB}\cdot\overrightarrow{AC}$ 是錯的。

接著看第二個選項：「若 $\angle BAC$ 是直角，則 $\angle BDC$ 是直角。」要判斷 $\angle BDC$ 是否為直角，我們可以先求出 $\cos\angle BDC$。

學生：$\cos\angle BDC=\frac{\overrightarrow{DB}\cdot\overrightarrow{DC}}{\left|\overrightarrow{DB}\right|\left|\overrightarrow{DC}\right|}$

老師：是的，你說得對。從前面的部分，我們知道

$\overrightarrow{DB}\cdot\overrightarrow{DC}=\overline{DA}^2+\overrightarrow{AB}\cdot\overrightarrow{AC}=\overline{DA}^2$（$\overrightarrow{AB}\cdot\overrightarrow{AC}=0$）。因此，$\cos\angle BDC=\frac{\overline{DA}^2}{\left|\overrightarrow{DB}\right|\left|\overrightarrow{DC}\right|}>0$。$\angle BDC$ 是直角嗎？

學生：不，$\angle BDC$ 是銳角。

老師： 正確。現在，我們可以用同樣的方式判斷選項 (3) 和 (4)。如果$\angle BAC$ 是銳角，那麼 $\overrightarrow{AB}\cdot\overrightarrow{AC}=|\overrightarrow{AB}||\overrightarrow{AC}|\cos\angle BAC>0$。

所以，$\cos\angle BDC=\frac{\overrightarrow{DB}\cdot\overrightarrow{DC}}{|\overrightarrow{DB}||\overrightarrow{DC}|}=\frac{\overline{DA}^2+\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{DB}||\overrightarrow{DC}|}>0$。當 $\angle BAC$ 是銳角，則 $\angle BDC$ 也是銳角 選項 (3) 是正確的。

請和你的夥伴一起判斷選項 (4)。

（幾分鐘後） 選項(4) 是正確的嗎？

學生： 不對，因為我們不知道 $\overline{DA}^2+\overrightarrow{AB}\cdot\overrightarrow{AC}$ 是正數還是負數。

老師： 太好了。當$\angle BAC$ 是鈍角時，我們知道 $\overrightarrow{AB}\cdot\overrightarrow{AC}<0$，但不知道 $\overline{DA}^2+\overrightarrow{AB}\cdot\overrightarrow{AC}$ 是多少，因此選項 (4) 是不正確的。

我們來看最後一個選項。$\overline{DA}^2+\overrightarrow{AB}\cdot\overrightarrow{AC}$ 是正數、負數還是其他呢？

學生： $\overline{DA}^2+\overrightarrow{AB}\cdot\overrightarrow{AC}$ 是正數！

老師： 答對了。

如果 $\overline{AB}<\overline{DA}$ 且 $\overline{AC}<\overline{DA}$，則 $|\overrightarrow{AB}\cdot\overrightarrow{AC}|=|\overrightarrow{AB}||\overrightarrow{AC}||\cos\angle BAC|\leq\overline{DA}^2$。

因此，$\overline{DA}^2+\overrightarrow{AB}\cdot\overrightarrow{AC}$ 永遠為正數，$\cos\angle BDC>0$，$\angle BDC$ 是銳角。選項 (5) 正確。答案是 (3) 和 (5)。

例題三

說明：這題是辨別四面體及運用內積求兩向量之夾角。

（英文）In the diagram, $\triangle ABC$ is a right triangle with $\angle ACB = 90°, \overline{AC} = 5,$ and $\overline{BC} = 6.$ If *ADEB* and *ADFC* are both rectangles, which of the following statement(s) is (are) true?

(1) There are two tetrahedrons if the object is cut along the plane *ACE*.

(2) The acute angle between the two planes *ADEB* and *ADFC* is greater than 45°.

(3) $\angle CEB < \angle AEB$

(4) $\tan\ \angle AEC < \sin\ \angle CEB$

(5) $\angle CEB < \angle AEC$

（中文）下圖為一個積木的示意圖，其中 ABC 為一直角三角形，$\angle ACB = 90°$，$\overline{AC} = 5$、$\overline{BC} = 6$，且 $ADEB$ 與 $ADFC$ 皆為矩形。試選出正確的選項。

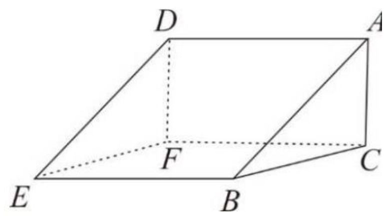
(1) 將此積木沿平面 ACE 切下，可切得兩個四面體

(2) 平面 $ADEB$ 與 $ADFC$ 所夾銳角大於 45°

(3) $\angle CEB < \angle AEB$

(4) $\tan\ \angle AEC < \sin\ \angle CEB$

(5) $\angle CEB < \angle AEC$

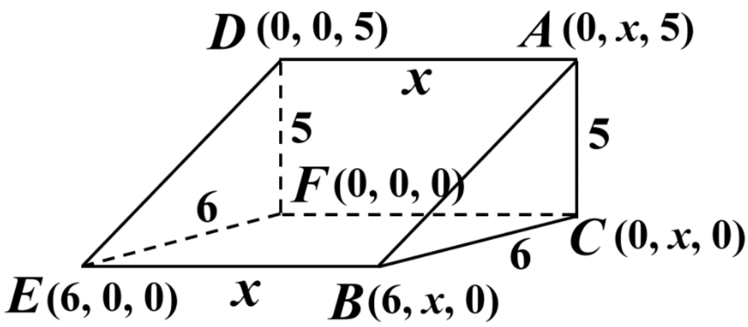


（111 年學測數學 A 多選題第 11 題）

Teacher: In option (1), the two objects *EABC* and *EADFC* are formed by cutting the object along the plane *ACE*,
Are they both tetrahedrons?

Student: No, the object *E-ADFC* has 5 different faces because *ADFC* is a rectangle.

Teacher: Very good. So, the option (1) is not correct.
There are different ways to determine whether the 2nd option is correct or not.
We can use the inner product to solve it here.
Let *F* be the origin (0, 0, 0) and $\overleftrightarrow{FE}, \overleftrightarrow{FC}, \overleftrightarrow{FD}$ as the *x*, *y*, *z* axes.
Assume $\overline{AC} = x,$ and we can label the coordinates of the other points.



The acute angle formed by the two planes $ADEB$ and $ADFC$ is $\angle BAC$.

$\cos\angle BAC = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|} = \frac{(6,0,-5)\cdot(0,0,-5)}{5\sqrt{61}} = \frac{5}{\sqrt{61}} < \frac{\sqrt{2}}{2}$

What can you conclude if $\cos\angle BAC < \cos 45°$

Student: $\angle BAC > 45°$.

Teacher: Yes, you are right. So, the option (2) is correct.

Of course, you can also find the answer if you directly use tan $\angle BAC$.

Because $\tan\angle BAC = \frac{6}{5} > 1$, we can get that $\angle BAC > 45°$.

The next option is to compare the measures of $\angle CEB$ and $\angle AEB$.

Find cos $\angle CEB$ and cos $\angle AEB$ now.

Student: $\cos\angle CEB = \frac{x}{\sqrt{36+x^2}}$ and $\cos\angle AEB = \frac{x}{\sqrt{61+x^2}}$.

Teacher: Good. Because both $\angle CEB$ and $\angle AEB$ are both acute angles.

We have $\angle CEB < \angle AEB$ if $\cos\angle CEB > \cos\angle AEB$.

The option (3) is correct.

Now, find $\tan\angle AEC$ and $\sin\angle CEB$ for option (4).

Student: $\tan\angle AEC < \sin\angle CEB$ because $\tan\angle AEC = \frac{5}{\sqrt{36+x^2}}$ and $\sin\angle CEB = \frac{6}{\sqrt{36+x^2}}$.

Teacher: Excellent. In option (5), we have $\sin\angle AEC = \frac{5}{\sqrt{61+x^2}} < \tan\angle AEC = \frac{5}{\sqrt{36+x^2}}$.

So, the option (5) is false because **sin $\angle AEC$** < tan $\angle AEC$ < **sin $\angle CEB$**.

Therefore, we have $\angle AEC < \angle CEB$. (Sine function of acute angles is strictly increasing.)

The answer is (2), (3) and (4).

老師： 選項(1)說：將此積木沿平面 ACE 切下，可切得兩個物體 $EABC$ 和 $EADFC$。它們都是四面體嗎？

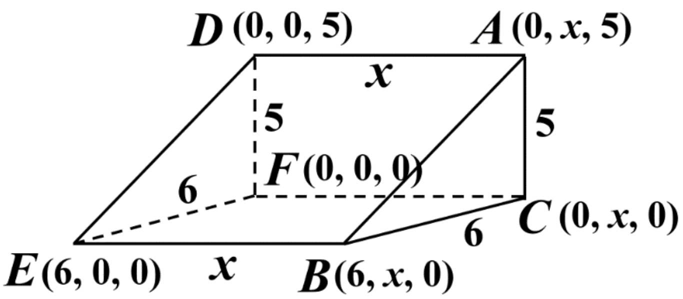
學生： 不是，因為 $ADFC$ 是一個矩形，所以 $E\text{-}ADFC$ 有 5 個不同的面。

老師： 非常好。所以，選項 (1) 是不正確的。

有不同的方法來判斷第二個選項是否正確，我們可以在這裡使用內積來判斷。

設 F 為原點 (0, 0, 0)，並將 $\overleftrightarrow{FE}$、$\overleftrightarrow{FC}$、$\overleftrightarrow{FD}$ 分別作為 x、y、z 軸。

假設 $\overline{AC} = x$，我們可以標上其他點的坐標。



由兩個平面 $ADEB$ 和 $ADFC$ 形成的銳角是 $\angle BAC$。

$\cos\angle BAC = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|} = \frac{(6,0,-5)\cdot(0,0,-5)}{5\sqrt{61}} = \frac{5}{\sqrt{61}} < \frac{\sqrt{2}}{2}$ 。

如果 $\cos\angle BAC < \cos 45°$，能得出什麼結論？

學生： $\angle BAC > 45°$。

老師： 是的，沒錯，選項 (2) 是正確的。當然，你也可以直接使用 $\tan\angle BAC$ 找到答案。 因為 $\tan\angle BAC = \frac{6}{5} > 1$，我們可以得到 $\angle BAC > 45°$。

下一個選項是比較 $\angle CEB$ 和 $\angle AEB$ 的大小。現在找出 $\cos\angle CEB$ 和 $\cos\angle AEB$。

學生： $\cos\angle CEB = \frac{x}{\sqrt{36+x^2}}$，$\cos\angle AEB = \frac{x}{\sqrt{61+x^2}}$。

老師： 很好。因為 $\angle CEB$ 和 $\angle AEB$ 都是銳角。$\cos\angle CEB > \cos\angle AEB$，則得出 $\angle CEB < \angle AEB$，選項 (3) 是正確的。

接著看選項 (4) 找出 $\tan\angle AEC$ 和 $\sin\angle CEB$。

學生： 因為 $\tan\angle AEC = \frac{5}{\sqrt{36+x^2}}$ 和 $\sin\angle CEB = \frac{6}{\sqrt{36+x^2}}$，所以 $\tan\angle AEC < \sin\angle CEB$。

老師： 太棒了！選項(5)中, 我們可以得到 $\sin\angle AEC = \frac{5}{\sqrt{61+x^2}} < \tan\angle AEC = \frac{5}{\sqrt{36+x^2}}$.所以選項 (5) 就是錯誤的，因為 $\sin\angle AEC < \tan\angle AEC < \sin\angle CEB$，得到 $\angle AEC < \angle CEB$。(銳角的正弦函數是嚴格遞增的。)

所以答案是 (2)、(3) 和 (4)。

單元四　外積與行列式
Cross Product and Determinant

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■ 前言 Introduction

本單元的內容包含空間向量的外積與三階行列式。首先運用外積來求兩不平行向量的公垂向量及平行四邊形的面積，並運用外積來求三向量所決定的平行六面體體積等；接下來練習三階行列式的性質及運用三階行列式來求平行六面體的體積。因外積是空間向量的特有運算，其計算方式較為繁雜，本單元將著重外積的運算及其運用。因學生已經在第三冊學過二階行列式等相關內容，建議老師可以先複習二階行列式等相關英文詞彙及內容，再介紹三階行列式的性質及運用三階行列式來求平行六面體的體積。最後以應用問題或學測題來加深學生們對本單元的了解。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
cross product / vector product	外積	**parallelepiped**	平行六面體
right-handed rule	右手法則	three-dimensional	三度空間的

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ **A/an _______ is _______ shape with _________**

例句：**A parallelepiped is** a three-dimensional **shape with** six parallelogram faces.

平行六面體是具有 6 個平行四邊形面的立體。

❷ **_______, with which ______.**

例句：This section introduces the determinant of a 3×3 matrix, **with which** we can find the volume of a parallelepiped.

本節介紹三階行列式，藉由三階行列式我們可以求出平行六面體的體積。

❸ **It is more concise to _________**

例句：In this question, **it** would **be more concise to** use the inner product to find the angle between two planes.

這一題用內積來求兩平面的夾角會更為簡潔。

■ 問題講解 Explanation of Problems

說明

In this section, we will cover the cross product and the determinant of a 3×3 matrix. The cross product of two vectors $\vec{a}$ and $\vec{b}$ (denoted by $\vec{a} \times \vec{b}$ and read as $\vec{a}$ cross $\vec{b}$) is defined in three-dimensional space (It doesn't exist in plane). Teachers can first discuss the area of a parallelogram formed by two vectors $\vec{a}$ and $\vec{b}$.

If θ is the angle between $\vec{a}(a_1, a_2, a_3)$ and $\vec{b}(b_1, b_2, b_3)$, then the area of the parallelogram

$$= |\vec{a}||\vec{b}|sin\theta = |\vec{a}||\vec{b}|\sqrt{1-cos^2\theta} = \sqrt{|\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2}$$

$$= \sqrt{({a_1}^2 + {a_2}^2 + {a_3}^2)({b_1}^2 + {b_2}^2 + {b_3}^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2}$$

$$= \sqrt{(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2}$$

$$= \sqrt{\begin{vmatrix} a_2 a_3 \\ b_2 b_3 \end{vmatrix}^2 + \begin{vmatrix} a_3 a_1 \\ b_3 b_1 \end{vmatrix}^2 + \begin{vmatrix} a_1 a_2 \\ b_1 b_2 \end{vmatrix}^2}$$

Then introduce the definition of the cross product $\vec{a} \times \vec{b} = \left(\begin{vmatrix} a_2 a_3 \\ b_2 b_3 \end{vmatrix}, \begin{vmatrix} a_3 a_1 \\ b_3 b_1 \end{vmatrix}, \begin{vmatrix} a_1 a_2 \\ b_1 b_2 \end{vmatrix}\right)$, where $\vec{\boldsymbol{a}} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$. The cross product of $\vec{a}$ and $\vec{b}$ (denoted by $\vec{a} \times \vec{b}$**)** is also a vector that is perpendicular to both $\vec{a}$ and $\vec{b}$. Remind students to note that $\vec{a} \times \vec{b}$ and $\vec{\boldsymbol{b}} \times \vec{\boldsymbol{a}}$ are two opposite vectors**.** In fact, $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$.

Second, we apply the magnitude of the cross product to find the area of a parallelogram or triangle. The last part of this section introduces the determinant of a 3×3 matrix and its properties. We learned the operations and properties of the second order determinants last semester. The third-order determinants also have similar operations and properties.

The third-order determinant is defined as follows:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2$$

The third-order determinants can also be expanded along any row or column.

If we expand the determinant along the first row, then

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2$$

$$= a_1\begin{vmatrix} b_2 c_2 \\ b_3 c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 c_2 \\ a_3 c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 b_2 \\ a_3 b_3 \end{vmatrix}.$$

We can use this way to simplify the calculation in the third-order determinants, and then apply the determinant of a 3×3 matrix to find the volume of a parallelepiped.

➿運算問題的講解 ➿

例題一

說明：本題為外積的運算及運用外積求平行四邊形的面積。

（英文）In space, there are two different vectors $\vec{a}=(1,-2,0)$ and $\vec{b}=(3,1,-2)$.

(1) Find $\vec{a}\times\vec{b}$ and $\vec{b}\times\vec{a}$.

(2) Find the area of the parallelogram formed by $\vec{a}$ and $\vec{b}$.

（中文）已知空間中兩向量 $\vec{a}=(1,-2,0)$、$\vec{b}=(3,1,-2)$，試求：

(1) $\vec{a}\times\vec{b}$及$\vec{b}\times\vec{a}$。

(2) 由向量 $\vec{a}$ 和 $\vec{b}$ 所形成平行四邊形的面積。

Teacher: The cross product $\vec{a}\times\vec{b}$ is defined as follows:

$\vec{a}\times\vec{b}=\left(\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix},\begin{vmatrix}a_3 & a_1\\ b_3 & b_1\end{vmatrix},\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}\right)$, where $\vec{a}=(a_1,a_2,a_3)$ and $\vec{b}=(b_1,b_2,b_3)$.

When $\vec{a}=(1,-2,0)$ and $\vec{b}=(3,1,-2)$,

$\vec{a}\times\vec{b}=\left(\begin{vmatrix}-2 & 0\\ 1 & -2\end{vmatrix},\begin{vmatrix}0 & 1\\ -2 & 3\end{vmatrix},\begin{vmatrix}1 & -2\\ 3 & 1\end{vmatrix}\right)=(4,2,7)$.

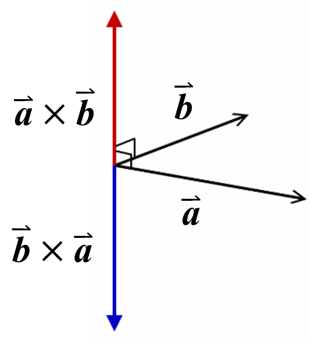
Can you use the same way to find $\vec{b}\times\vec{a}$ now?

Student: Yes, $\vec{b}\times\vec{a}=(-4,-2,-7)$.

Teacher: Very good.

Also, please notice that $\vec{a}\times\vec{b}\neq\vec{b}\times\vec{a}$.

By using the right-handed rule, the cross products of $\vec{a}\times\vec{b}$ and $\vec{b}\times\vec{a}$ are two opposite vectors which are both perpendicular to the vectors $\vec{a}$ and $\vec{b}$.



$$\vec{a}\times\vec{b}=-(\vec{b}\times\vec{a})$$

In Part (2), the area of the parallelogram A is equal to the magnitude of $\vec{a}\times\vec{b}$ (denoted by $|\vec{a}\times\vec{b}|$).

If θ is the angle between the two vectors $\vec{a}$ and $\vec{b}$.

The area A = $|\vec{a}||\vec{b}|sin\theta = |\vec{a}||\vec{b}|\sqrt{1-cos^2\theta} = \sqrt{|\vec{a}|^2|\vec{b}|^2-(\vec{a}\cdot\vec{b})^2}$

$$= \sqrt{({a_1}^2+{a_2}^2+{a_3}^2)({b_1}^2+{b_2}^2+{b_3}^2)-(a_1b_1+a_2b_2+a_3b_3)^2}$$

$$= \sqrt{(a_2b_3-a_3b_2)^2+(a_3b_1-a_1b_3)^2+(a_1b_2-a_2b_1)^2}$$

$$= \sqrt{\begin{vmatrix}a_2 a_3\\ b_2 b_3\end{vmatrix}^2+\begin{vmatrix}a_3 a_1\\ b_3 b_1\end{vmatrix}^2+\begin{vmatrix}a_1 a_2\\ b_1 b_2\end{vmatrix}^2}$$

$$= |\vec{a}\times\vec{b}|$$

Can you find the area of the parallelogram by finding $|\vec{a}\times\vec{b}|$?

Student: Yes, it is $\sqrt{69}$.

Teacher: Well done.

老師： 外積 $\vec{a}\times\vec{b}$ 的定義如下：

$\vec{a}\times\vec{b} = \left(\begin{vmatrix}a_2 a_3\\ b_2 b_3\end{vmatrix},\begin{vmatrix}a_3 a_1\\ b_3 b_1\end{vmatrix},\begin{vmatrix}a_1 a_2\\ b_1 b_2\end{vmatrix}\right)$，其中$\vec{a}=(a_1,a_2,a_3)$，$\vec{b}=(b_1,b_2,b_3)$。

當$\vec{a}=(1,-2,0)$且$\vec{b}=(3,1,-2)$時，

$$\vec{a}\times\vec{b} = \left(\begin{vmatrix}-2 & 0\\ 1 & -2\end{vmatrix},\begin{vmatrix}0 & 1\\ -2 & 3\end{vmatrix},\begin{vmatrix}1 & -2\\ 3 & 1\end{vmatrix}\right) = (4,2,7)$$

你能用同樣的方式找出 $\vec{b}\times\vec{a}$ 嗎？

學生： $\vec{b}\times\vec{a}=(-4,-2,-7)$ 非常好。

老師： 另外，請注意$\vec{a}\times\vec{b}\neq\vec{b}\times\vec{a}$。 利用右手定則，$\vec{a}\times\vec{b}$ 和 $\vec{b}\times\vec{a}$ 的外積是兩個互為相反的向量，它們都垂直於向量 $\vec{a}$ 和 $\vec{b}$。

$$\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$$

第(2)小題，平行四邊形 A 的面積等於$\vec{a}\times\vec{b}$的大小（表示為$|\vec{a}\times\vec{b}|$）。

如果 θ 是向量 $\vec{a}$ 、 $\vec{b}$ 間的夾角，面積則是 $|\vec{a}||\vec{b}|sin\theta$

$= |\vec{a}||\vec{b}|\sqrt{1-cos^2\theta} = \sqrt{|\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2}$

$= \sqrt{({a_1}^2+{a_2}^2+{a_3}^2)({b_1}^2+{b_2}^2+{b_3}^2)-(a_1b_1+a_2b_2+a_3b_3)^2}$

$= \sqrt{(a_2b_3-a_3b_2)^2+(a_3b_1-a_1b_3)^2+(a_1b_2-a_2b_1)^2}$

$= \sqrt{\begin{vmatrix} a_2 a_3 \\ b_2 b_3 \end{vmatrix}^2 + \begin{vmatrix} a_3 a_1 \\ b_3 b_1 \end{vmatrix}^2 + \begin{vmatrix} a_1 a_2 \\ b_1 b_2 \end{vmatrix}^2}$

$= |\vec{a} \times \vec{b}|$

現在算出 $|\vec{a} \times \vec{b}|$ 求平行四邊形的面積

學生： 是 $\sqrt{69}$。

老師： 很好。

例題二

說明：本題為運用外積求三角形的面積。

（英文）In space, $A(0,1,0)$, $B(3,5,0)$ and $C\,(-1,3,2)$ are three noncollinear points. Find the area of $\triangle ABC$.

（中文）已知空間中不共線三點 $A(0,1,0)$、$B(3,5,0)$ 及$C\,(-1,3,2)$，試求 $\triangle ABC$ 的面積。

Teacher: The area of $\triangle ABC$ is equal to half of the area of the parallelogram formed by the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.The area of $\triangle ABC = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$.

Find the vectors of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ now.

Student: $\overrightarrow{AB} = (3,4,0)$ and $\overrightarrow{AC} = (-1,2,2)$.

Teacher: Very good. What is the outer product $\overrightarrow{AB} \times \overrightarrow{AC}$?

Student: The outer product is (8, −6, 8).

Teacher: Excellent.

So, the area of $\triangle ABC = \frac{1}{2}\sqrt{164} = 2\sqrt{41}$

老師：　$\triangle ABC$ 的面積就是由向量 $\overrightarrow{AB}$ 和 $\overrightarrow{AC}$ 形成的平行四邊形的面積的一半。

$\triangle ABC = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$。現在找出 $\overrightarrow{AB}$ 和 $\overrightarrow{AC}$ 的向量。

學生：　$\overrightarrow{AB} = (3,4,0)$、$\overrightarrow{AC} = (-1,2,2)$。

老師：　非常好。外積 $\overrightarrow{AB} \times \overrightarrow{AC}$ 是多少？

學生：　外積是 $(8, -6, 8)$。

老師：　不錯。因此，$\triangle ABC = \frac{1}{2}\sqrt{164} = 2\sqrt{41}$。

例題三

說明：本題是運用外積求垂直向量。

（英文）In space, there are two nonzero vectors $\vec{a} = (2,1,0)$ and $\vec{b} = (0,-2,2)$.

Find a vector $\vec{c}$ if $\vec{c}$ is perpendicular to $\vec{a}$ and $\vec{b}$ with $|\vec{c}| = 12$.

（中文）已知空間中兩向量$\vec{a} = (2,1,0)$、$\vec{b} = (0,-2,2)$，若向量 $\vec{c}$ 垂直於向量 $\vec{a}$ 與向量 $\vec{b}$且$|\vec{c}|$=12，試求向量$\vec{c}$。

Teacher:　$\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$ and $\vec{b}$. $\vec{c} = k(\vec{a} \times \vec{b})$

Find $\vec{a} \times \vec{b}$ now.

Student:　$\vec{a} \times \vec{b} = (2,-4,-4)$

Teacher:　Very good.

$|\vec{a} \times \vec{b}| = 6$. $|\vec{c}| = |k(\vec{a} \times \vec{b})| = 12$.

What is the value of k?

Student:　$k = 2$.

Teacher:　Well, $k = 2$ is correct. But there is another solution “$k = -2$”.

So, $\vec{c} = 2\cdot(2,-4,-4) = (4,-8,-8)$ or $\vec{c} = -2\cdot(2,-4,-4) = (-4,8,8)$.

The solution is $(4,-8,-8)$ or $(-4,8,8)$.

老師： $\vec{a} \times \vec{b}$ 是垂直於 $\vec{a}$ 和 $\vec{b}$ 的向量。令 $\vec{c} = k(\vec{a} \times \vec{b})$。現在找出 $\vec{a} \times \vec{b}$。

學生： $\vec{a} \times \vec{b} = (2, -4, -4)$ 。

老師： 很好。$|\vec{a} \times \vec{b}| = 6$，而 $|\vec{c}| = |k(\vec{a} \times \vec{b})| = 12$。求 k 是多少？

學生： $k = 2$。

老師： 嗯，$k = 2$ 是正確的，但還有另一個解是「$k = -2$」喔！所以，$\vec{c} = 2\cdot (2, -4, -4) = (4, -8, -8)$ 或 $\vec{c} = -2\cdot (2, -4, -4) = (-4, 8, 8)$。
答案是 $(4, -8, -8)$ 或 $(-4, 8, 8)$。

例題四

說明：本題是運用外積求四面體的體積。

（英文）In space, E is a plane which passes through the three points $A(0, -1, -1)$, $B(1, -1, -2)$, and $C(0, 1, 0)$. H is a point and $\overrightarrow{AH} = \frac{2}{3}\overrightarrow{AB} - \frac{1}{3}\overrightarrow{AC} + 3(\overrightarrow{AB} \times \overrightarrow{AC})$. Find the volume of the tetrahedron $ABCH$.

(The volume of a tetrahedron = $\frac{1}{3} \times$ base area $\times$ height.)

（中文）坐標空間中，令 E 為通過三點 $A(0, -1, -1)$、$B(1, -1, -2)$、$C(0, 1, 0)$ 的平面。假設 H 為空間中一點，且滿足 $\overrightarrow{AH} = \frac{2}{3}\overrightarrow{AB} - \frac{1}{3}\overrightarrow{AC} + 3(\overrightarrow{AB} \times \overrightarrow{AC})$。根據上述，試回答下列問題。

(1) 試求四面體 $ABCH$ 的體積。（註：四面體體積為三分之一的底面積乘以高）

（110 年指考數甲非選題第 1 題）

Teacher: $A(0, -1, -1)$, $B(1, -1, -2)$, and $C(0, 1, 0)$ are the vertices of the base plane of the tetrahedron.
Find $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AB} \times \overrightarrow{AC}$.

Student: $\overrightarrow{AB} = (1, 0, -1)$, $\overrightarrow{AC} = (0, 2, 1)$, and $\overrightarrow{AB} \times \overrightarrow{AC} = (2, -1, 2)$.

Teacher: Yes, you are correct.

The area of the base triangle is equal to $\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$. The volume of a tetrahedron

$=\frac{1}{3}\times$ base area $\times$ height

$=\frac{1}{6}\times$ the volume of the parallelepiped (formed by $\overrightarrow{AB}$,$\overrightarrow{AC}$, and $\overrightarrow{AH}$)

(The base area of the triangle is one-half of the area of the parallelogram formed by $\overrightarrow{AB}$ and $\overrightarrow{AC}$.)

$=\frac{1}{6}\times|\overrightarrow{AH}\cdot(\overrightarrow{AB}\times\overrightarrow{AC})|$.

By the given information, $\overrightarrow{AH}=\frac{2}{3}\overrightarrow{AB}-\frac{1}{3}\overrightarrow{AC}+3(\overrightarrow{AB}\times\overrightarrow{AC})$.

$\overrightarrow{AH}\cdot(\overrightarrow{AB}\times\overrightarrow{AC})=3|\overrightarrow{AB}\times\overrightarrow{AC}|^2=27$

Because $\overrightarrow{AB}\times\overrightarrow{AC}$ is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$\overrightarrow{AB}\cdot(\overrightarrow{AB}\times\overrightarrow{AC})=0$ and $\overrightarrow{AC}\cdot(\overrightarrow{AB}\times\overrightarrow{AC})=0$

What is the volume of the tetrahedron?

Student: $\frac{9}{2}$.

Teacher: Yes, you are correct.

老師： $A(0,\ -1,\ -1)$、$B(1,\ -1,\ -2)$ 和 $C(0,1,0)$ 是四面體底面的頂點。求 $\overrightarrow{AB}$、$\overrightarrow{AC}$ 和 $\overrightarrow{AB}\times\overrightarrow{AC}$。

學生： $\overrightarrow{AB}=(1,\ 0,-1)$、$\overrightarrow{AC}=(0,2,1)$，$\overrightarrow{AB}\times\overrightarrow{AC}=(2,-1,2)$。

老師： 是的，沒錯。

底面三角形的面積等於 $\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$。四面體的體積等於 $\frac{1}{3}\times$ 底面積 $\times$ 高，等於

$\frac{1}{6}\times$平行六面體的體積（由 $\overrightarrow{AB}$、$\overrightarrow{AC}$ 和 $\overrightarrow{AH}$ 所形成）

（三角形的底面積為 $\overrightarrow{AB}$ 和 $\overrightarrow{AC}$ 所形成的平行四邊形的一半）。

$=\frac{1}{6}\times|\overrightarrow{AH}\cdot(\overrightarrow{AB}\times\overrightarrow{AC})|$。

老師： 根據題目給的資訊，$\overrightarrow{AH}=\frac{2}{3}\overrightarrow{AB}-\frac{1}{3}\overrightarrow{AC}+3(\overrightarrow{AB}\times\overrightarrow{AC})$。

$\overrightarrow{AH}\cdot(\overrightarrow{AB}\times\overrightarrow{AC})=3(\overrightarrow{AB}\times\overrightarrow{AC})\cdot(\overrightarrow{AB}\times\overrightarrow{AC})=27$。因為 $\overrightarrow{AB}\times\overrightarrow{AC}$ 垂直於 $\overrightarrow{AB}$ 和 $\overrightarrow{AC}$，所以 $\overrightarrow{AB}\cdot(\overrightarrow{AB}\times\overrightarrow{AC})=0$，$\overrightarrow{AC}\cdot(\overrightarrow{AB}\times\overrightarrow{AC})=0$。

四面體的體積是多少？

學生： $\frac{9}{2}$。

老師： 是的，你說得對。

例題五

說明：本題是三階行列式的運算。

（英文）Find the third-order determinants.

(1) $\begin{vmatrix}-2 & -1 & 2\\ -2 & 1 & 4\\ 0 & 3 & -1\end{vmatrix}$ (2) $\begin{vmatrix}-2 & -1 & 2\\ 1 & 0 & 3\\ 1 & 2 & -3\end{vmatrix}+\begin{vmatrix}2 & 1 & 5\\ 1 & 0 & 3\\ 1 & 2 & -3\end{vmatrix}$

（中文）試求下列三階行列式的值

(1) $\begin{vmatrix}-2 & -1 & 2\\ -2 & 1 & 4\\ 0 & 3 & -1\end{vmatrix}$ (2) $\begin{vmatrix}-2 & -1 & 2\\ 1 & 0 & 3\\ 1 & 2 & -3\end{vmatrix}+\begin{vmatrix}2 & 1 & 5\\ 1 & 0 & 3\\ 1 & 2 & -3\end{vmatrix}$

Teacher: We learned the operations and properties of the second order determinants last semester.

The third-order determinants also have similar operations and properties.

The third-order determinant is defined as follows:

$$\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}=a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_3b_2c_1-a_2b_1c_3-a_1b_3c_2$$

Find out the value of the determinant in Part (1).

Student: It is 16.

Teacher: Yes, you are correct.

Let's see the next question.

$$\begin{vmatrix} -2 & -1 & 2 \\ 1 & 0 & 3 \\ 1 & 2 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 5 \\ 1 & 0 & 3 \\ 1 & 2 & -3 \end{vmatrix}$$

$$= \begin{vmatrix} -2+2 & -1+1 & 2+5 \\ 1 & 0 & 3 \\ 1 & 2 & -3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 7 \\ 1 & 0 & 3 \\ 1 & 2 & -3 \end{vmatrix} = 7\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = 14$$

Do you have any questions so far?

Student: No.

Teacher: Ok.

老師： 上學期我們學了二階行列式的運算和性質，而三階行列式也有類似的運算和性質，它的定義如下：

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2$$

算出第(1)小題的行列式值。

學生： 算出來是 16。

老師： 沒錯，答對了。我們來看下一小題。

$$\begin{vmatrix} -2 & -1 & 2 \\ 1 & 0 & 3 \\ 1 & 2 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 5 \\ 1 & 0 & 3 \\ 1 & 2 & -3 \end{vmatrix}$$

$$= \begin{vmatrix} -2+2 & -1+1 & 2+5 \\ 1 & 0 & 3 \\ 1 & 2 & -3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 7 \\ 1 & 0 & 3 \\ 1 & 2 & -3 \end{vmatrix} = 7\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = 14$$

到目前有什麼問題嗎？

學生： 沒有。

老師： 好的。

ꕤ 應用問題 / 學測指考題 ꕤ

例題一

說明：這題是運用外積求解。

（英文）In the diagram, *OABC-DEFG* is a cube, which of the following vectors is parallel to the cross product $\overrightarrow{AD}\times\overrightarrow{AG}$?

(1) $\overrightarrow{AE}$

(2) $\overrightarrow{BE}$

(3) $\overrightarrow{CE}$

(4) $\overrightarrow{DE}$

(5) $\overrightarrow{OE}$

（中文）如圖，*OABC-DEFG* 為一正方體，試問向量外積 $\overrightarrow{AD}\times\overrightarrow{AG}$ 與下列哪一個向量平行？

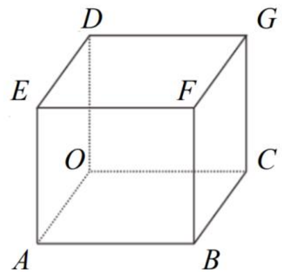
(1) $\overrightarrow{AE}$

(2) $\overrightarrow{BE}$

(3) $\overrightarrow{CE}$

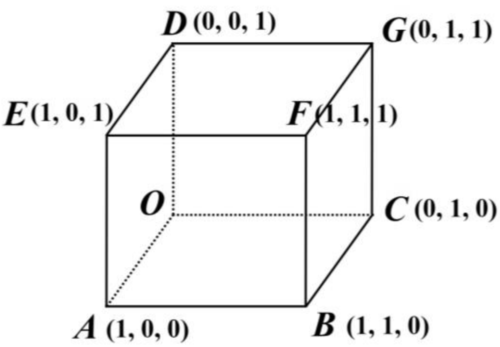
(4) $\overrightarrow{DE}$

(5) $\overrightarrow{OE}$



（113 年學測數學 A 單選題第 2 題）

Teacher: Refer to the diagram, we can label the coordinates as follows:



The two vectors $\overrightarrow{AD}$ and $\overrightarrow{AG}$ are $(-1, 0, 1)$ and $(-1, 1, 1)$.

Find the cross product $\overrightarrow{AD}\times\overrightarrow{AG}$.

Student: $\overrightarrow{AD}\times\overrightarrow{AG}=(-1, 0, -1)$

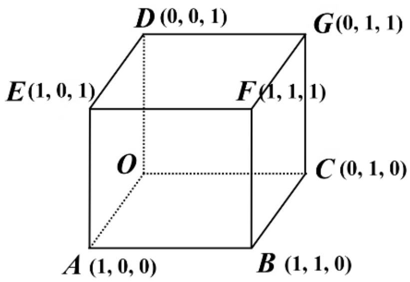
Teacher: Yes, it is correct.

Now, find out the five vectors and see which one is parallel to $\overrightarrow{AD} \times \overrightarrow{AG}$.

Student: The answer is (5) $\overrightarrow{OE}$.

Teacher: Well done.

老師： 我們可以將題目的圖形標示坐標如下：



兩向量 $\overrightarrow{AD}$ 和 $\overrightarrow{AG}$ 分別為 $(-1, 0, 1)$ 和$(-1, 1, 1)$。求外積 $\overrightarrow{AD} \times \overrightarrow{AG}$。

學生： $\overrightarrow{AD} \times \overrightarrow{AG} = (-1, 0, -1)$

老師： 是的，答對了。

現在，找出五個向量，看哪一個與 $\overrightarrow{AD} \times \overrightarrow{AG}$ 平行。

學生： 答案是 (5) $\overrightarrow{OE}$。

老師： 做得好。

例題二

說明：這題是行列式的運算問題。

（英文）In coordinate space, there are three noncollinear points P, Q and R on the plane $2x-3y+5z=\sqrt{7}$. Let $\overrightarrow{PQ}=(a_1,b_1,c_1)$ and $\overrightarrow{PR}=(a_2,b_2,c_2)$.

Which of the following determinants has the greatest absolute value?

(1) $\begin{vmatrix} -1 & 1 & 1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$ (2) $\begin{vmatrix} 1 & -1 & 1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$ (3) $\begin{vmatrix} 1 & 1 & -1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$

(4) $\begin{vmatrix} -1 & -1 & 1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$ (5) $\begin{vmatrix} -1 & -1 & -1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$

（中文）已知坐標空間中 P、Q、R 為平面 $2x-3y+5z=\sqrt{7}$ 上不共線三點。令 $\overrightarrow{PQ}=(a_1,b_1,c_1)$，$\overrightarrow{PR}=(a_2,b_2,c_2)$。試選出下列行列式中絕對值為最大的選項。

(1) $\begin{vmatrix} -1 & 1 & 1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$ (2) $\begin{vmatrix} 1 & -1 & 1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$ (3) $\begin{vmatrix} 1 & 1 & -1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$

(4) $\begin{vmatrix} -1 & -1 & 1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$ (5) $\begin{vmatrix} -1 & -1 & -1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$

（112 年學測數學 A 單選題第 5 題）

Teacher: $\overrightarrow{PQ}\times\overrightarrow{PR}$ is parallel to the normal vector of the plane.

So, $\overrightarrow{PQ}\times\overrightarrow{PR}=k(2,-3,5)$.

$$\overrightarrow{PQ}\times\overrightarrow{PR}=\left(\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix},\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix},\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\right)$$

$=k(2,-3,5)=(2k,-3k,5k)$

Besides, the third-order determinant can be written as the inner product form.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}-b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}+c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$=a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}+b_1\begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix}+c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= (a_1, b_1, c_1)\cdot\left(\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix}, \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}\right)$$

The first determinant $\begin{vmatrix} -1 & 1 & 1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = (-1, 1, 1)\cdot\left(\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\right)$

$= (-1, 1, 1)\cdot(2k, -3k, 5k) = 0$

The 2nd determinant $\begin{vmatrix} 1 & -1 & 1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = (1, -1, 1)\cdot(2k, -3k, 5k) = 10k$

The 3rd determinant $\begin{vmatrix} 1 & 1 & -1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = (1, 1, -1)\cdot(2k, -3k, 5k) = -6k$

Find the values of the other determinants now.

(Wait for a few minutes.)

What are the 4th and 5th determinants?

Student: $\begin{vmatrix} -1 & -1 & 1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = (-1, -1, 1)\cdot(2k, -3k, 5k) = 6k$

$\begin{vmatrix} -1 & -1 & -1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = (-1, -1, -1)\cdot(2k, -3k, 5k) = -4k$

Teacher: Good. They are correct.

Which one has the greatest absolute value?

Student: The second one.

Teacher: Yes, you are right.

Compare the absolutes of these determinants: 0, $10|k|$,$6|k|$, $6|k|$, and $4|k|$.

The greatest one is $10|k|$.

So, the answer is (2) $\begin{vmatrix} 1 & -1 & 1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$.

老師： $\overrightarrow{PQ} \times \overrightarrow{PR}$ 與平面的法向量平行，因此，$\overrightarrow{PQ} \times \overrightarrow{PR} = k\,(2, -3, 5)$。

所以，

$\overrightarrow{PQ} \times \overrightarrow{PR} = \left(\begin{vmatrix} b_1 c_1 \\ b_2 c_2 \end{vmatrix}, \begin{vmatrix} c_1 a_1 \\ c_2 a_2 \end{vmatrix}, \begin{vmatrix} a_1 b_1 \\ a_2 b_2 \end{vmatrix} \right) = k\,(2,\ -3, 5) = (2k,\ -3k, 5k)$。此外，三階行列式可以寫成內積形式。

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 c_2 \\ b_3 c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 c_2 \\ a_3 c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 b_2 \\ a_3 b_3 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 c_2 \\ b_3 c_3 \end{vmatrix} + b_1 \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 b_2 \\ a_3 b_3 \end{vmatrix}$$

$$= (a_1,\ b_1,\ c_1) \cdot \left(\begin{vmatrix} b_2 c_2 \\ b_3 c_3 \end{vmatrix}, \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix}, \begin{vmatrix} a_2 b_2 \\ a_3 b_3 \end{vmatrix} \right)$$

選項(1)行列式 $\begin{vmatrix} -1 & 1 & 1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = (-1, 1, 1) \cdot \left(\begin{vmatrix} b_1 c_1 \\ b_2 c_2 \end{vmatrix}, \begin{vmatrix} c_1 a_1 \\ c_2 a_2 \end{vmatrix}, \begin{vmatrix} a_1 b_1 \\ a_2 b_2 \end{vmatrix} \right)$

$= (-1, 1, 1) \cdot (2k,\ -3k, 5k) = 0$。

選項(2)行列式 $\begin{vmatrix} 1 & -1 & 1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = (1,\ -1, 1) \cdot (2k,\ -3k, 5k) = 10k$。

選項(3)行列式 $\begin{vmatrix} 1 & 1 & -1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = (1, 1,\ -1) \cdot (2k,\ -3k, 5k) = -6k$。

現在找出其他行列式的值。（等待幾分鐘。）選項(4)、(5)分別是多少呢？

學生： $\begin{vmatrix} -1 & -1 & 1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = (-1,\ -1, 1) \cdot (2k,\ -3k, 5k) = 6k$

$\begin{vmatrix} -1 & -1 & -1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = (-1,\ -1,\ -1) \cdot (2k,\ -3k, 5k) = -4k$

老師： 很好，正確。哪一個絕對值最大？

學生： 第二個。

老師： 沒錯。比較這些行列式的絕對值：0、10$|k|$、6$|k|$、6$|k|$、和 4$|k|$，最大的是 10$|k|$。

所以，答案是 (2) $\begin{vmatrix} 1 & -1 & 1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$。

例題三

說明：本題是有關外積與平行六面體體積的應用問題。

（英文）In space, $A(-1, 2, 1)$, $B(-4, 1, 3)$, $C(2, 0, -3)$ are three vertices of the base plane of a parallelepiped, and one of the vertices on the other plane is located on the x-y plane which is 1 unit from the origin O. What is the maximum volume of a parallelepiped that satisfies the above conditions?

（中文）坐標空間中一平行六面體，某一底面的其中三頂點為 $(-1, 2, 1)$、$(-4, 1, 3)$、$(2, 0, -3)$，另一面之一頂點在 xy 平面上且與原點距離為 1。滿足前述條件之平行六面體中，最大體積為 ＿＿＿＿＿？

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Teacher: A $(-1, 2, 1)$, B $(-4, 1, 3)$, and C $(2, 0, -3)$ are the vertices on the base plane of the parallelepiped.

The area of the base parallelogram is equal to $|\overrightarrow{AB} \times \overrightarrow{AC}|$

Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$, and then the area of the parallelogram.

Student: $\overrightarrow{AB} = (-3, -1, 2)$, $\overrightarrow{AC} = (3, -2, 4)$, and the area is $\sqrt{181}$.

Teacher: Yes, you are correct.

Assume $P(\cos\theta, \sin\theta, 0)$ is the vertex which is on the x-y plane and 1 unit from the origin O.

The volume of the parallelepiped is equal to the vertical distance of P to the base plane times the base area.

One of the normal vectors of the base plane is $\overrightarrow{AB} \times \overrightarrow{AC} = (8, -6, 9)$.

So, the equation of the base plane with normal vector $(8, -6, 9)$ and passing through the point $A(-1, 2, 1)$ is $8(x+1) - 6(y-2) + 9(z-1) = 0$.

The standard form of the plane equation is $8x - 6y + 9z = 11$.

The distance of point P to the base plane

$= \frac{|8\cos\theta - 6\sin\theta - 11|}{\sqrt{8^2 + (-6)^2 + 9^2}} = \frac{|8\cos\theta - 6\sin\theta - 11|}{\sqrt{181}}$.

We need to know the range of $(8\cos\theta - 6\sin\theta)$ before we find the greatest value of $|8\cos\theta - 6\sin\theta - 11|$.

So, what is the range of $(8\cos\theta - 6\sin\theta)$?

Student:　The range of $(8\cos\theta-6\sin\theta)$ is between 10 and -10.

Teacher:　Yes, you are correct.

$(8\cos\theta-6\sin\theta)=10\left(\frac{4}{5}\cos\theta-\frac{3}{5}\sin\theta\right)=10\cos(\alpha+\theta)$, where $\cos\alpha=\frac{4}{5}$ and $\sin\alpha=\frac{3}{5}$.

Hence, the greatest value of $|8\cos\theta-6\sin\theta-11|$ is 21.

The greatest volume $=\sqrt{181}\cdot\frac{21}{\sqrt{181}}=21$.

老師：A 點 $(-1,2,1)$、B 點 $(-4,1,3)$、和 C 點 $(2,0,-3)$ 是平行六面體底面的頂點。底面平行四邊形的面積是 $|\overrightarrow{AB}\times\overrightarrow{AC}|$ 找出 $\overrightarrow{AB}$ 和 $\overrightarrow{AC}$，然後計算平行四邊形的面積。

學生：$\overrightarrow{AB}=(-3,-1,2)$，$\overrightarrow{AC}=(3,-2,4)$，面積為 $\sqrt{181}$.

老師：是的，沒錯。假設 $P(\cos\theta,\sin\theta,0)$ 是位於 x-y 平面上、距離原點 O 1 單位的頂點。平行六面體的體積等於 P 到底面平面的垂直距離乘以底面積。

底面的其中一個法向量為 $\overrightarrow{AB}\times\overrightarrow{AC}=(8,-6,9)$，因此，通過點 A$(-1,2,1)$、法向量為$(8,-6,9)$ 的底面平面方程式為 $8(x+1)-6(y-2)+9(z-1)=0$。

標準式為 $8x-6y+9z=11$。

點 P 到底面平面的距離$=\frac{|8\cos\theta-6\sin\theta-11|}{\sqrt{8^2+(-6)^2+9^2}}=\frac{|8\cos\theta-6\sin\theta-11|}{\sqrt{181}}$。

我們需要知道 $(8\cos\theta-6\sin\theta)$ 的範圍，才能找到 $|8\cos\theta-6\sin\theta-11|$ 的最大值。那，$(8\cos\theta-6\sin\theta)$ 的範圍是多少？

學生：$(8\cos\theta-6\sin\theta)$ 的範圍在 10 到 -10 之間。

老師：是的，正確。$(8\cos\theta-6\sin\theta)=10\left(\frac{4}{5}\cos\theta-\frac{3}{5}\sin\theta\right)=10\cos(\alpha+\theta)$，其中 $\cos\alpha=\frac{4}{5}$ 且 $\sin\alpha=\frac{3}{5}$。因此，$|8\cos\theta-6\sin\theta-11|$ 的最大值為 21，最大體積 $=\sqrt{181}\cdot\frac{21}{\sqrt{181}}=21$。

單元五　平面方程式 Planes in Space

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■ 前言 Introduction

如何找到空間中的平面方程式呢？我們可以由平面上的一點及與垂直於平面的非零向量（該平面的法向量）來找到空間中的平面方程式。那如何求得平面的法向量呢？這時我們可以在平面上找到兩個非平行的向量，再求它們的外積就可以找到。法向量除了能幫我們找到平面的方程式，也可以利用兩平面的法向量所夾的角來找到兩平面的夾角。最後我們會計算空間中點到平面的距離，及兩平行平面的距離。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
plane	平面	**angle between two planes**	**兩平面夾角**
normal vector	**法向量**	distance	距離
equation of a plane	平面方程式	**orthogonal**	**垂直／正交**
midpoint	中點	parallel	平行
cross product	**外積**	Cauchy’s inequality	柯西不等式
inner product	**內積**	minimum	最小值

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ Find a normal vector of ________.

例句：**Find a normal vector of** plane xy.

找出平面 xy 的一個法向量。

❷ If plane ____ passes through_____ with normal vector $\vec{n}$ = _______, then the equation of plane ____ is _______________.

例句：**If plane** E **passes through** $A(3,4,6)$ **with normal vector** $\vec{n}=(1,2,3)$, **then the equation of plane** E **is** $1(x-3)+2(y-4)+3(z-6)=0$.

若平面 E 過 $A(3,4,6)$ 且法向量為 $\vec{n}=(1,2,3)$, 則平面 E 的方程式可寫成 $1(x-3)+2(y-4)+3(z-6)=0$

❸ Given two points ____ and ____, find the perpendicular bisector of _________.

例句：**Give two points** $A(1,2,5)$ **and** $B(0,1,2)$, **find the perpendicular bisector of** $\overline{AB}$.

已知 $A(1,2,5)$，$B(0,1,2)$，試求 $\overline{AB}$ 的垂直平分面方程式。

❹ Given a plane _____ through three nonlinear points _______ and _______, find the equation of plane _______.

例句：**Given a plane** E **through three nonlinear points** $A(1,2,0)$, $B(0,3,-1)$ **and** $C(-2,-3,-1)$, **find the equation of plane** E

平面 E 通過不共線三點 $A(1,2,0)$、$B(0,3,-1)$、$C(-2,-3,-1)$，求平面 E 的方程式。

❺ Find the angle between plane ________ and plane _________.

例句：**Find the angle between plane** $E_1: x + 2y + 3z = 9$ **and plane** $E_2: x - y + 7z = -3$.

求兩平面 $E_1: x + 2y + 3z = 9$ 和 $E_2: x - y + 7z = -3$ 的夾角。

❻ Find the distance from point ________ to plane _________.

例句：**Find the distance from** $P(1, 2, 3)$ **to plane** $E: 3x - 4y + 6z = -3$.

求點$P(1, 2, 3)$ 到平面 $E: 3x - 4y + 6z = -3$ 的距離。

❼ Find the distance between two parallel planes ________ and _________.

例句：**Find the distance between two parallel planes** $E_1: x + 2y + 3z = 9$ **and** $E_2: 2x + 4y + 6z = 5$

求兩平行平面 $E_1: x + 2y + 3z = 9$ 和 $E_2: 2x + 4y + 6z = 5$ 的距離。

■ 問題講解 Explanation of Problems

說明

To write the equation of a plane in space, we need to find its normal vector. What is a normal vector? A normal vector to plane E is any nonzero vector that is perpendicular to plane E.
We can find the equation of a plane E if we know its normal vector and one point on E.

Normal vectors can also be used to tell whether two planes are parallel or not. If one plane's normal vector is parallel to another plane's, then those two planes are parallel. If one plane's normal vector is not parallel to another plane's, then we can use their normal vectors to find the angle between those two planes by finding the angle between their normal vectors.

We have learned how to find the distance from a point to a line on the coordinate plane before. How do we find the distance from a point P to a plane E in space? First, we can choose any point A on plane E. Second, we find $\overrightarrow{AP}$ and calculate its projection on plane E 's normal vector $\vec{n}$, called it $proj_{\vec{n}}\overrightarrow{AP}$. The last part to be found is the magnitude of this vector $proj_{\vec{n}}\overrightarrow{AP}$ and deduce the formula to it.

Since the normal vector plays such an important role in this lesson, is there any way we can use it to find a plane's normal vector? The answer to this question is *cross product*. By using the cross product of two nonparallel vectors we find on this plane, we can find a vector orthogonal to those two nonparallel vectors at the same time which surely is orthogonal to the plane. So that vector will perfectly serve as the normal vector we need. And any scalar multiple of the vector can be a normal vector to this plane.

運算問題的講解

例題一

說明：本題是用平面的法向量及過平面上一點求出平面方程式。

（英文）Given plane E passing through $A(3,4,6)$ with normal vector $\vec{n}=(1,2,3)$, find the equation of plane E.

（中文）若平面 E 過 $A(3,4,6)$ 且法向量為 $\vec{n}=(1,2,3)$，試求平面 E 的方程式。

Teacher: Once we know the normal vector and a point of a plane, we can find the equation of a plane. What is a normal vector of a plane?

Student: A normal vector is a vector orthogonal to a plane.

Teacher: Great! What is the given normal vector here?

Student: It is $\vec{n}=(1,2,3)$.

Teacher: What is the given point here?

Student: It is $A(3,4,6)$.

Teacher: Then we can use the point-normal-vector form of the plane, $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ to express this plane, where (a,b,c) is the normal vector and (x_0,y_0,z_0) is a point on this plane. Now, you can plug the normal vector $\vec{n}=(1,2,3)$ and $A(3,4,6)$ to plane E.

Student: The equation of plane E is $1(x-3)+2(y-4)+3(z-6)=0.$

Teacher: Excellent. Simplify $1(x-3)+2(y-4)+3(z-6)=0$ to the general form. We can get $x+2y+3z-29=0$

老師： 當我們知道該平面的一個法向量及平面上的一點時，我們就能寫出該平面的方程式了。還記得法向量是什麼嗎？

學生： 一平面的法向量是與該平面垂直的向量。

老師： 很好。那題目給的法向量是什麼呢？

學生： 是 $\vec{n}=(1,2,3)$

老師： 題目給的平面上一點為何？

學生： 是 $A(3,4,6)$。

老師： 那麼我們可以利用點法式將該法向量 $\vec{n}=(1,2,3)$ 及點$A(3,4,6)$ 帶入

$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$。

學生： 平面 E 的方程式為 $1(x-3)+2(y-4)+3(z-6)=0.$

老師： 很棒！我們可以方程式整理為平面的標準式如下 $x+2y+3z-29=0$

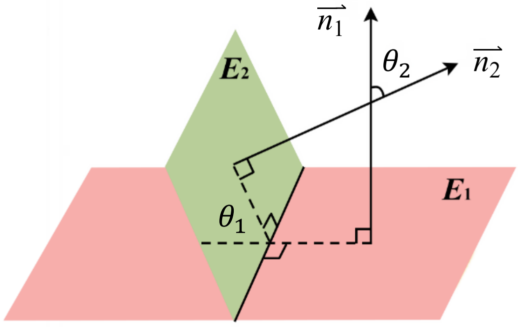
例題二

說明：本題是用兩平面的法向量求出此兩平面的夾角。

（英文）Find the angle between plane $E_1: x+2y+3z=9$ and plane $E_2: 2x-3y-z=-3.$

（中文）求兩平面 $E_1: x+2y+3z=9$ 和 $E_2: 2x-3y-z=-3$ 的夾角。

Teacher: How do we find the angle between two planes? Let's take a look at the graph below. Let $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$ be the normal vectors of $E_1$and E_2, respectively. Let θ_1 be the angle between E_1 and E_2, and θ_2 be the angle between $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$, Compare θ_1 and θ_2.



Student: They are equal because they are supplementary to the same angle.

Teacher: Good job. To find the angle between two planes, all we need to do is to find the angle between their normal vectors. From the previous lesson, we know the angle between two vectors can be found by applying the formula

$\cos\theta = \dfrac{\overrightarrow{n_1}\cdot\overrightarrow{n_2}}{|\overrightarrow{n_1}|\cdot|\overrightarrow{n_2}|}$. What is the normal vector $\overrightarrow{n_1}$ of $E_1: x+2y+3z=9$?

Student: $\overrightarrow{n_1}$ is $(1,2,3)$.

Teacher: Good. What is the normal vector $\overrightarrow{n_2}$ of $E_2: 2x-3y-z=-3$?

Student: $\overrightarrow{n_2}$ is $(2,-3,-1)$.

Teacher: Plug $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$ into $\cos\theta = \dfrac{\overrightarrow{n_1}\cdot\overrightarrow{n_2}}{|\overrightarrow{n_1}|\cdot|\overrightarrow{n_2}|}$.

Then $\cos\theta = \dfrac{1\cdot 2+2\cdot(-3)+3\cdot(-1)}{\sqrt{1^2+2^2+3^3}\cdot\sqrt{2^2+(-3)^2+(-1)^2}} = \dfrac{-7}{\sqrt{14}\sqrt{14}} = \dfrac{-1}{2}$.

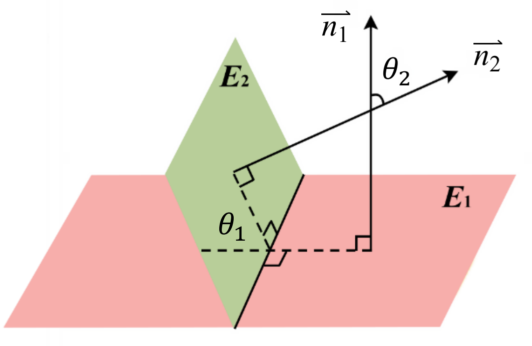
What is θ?

Student: $\theta = 120^\circ$. So the angle between E_1 and E_2 is 120°.

Teacher: Be cautious that the supplementary angle of θ is also the angle between these two planes. So the answer is 120° and 60°.

Student: Hope I won't forget to do it next time.

老師： 我們可以利用兩平面的法向量去求得兩平面的夾角。如下圖 θ_1 為平面 E_1 和 E_2 的夾角，θ_2 為兩平面法向量 $\overrightarrow{n_1}$ 和 $\overrightarrow{n_2}$ 的夾角。試比較 θ_1 和 θ_2。



學生： θ_1 和 θ_2 相等，因為它們和同一角的互補。

老師： 厲害哦！所以我們發現只要找到兩法向量的夾角就能找到兩平面的夾角了。之前我們有學到用這個公式來找兩向量的夾角:

$$\cos\theta = \frac{\overline{n_1}\cdot\overline{n_2}}{|\overline{n_1}|\cdot|\overline{n_2}|}.$$ 平面 $E_1: x+2y+3z=9$ 的法向量 $\overline{n_1}$ 為何？

學生： $\overline{n_1}$ 是 $(1,2,3)$。

老師： 很好。平面 $E_2: 2x-3y-z=-3$ 的法向量 $\overline{n_2}$ 為何？

學生： $\overline{n_2}$ 是 $(2,-3,-1)$。

老師： 將 $\overline{n_1}$ 和 $\overline{n_2}$ 代入 $\cos\theta = \frac{\overline{n_1}\cdot\overline{n_2}}{|\overline{n_1}|\cdot|\overline{n_2}|}$. 我們得到

$$\cos\theta = \frac{1\cdot 2+2\cdot(-3)+3\cdot(-1)}{\sqrt{1^2+2^2+3^3}\cdot\sqrt{2^2+(-3)^2+(-1)^2}} = \frac{-7}{\sqrt{14}\sqrt{14}} = \frac{-1}{2}$$

θ 是多少？

學生： $\theta = 120^\circ$。所以兩平面 E_1 和 E_2 的夾角為 120°。

老師： 記得同時要回答 120° 的餘角，因為該角也會是這兩平面的夾角。答案為 120° 及 60°。

學生： 希望我下次不會忘記這件事。

例題三

說明：本題是要求出一點到另一平面的距離。

（英文）Find the distance from $P(1, 2, 3)$ to plane $E: 3x - 4y + 6z = -3$

（中文）求點$P(1, 2, 3)$ 到平面 $E: 3x - 4y + 6z = -3$ 的距離。

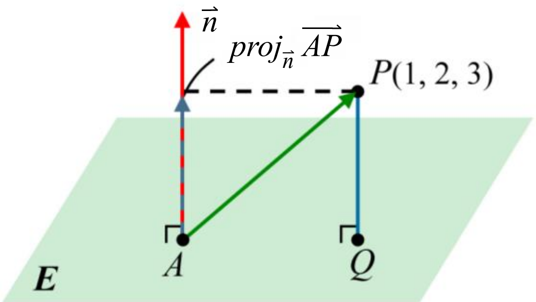
Teacher: How do we find the distance from a point P (not on E) to a plane E? Let's take a look at the graph below. Let $\vec{n} = (a, b, c)$ be the normal vector of plane $E: ax + by + cz + d = 0$ and $A(x_1, y_1, z_1)$ be any point on plane E, the distance from P to plane E equals the magnitude of the projection of $\overrightarrow{AP}$ on $\vec{n}$ (denoted as $proj_{\vec{n}}\overrightarrow{AP}$). Given $A(x_1, y_1, z_1)$ and $\vec{n} = (a, b, c)$, the equation of plane E is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ and we get $d = -(ax_1 + by_1 + cz_1)$. From the previous lesson, we know the magnitude of $proj_{\vec{n}}\overrightarrow{AP}$ is

$$\left|proj_{\vec{n}}\overrightarrow{AP}\right|$$

$$= \left|\frac{\overrightarrow{AP} \cdot \vec{n}}{\left|\vec{n}\right|^2} \cdot \vec{n}\right| = \left|\frac{\overrightarrow{AP} \cdot \vec{n}}{\vec{n}}\right| = \frac{\left|(x_0 - x_1, y_0 - y_1, z_0 - z_1) \cdot (a, b, c)\right|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{\left|ax_0 + by_0 + cz_0 - (ax_1 + by_1 + cz_1)\right|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{\left|ax_0 + by_0 + cz_0 + d\right|}{\sqrt{a^2 + b^2 + c^2}}$$



Teacher: What is left to do is to plug the point into this formula.

What is the normal vector $\vec{n}$ of $E: 3x - 4y + 6z = -3$?

Student: $\vec{n}$ is $(3,-4,6)$.

Teacher: What is the given point?

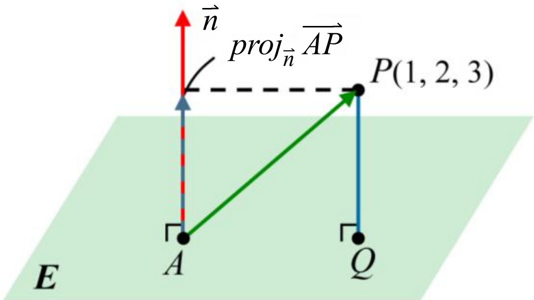
Student: It is $P(1, 2, 3)$.

Teacher: Plug $\vec{n}$ and P into $\dfrac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$.

Then the distance is $\dfrac{|3\cdot1+(-4)\cdot2+6\cdot3+3|}{\sqrt{3^2+(-4)^2+6^2}}=\dfrac{16}{\sqrt{61}}$

老師： 如果要求得一點到另一平面的距離，我們可以利用該點到平面上一點的向量對該平面法向量的正射影去求得，進而可以導出距離公式如下

$$\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$



老師： 剩下的就是代入公式，先找出 $E:3x-4y+6z=-3$的法向量。

學生： 法向量$\vec{n}$為$(3,-4,6)$.

老師： 題目所給的點為何？

學生： $P(1, 2, 3)$。

老師： 將法向量及 $P(1, 2, 3)$ 代入公式 $\dfrac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$

得到$\dfrac{|3\cdot1+(-4)\cdot2+6\cdot3+3|}{\sqrt{3^2+(-4)^2+6^2}}=\dfrac{16}{\sqrt{61}}$

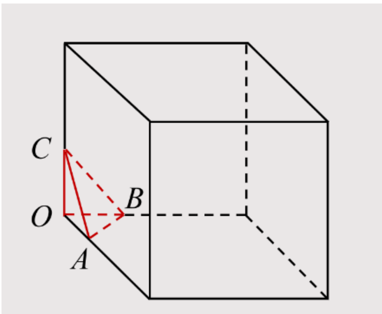
꧂ 應用問題 / 學測指考題 ꧁

例題一

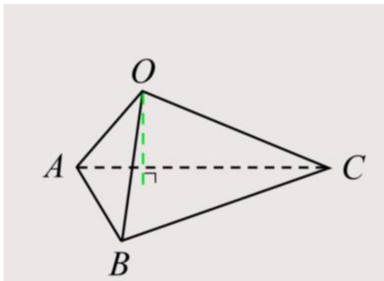
說明：這題利用點到平面的距離來求解。

（英文）We cut a triangular pyramid $O-ABC$ out of a cube. Assume the side length of the cube is 6 cm, $\overline{OA}=1$ cm, $\overline{OB}=1$ cm and $\overline{OC}=2$ cm, if we place this pyramid on the ground by using triangle ABC as its base, find how high it is from O to the base.

（中文）在一個邊長為 6 公分的正立方體中，切下四面體 $O-ABC$，如圖 1，其中 $\overline{OA}=1$ 公分，$\overline{OB}=1$ 公分，$\overline{OC}=2$ 公分，如果把四面體放在地面上，以面 ABC 為底面，如圖 2，請問此時四面體有多高？



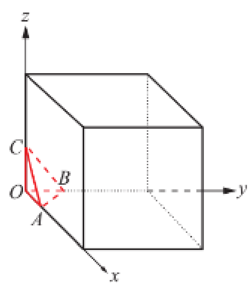
(圖 1)



(圖 2)

（圖改編自南一高二下 84 頁）

Teacher: To begin with, we place this cube in space by setting O as the origin, $\overrightarrow{OA}$ as the positive direction of the x-axis, $\overrightarrow{OB}$ as the positive direction of the y-axis, and $\overrightarrow{OZ}$ as the positive direction of the z-axis. Given $\overline{OA}=1$ cm, $\overline{OB}=1$ cm and $\overline{OC}=2$ cm, tell me what the coordinates of A, B and C are?



Student: $A(1,0,0), B(0,1,0), C(0,0,2)$.

Teacher: To find how high this pyramid is, we find the distance from O to plane ABC. Since we already know the three intercepts of plane ABC through, we can use the intercept form: $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ to find the equation of plane ABC.

Plug $A(1,0,0),B(0,1,0),C(0,0,2)$ into the intercept form by replacing $a=1,b=1$ and $c=2$.

Student: I get $\frac{x}{1}+\frac{y}{1}+\frac{z}{2}=1$

Teacher: For the convenience of calculation, I will simplify $\frac{x}{1}+\frac{y}{1}+\frac{z}{2}=1$ as $2x+2y+z-2=0$. Now, to find the point $O(0,0,0)$ to a plane $2x+2y+z=2$ we cam apply the distance formula from a point to a plane: $\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$.

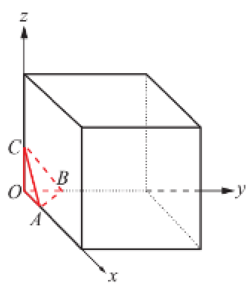
What is the normal vector of plane $ABC: 2x+2y+z=2$?

Student: $\vec{n}=(2,2,1)$.

Teacher: Plug $O(0,0,0)$ and $\vec{n}=(2,2,1)$ into the formula, we get

$$\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}=\frac{|0\cdot2+0\cdot2+0\cdot1-2|}{\sqrt{2^2+2^2+1^2}}=\frac{2}{3}\text{ cm.}$$

老師： 我們在正立方體上建立坐標系統，以 O 為原點，$\overrightarrow{OA}$ 為 x 軸正向，$\overrightarrow{OB}$ 為 y 軸正向，$\overrightarrow{OZ}$ 為 z 軸正向。已知 $\overline{OA}=1$公分，$\overline{OB}=1$公分，$\overline{OC}=2$ 公分，現在找出 A、B 和 C 的坐標。



學生： $A(1,0,0),B(0,1,0),C(0,0,2)$。

老師：要找到 O 到平面 ABC 的距離，我們可以利用點到平面的距離公式。在此之前我們可以先用平面的截距式 $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ 來找平面 ABC 的方程式。

將 $A(1,0,0), B(0,1,0), C(0,0,2)$ 代入截距式中。

學生：我得到 $\frac{x}{1}+\frac{y}{1}+\frac{z}{2}=1$。

老師：為了方便計算，可將方程式 $\frac{x}{1}+\frac{y}{1}+\frac{z}{2}=1$ 化簡成 $2x+2y+z=2$。現在可以找點 $O(0,0,0)$ 到平面 $2x+2y+z=2$ 的距離了。先找出平面 $ABC:2x+2y+z=2$ 的法向量。

學生：$\vec{n}=(2,2,1)$.

老師：最後把 $O(0,0,0)$ 及 $\vec{n}=(2,2,1)$ 代入公式，得到

$$\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}=\frac{|0\cdot2+0\cdot2+0\cdot1-2|}{\sqrt{2^2+2^2+1^2}}=\frac{2}{3}$$ 。

例題二

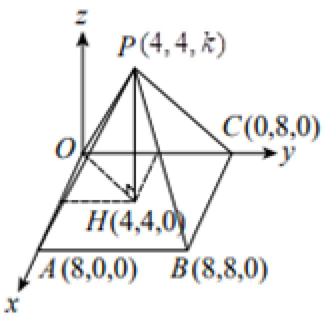
說明：這題利用兩向量的外積求得一組平面的法向量。

（英文）A square is on xy plane with vertices $O(0,0,0), A(8,0,0), B(8,8,0), C(0,8,0)$. P is a point above the xy plane and equidistant from $O, A, B,$ and C. Given $PA=6$, find the equation of a plane through A, B and P in terms of $x+by+cz=d$.

（中文）坐標空間中 xy 平面上有一正方形，其頂點為 $O(0,0,0)$、$A(8,0,0)$、$B(8,8,0)$、$C(0,8,0)$，另一點 P 在 xy 平面的上方，且與 O、A、B、C 四點的距離皆等於 6。求通過 A、B、P 三點的平面方程式 $x+by+cz=d$。

（改編自 98 年學測選填題 G)

Teacher: Since P is equidistant from $O, A, B,$ and C, P is on the normal vector through the center of square $OABC$. Where is the center H of square $OABC$?



Student: H is $(4,4,0)$.

Teacher: Assume the coordinate of P is $(4,4,k)$, let's use distance formula to find k.

Given $PA=6$, we get,

$$\sqrt{(4-8)^2+(4-0)^2+(k-0)^2}=6$$

Square both sides.

$16+16+k^2=36$

Simplify.

$k^2=4$

Since P is above xy plane, we only keep $k=2$. Next let's find the equation of a plane E through A, B and P. First, find $\overline{AB}$.

Student: $\overline{AB}=(0,8,0)$.

Teacher: Good. Then, find $\overrightarrow{AP}$.

Student: $\overrightarrow{AP}=(-4,4,2)$.

Teacher: Great! Here comes the most important part of solving this question. How do we find a vector orthogonal to $\overrightarrow{AP}$ and $\overrightarrow{AB}$?

Student: By finding the cross product of $\overrightarrow{AP}$ and $\overrightarrow{AB}$.

Teacher: Excellent! Compute

$$\overrightarrow{AB}\times\overrightarrow{AP}$$
$$=\left(\begin{vmatrix}8 & 0\\ 4 & 2\end{vmatrix},\begin{vmatrix}0 & 0\\ 2 & -4\end{vmatrix},\begin{vmatrix}0 & 8\\ -4 & 4\end{vmatrix}\right)$$
$$=(16,0,-32)$$
$$=16(1,0,-2)$$

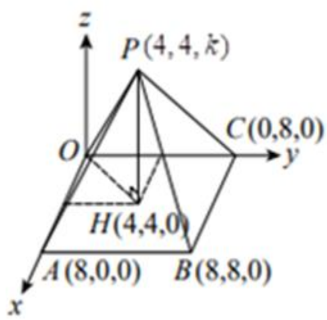
Observe that the question asks us to write the equation of the plane as $x+by+cz=d$, so we can consider the normal vector of plane as $\vec{n}=(1,0,-2)$.

Student: Therefore, I can plug $A(8,0,0)$ and $\vec{n}=(1,0,-2)$ into point-normal vector form

$$1(x-8)+0(y-0)-2(z-0)=0$$
$$x-2z=9$$

Teacher: Awesome!

老師： 因為 P 與 $O, A, B,$ 和 C 等距，P 在過正方形 $OABC$ 中心的法向量上面。試找到 $OABC$ 的中心 H？



學生： H 坐標為 $(4,4,0)$。

老師： 設 P 點坐標為 $(4,4,k)$，可利用距離公式求得 k。

已知 $PA=6$，

$$\sqrt{(4-8)^2+(4-0)^2+(k-0)^2}=6$$

兩邊平方得到 $16+16+k^2=36$。

接著化簡，$k^2=4$

因為 P 在 xy 平面的上面，$k=2$. 接下來我們來找平面 E 過 A、B 和 P 的方程式。先求出$\overrightarrow{AB}$。

學生： $\overrightarrow{AB}=(0,8,0)$。

老師： 很好，接下來求出 $\overrightarrow{AP}$。

學生： $\overrightarrow{AP}=(-4,4,2)$。

老師： 太棒了！那如何找到同時與 $\overrightarrow{AP}$ 和 $\overrightarrow{AB}$ 正交的向量呢？

學生： 我們可以用$\overrightarrow{AP}$和 $\overrightarrow{AB}$外積求得。

老師： 厲害啊！計算這兩向量的外積如下：

$$\begin{aligned}&\overrightarrow{AB}\times\overrightarrow{AP}\\&=\left(\begin{vmatrix}8&0\\4&2\end{vmatrix},\begin{vmatrix}0&0\\2&-4\end{vmatrix},\begin{vmatrix}0&8\\-4&4\end{vmatrix}\right)\\&=(16,0,-32)\\&=16(1,0,-2)\end{aligned}$$

我們同時觀察到題目所給的方程式 $x+by+cz=d$ 中，x 係數為 1，所以我們可以取 $\vec{n}=(1,0,-2)$ 當作該平面的法向量。

學生： 所以我可以將 $A(8,0,0)$ 及 $\vec{n}=(1,0,-2)$ 代入點法式

$$1(x-8)+0(y-0)-2(z-0)=0$$
$$x-2z=9$$

老師： 你做對了！

單元六　空間中的直線
Lines in Space

國立新竹科學園區實驗高級學校　陳立業老師

■ 前言 Introduction

空間中的直線可以用跟此直線平行的一個方向向量及線上一點來表示，而表示的方式除了有直線的參數式、比例式，也可以用兩平面的聯立方程式。本節接下來討論直線與平面的關係，我們可以由直線與平面的方程式來決定該直線與該平面是否交於一點，平行或重合呢？接著討論的是直線與直線的關係，也是透過兩直線的方程式來決定此兩直線是否交於一點，平行或歪斜呢？最後是如何計算空間中點到直線的距離，兩平行線的距離及兩歪斜線的距離。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
nonzero vector	**非零向量**	**projection point**	**投影點**
parallel	平行	skew	歪斜
direction vector	方向向量	coincide	重合
parameter	參數	common perpendicular	公垂線
dot product	內積	common perpendicular segment	公垂線段
cross product	**外積**	**foot of perpendicular**	**垂足**

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ If _____ and ______ have coordinates ____ and ____, find the parametric(symmetric) equation of line _____.

例句：**If** A **and** B **have coordinates** $A(1, 2, 3)$ **and** $B(2, -1, 4)$, **find the parametric (symmetric) equation of line** AB.

設 A、B 兩點的坐標為 $A(1, 2, 3)$、$B(2, -1, 4)$，求直線 AB 的參數式(比例式)。

❷ Given two planes _____ and ______ intersecting at line_____, find the parametric equation of line ______

例句：**Given two planes** $E_1: x + 3y - 2z + 5 = 0$ **and** $E_2: 3x + 4y - z + 2 = 0$ **intersecting at line** L, **find the parametric equation of line** L.

若兩平面 $E_1: x + 3y - 2z + 5 = 0$ 和 $E_2: 3x + 4y - z + 2 = 0$ 交於直線 L，求出直線 L 的參數式。

❸ Given a plane ____ and a line ____, tell whether the line intersects, is parallel to or lies on plane_____.

例句：**Given a plane** $E_1: 3x + y - 2z + 7 = 0$ **and a line** $L_1: \begin{cases} x = 2 + t \\ y = 3 - t \\ z = 1 + 4t \end{cases}$, **tell whether the line** L_1 **intersects, is parallel to or lies on plane** E_1.

已知平面 $E_1: 3x + y - 2z + 7 = 0$ 和直線 $L_1: \begin{cases} x = 2 + t \\ y = 3 - t \\ z = 1 + 4t \end{cases}$，試判別直線 L_1 與平面 E_1 的關係。

❹ Find the point of projection of ______ on plane _______.

例句：**Find the point of projection of** $A(1, 0, 4)$ **on plane** $E: 3x - y + 2z - 3 = 0$.

求 $A(1, 0, 4)$ 在平面 $E: 3x - y + 2z - 3 = 0$ 的投影點。

❺ Given plane_____ containing a point _____ and a line_______, find the equation of plane ______.

例句：**Given plane** E **containing** $A(2, 7, 4)$ **and** $L: \frac{x-1}{3} = \frac{y-6}{1} = \frac{z+2}{2}$**, find the equation of plane** E**.**

已知平面 E 包含點 $A(2, 7, 4)$ 及直線 $L: \frac{x-1}{3} = \frac{y-6}{1} = \frac{z+2}{2}$，試求平面 E 的方程式。

❻ Tell the relationship between two lines _______ and _________.

例句：**Tell the relationship between two lines** $L_1: \frac{x-1}{3} = \frac{y-6}{1} = \frac{z+2}{2}$ **and** $L_2: \frac{x+3}{1} = \frac{y+2}{3} = \frac{z-1}{2}$**.**

判別兩直線 $L_1: \frac{x-1}{3} = \frac{y-6}{1} = \frac{z+2}{2}$ 及 $L_2: \frac{x+3}{1} = \frac{y+2}{3} = \frac{z-1}{2}$ 的關係。

❼ Find the distance between a point _______ and a line ___________.

例句：**Find the distance between** $A(1, 1, 5)$ **and** $L: \frac{x+3}{1} = \frac{y+2}{3} = \frac{z-1}{2}$**.**

求點 $A(1, 1, 5)$ 到直線 $L: \frac{x+3}{1} = \frac{y+2}{3} = \frac{z-1}{2}$ 的距離。

■ 問題講解 Explanation of Problems

❧ 說明 ☙

In the previous lesson, we used a normal vector and a point to find an equation of the plane. Now, we also use a vector, the direction vector, and a point to find an equation of a line. There are three ways to express the equation of a line in space which are parametric equation, symmetric equation and the intersection of two planes.

There are three scenarios for a line and a plane in space. First, a line can intersect a plane at one point. Second, a line can be parallel to a plane which also means there is no intersection between them. Third, a line can lie on a plane or the plane contains this line which means there are infinitely many intersections between them. To decide which of the relationships these lines and planes have.

There are four scenarios for a line and another line in space. If two lines are on the same plane, then they can either intersect, parallel, or overlap(coincide). If two liners are not on the same plane, we say they are skewed to each other. To decide which of the relationships these two lines have, we plug the equation of a line into another one to check how many solutions we find along with checking to see if their direction vectors are in proportion.

How do we find the distance from a point A to a line L_1? Construct line L_2 perpendicular to L_1 through A and name the intersection (also known as the foot of perpendicular) as B. Then the length of $\overline{AB}$ is the distance from a point A to a line L_1. To find the distance from two skew lines takes more effort. To begin with, we need to find a common perpendicular to those skew lines. Next, we find the intersections between the perpendicular and two skew lines. In the end, the distance between two intersections is the distance between these two skew lines.

運算問題的講解

例題一

說明：本題是用直線上的兩點先求出方向向量再來求得直線的方程式。

（英文）If A and B have coordinates $A(1, 0, 2)$ and $B(2, -1, 4)$, find the parametric equation of line AB.

（中文）設 A、B 兩點的坐標為 $A(1, 0, 2)$、$B(2, -1, 4)$，求直線 AB 的參數式。

Teacher: Once we know the direction vector and a point of a line, we can find the equation of a line. What is the direction vector of a line?

Student: A direction vector is a vector parallel to the line.

Teacher: Great! Since A and B are on $\overleftrightarrow{AB}$, we can use $\overrightarrow{AB}$ as a direction vector of $\overleftrightarrow{AB}$. What is $\overrightarrow{AB}$?

Student: $\overrightarrow{AB} = (1, -1, 2)$.

Teacher: What is the given point here?

Student: There are two given points, $A(1, 0, 2)$ and $B(2, -1, 4)$. Which one should I use?

Teacher: Either A or B is fine. We can use the parametric equation form of the line,

$L: \begin{cases} x = x_0 + at, \\ y = y_0 + bt, \\ z = z_0 + ct, \end{cases}$ to express this line, where (a, b, c) is a direction vector and (x_0, y_0, z_0) is a point on this line. Now, you can plug the direction vector $\overrightarrow{AB} = (1, -1, 2)$ and $A(1, 0, 2)$ to line L.

Student: The equation is $L: \begin{cases} x = 1 + t \\ y = 0 - t \\ z = 2 + 2t \end{cases}$.

Teacher: Excellent.

老師：當我們知道該直線的一個方向向量及直線上的一點時，我們就能寫出該直線的方程式了。直線的方向向量是什麼嗎？

學生：直線的方向向量是與該線平行的向量。

老師：　很好！因為 A 和 B 在直線 $\overleftrightarrow{AB}$ 上，我們可以用向量 $\overline{AB}$ 當作 $\overleftrightarrow{AB}$ 的一個方向向量。請算出向量 $\overline{AB}$？

學生：　$\overline{AB}=(1,-1,2)$。

老師：　題目所給的直線上的點為何？

學生：　有兩個點 $A(1,0,2)$ 和 $B(2,-1,4)$。我該用哪一個呢？

老師：　A 或 B 都可以。接下來可以利用直線的參數式，

$L:\begin{cases} x=x_0+at, \\ y=y_0+bt, \\ z=z_0+ct, \end{cases}$，其中 (a,b,c) 為直線的方向向量而 (x_0,y_0,z_0) 為直線上任一點。

現在我們可將 $\overline{AB}=(1,-1,2)$ 及 $A(1,0,2)$ 代入此參數式中。

學生：　代入後我得到 $L:\begin{cases} x=1+t \\ y=0-t \\ z=2+2t \end{cases}$。

老師：　你答對了。

例題二

說明：本題是判斷直線與平面的關係。

（英文）Given a plane $E_1: 3x+y-2z+7=0$ and a line $L_1:\begin{cases} x=2+t \\ y=3-t \\ z=1+4t \end{cases}$, tell whether the line L_1 intersects, is parallel to or lies on plane E_1.

（中文）已知平面 $E_1: 3x+y-2z+7=0$ 和直線 $L_1:\begin{cases} x=2+t \\ y=3-t \\ z=1+4t \end{cases}$，試判別直線 L_1 與平面 E_1 是交於一點、平行、還是直線落在平面上。

Teacher: Generally speaking, we plug the parametric equation of a line into the equation of plane to solve for the parameter t. There are three possible scenarios,

1. If there is only one solution of the parameter t, then the line intersects the plane at one point.
2. If there is no real solution, then the line is parallel to the plane.
3. If there are infinitely many solutions, then the line lies on the plan.

Assume $P(2+t,3-t,1+4t)$ is on L_1 and plug that into $E_1:3x+y-2z+7=0$, what do you get?

Student: I get $3(2+t)+(3-t)-2(1+4t)+7=0$

Teacher: Solve for t.

Student: $6+3t+3-t-2-8t+7=0$

$-6t=-14$

$t=\dfrac{-14}{-6}=\dfrac{7}{3}$

Teacher: How many solutions are there?

Student: There is only one solution. So the L_1 intersect E at one point.

Teacher: Well done!

老師： 一般來說我們將直線 L 的參數式代入平面 E 的方程式後，來解參數 t ：

1、若 t 只有一個實數解，則該直線與平面交於一點。

2、若 t 無實數解，則該直線與平面平行。

3、若 t 為任意實數，則該直線落在平面上。

假設 $P(2+t,3-t,1+4t)$ 為 L_1 上任一點，將 P 點代入平面 $E_1:3x+y-2z+7=0$

你得到什麼？

學生： 我得到 $3(2+t)+(3-t)-2(1+4t)+7=0$

老師： 試著將 t 解出來。

學生： $6+3t+3-t-2-8t+7=0$

$-6t=-14$

$t=\dfrac{-14}{-6}=\dfrac{7}{3}$

老師： 有多少解呢？

學生： 只有一解，所以 L_1 交平面 E 於一點。

老師： 答對了！

例題三

說明：本題是要求出一點到另一直線的距離。

（英文）Find the distance from $P(1, 2, 3)$ to line $L: \frac{x+3}{1} = \frac{y+2}{3} = \frac{z-1}{2}$.

（中文）求點 $P(1, 2, 3)$ 到直線 $L: \frac{x+3}{1} = \frac{y+2}{3} = \frac{z-1}{2}$ 的距離。

Teacher: How do we find the distance from a point P (not on L) to a line L? First, we draw line L_1 perpendicular to L through P, and name the foot of perpendicular as B. Then the length of $\overline{PB}$ is the distance from a point P to a line L.

Assume $B(-3+t, -2+3t, 1+2t)$ is on L, find $\overrightarrow{PB}$.

Student: $\overrightarrow{PB} = (-3+t-1, -2+3t-2, 1+2t-3) = (-4+t, -4+3t, -2+2t)$

Teacher: Since $\overrightarrow{PB}$ is orthogonal to L, what is the dot product of $\overrightarrow{PB}$ and the direction vector of L?

Student: The dot product is 0.

Teacher: Good! So we can use $(-4+t, -4+3t, -2+2t)\cdot(1,3,2) = 0$ to solve for t.

$$(-4+t)\cdot 1 + (-4+3t)\cdot 3 + (-2+2t)\cdot 2 = 0$$
$$-4+t-12+9t-4+4t = 0$$
$$14t = 20$$
$$t = \frac{10}{7}$$

Now, plug $t = \frac{10}{7}$ into $B(-3+t, -2+3t, 1+2t)$.

Student: B has coordinates

$$B\left(-3+\frac{10}{7}, -2+3\cdot\frac{10}{7}, 1+2\cdot\frac{10}{7}\right)$$
$$= \left(-\frac{1}{7}, \frac{16}{7}, \frac{27}{7}\right).$$

Teacher: Here comes the last part, using the distance formula to find $\overline{PB}$. Would you like to do this part by yourself?

Student: Yes, I can do it.

$$\overline{PB}=\sqrt{\left(1-\left(-\frac{1}{7}\right)\right)^2+\left(2-\frac{16}{7}\right)^2+\left(3-\frac{27}{7}\right)^2}$$
$$=\sqrt{\frac{56}{49}+\frac{4}{49}+\frac{36}{49}}$$
$$=\sqrt{\frac{96}{49}}$$
$$=\frac{4}{7}\sqrt{6}$$

Teacher: You are awesome!

老師：如果要求得一點 P 到另一直線 L 的距離，我們先找到過該點與直線 L 的垂線 L_1，並找到垂足 B。那麼 $\overline{PB}$ 的長度即為點 P 到另一直線 L 的距離。假設直線 L 上的垂足 $B(-3+t, -2+3t, 1+2t)$，先求出 $\overrightarrow{PB}$。

學生：$\overrightarrow{PB}=(-3+t-1,-2+3t-2,1+2t-3)=(-4+t,-4+3t,-2+2t)$

老師：因為$\overline{PB}$與直線L正交，$\overrightarrow{PB}$和直線L方向向量的內積為何？

學生：它們的內積為 0。

老師：很好。接下來用內積為 0，$(-4+t,-4+3t,-2+2t)\cdot(1,3,2)=0$ 來找t.

$$(-4+t)\cdot 1+(-4+3t)\cdot 3+(-2+2t)\cdot 2=0$$
$$-4+t-12+9t-4+4t=0$$
$$14t=20$$
$$t=\frac{10}{7}$$

我們可以將$t=\frac{10}{7}$代入$B(-3+t,-2+3t,1+2t)$.

學生：B 坐標為

$$B\left(-3+\frac{10}{7},-2+3\cdot\frac{10}{7},1+2\cdot\frac{10}{7}\right)$$
$$=\left(-\frac{1}{7},\frac{16}{7},\frac{27}{7}\right)$$

老師：　最後，用兩點距離公式來求 $\overline{PB}$ 。你要試著做做看嗎？

學生：　好的，我可以。

$$\overline{PB}=\sqrt{\left(1-\left(-\frac{1}{7}\right)\right)^2+\left(2-\frac{16}{7}\right)^2+\left(3-\frac{27}{7}\right)^2}$$

$$=\sqrt{\frac{56}{49}+\frac{4}{49}+\frac{36}{49}}$$

$$=\sqrt{\frac{96}{49}}$$

$$=\frac{4}{7}\sqrt{6}$$

老師：　太棒了！

≈ 應用問題 / 學測指考題 ≈

例題一

說明：這題利用直線的參數式及配方法來求解。

（英文）There are two fighter jets, Jet X and Jet Y, flying in formation. Assume that both jets fly straight with constant speed in the air. The initial position for Jet X is at $A(1, 2, -1)$ and the initial position for Jet Y is at $C(0, 8, 4)$. Jet X flies toward $B(5, 8, -3)$ after 2 seconds and Jet Y flies toward $D(2, 10, 3)$ after 1 second.

(a) Where are Jets X and Y after t seconds?

(b) When will Jet X be closest to Jet Y?

（中文）有兩架飛機在空中進行分列式表演，在某個小範圍的空域中，兩架飛機沿直線等速飛行。根據塔台的觀測資料顯示：在同一個時刻，甲飛機一開始 $A(1, 2, -1)$，2 秒之後飛到 $B(5, 8, -3)$；乙飛機一開始在 $C(0, 8, 4)$，1 秒之後飛到 $D(2, 10, 3)$。

(1)若 t 秒後甲飛機、乙飛機分別飛到 P、Q 兩個位置，請用 t 表示 P、Q 兩個位置的坐標。

(2)試問當甲、乙兩架飛機飛行了幾秒後，距離會最近。

Teacher: Since both jets fly straight in the air, we can assume Jet X flies on line $\overleftrightarrow{AC}$ and Jet Y flies on line $\overleftrightarrow{BD}$. Because we also need to tell the position of each jet at a specific time t, it will be convenient to use parametric equations to express both lines. To begin with, what are the direction vectors for both lines?

Student: I will use $\overrightarrow{AB}=(5-1,8-2,-3-(-1))=(4,6,-2)$ as the direction vector for line $\overleftrightarrow{AC}$ and $\overrightarrow{CD}=(2-0,10-8,3-4)=(2,2,-1)$ as the direction vector for line $\overleftrightarrow{BD}$.

Teacher: Let's take a look at the movement of Jet X on $\overrightarrow{AB}$ within 2 seconds. The direction vector $(4,6,-2)$ indicates the plane moves 4 units in the positive x, 6 units in the positive y and 2 units in the negative z in 2 seconds. That also means the plane will move 2 units in the positive x, 3 units in the positive y, and 1 unit in the negative z in 1 second. We can use the direction vector to indicate the movement of Jet X in 1 second. Therefore, we can use $(2,3,-1)$ to indicate the movement of Jet X in 1 second. Treat t as time in second, the parametric equation for $\overleftrightarrow{AB}$ can be written as $\begin{cases} x=1+2t \\ y=2+3t, t\in\mathbb{R} \\ z=-1-t \end{cases}$. What is the parametric equation for $\overleftrightarrow{CD}$?

Student: The parametric equation for $\overleftrightarrow{CD}$ is

$$\begin{cases} x=0+2t \\ y=8+2t, t\in\mathbb{R}. \\ z=4-t \end{cases}$$

Teacher: Good! Assume $P(1+2t,2+3t,-1-t)$ is the position of Jet X after t seconds on $\overleftrightarrow{AB}$ and $Q(2t,8+2t,4-t)$ is the position of Jet Y after t seconds on $\overleftrightarrow{CD}$, what is the distance of $\overline{PQ}$? Write your answer in terms of t.

Student: $\overline{PQ}$

$$=\sqrt{(1+2t-2t)^2+(2+3t-8-2t)^2+(-1-t-4+t)^2}$$
$$=\sqrt{1+t^2-12t+36+25}$$
$$=\sqrt{t^2-12t+62}$$

Teacher: To find the shortest distance from P to Q, we complete the square for the quadratic polynomial within the square root.

$$t^2-12t+62$$
$$=t^2-12t+36+62-36$$
$$=(t-6)^2+26$$

That is, when $t=6$, P is closest to Q.

老師： 由題目得知兩架飛機都在空中直線飛行，假設飛機 X 飛在直線 $\overleftrightarrow{AC}$ 上而飛機 Y 飛在直線 $\overleftrightarrow{BD}$ 上。如果要表示兩架飛機在 t 秒後的位置，我們可以用直線的參數式來表示。首先，請問兩直線的方向向量為何？

學生： 我想用 $\overline{AB}=(5-1,8-2,-3-(-1))=(4,6,-2)$ 當作 $\overleftrightarrow{AC}$ 的方向向量。

$\overline{CD}=(2-0,10-8,3-4)=(2,2,-1)$ 當作 $\overleftrightarrow{BD}$ 的方向向量.

老師： 但是 (4, 6, －2) 代表飛機 X 兩秒內在 x 軸正向上移動了 4 單位，y 軸正向上移動了 6 單位，z 軸負向上移動了 2 單位。也就是說飛機 X 一秒內在 x 軸正向上移動了 2 單位，y 軸正向上移動了 3 單位，z 軸負向上移動了 1 單位。所以我們可以用 $(2,3,-1)$ 來表示飛機 X 在 1 秒位置的改變。直線 $\overleftrightarrow{AB}$ 上任一點 P 在 t 秒之後的位置可以表示成 $\begin{cases} x=1+2t \\ y=2+3t, t\in\mathbb{R} \\ z=-1-t \end{cases}$. 試求出直線 $\overleftrightarrow{CD}$ 的參數式？

學生： 直線 $\overleftrightarrow{CD}$ 的參數式為：

$$\begin{cases} x=0+2t \\ y=8+2t, t\in\mathbb{R}. \\ z=4-t \end{cases}$$

老師： 很好。設 $P(1+2t,2+3t,-1-t)$ 是飛機 X 在 t 秒後的位置，$Q(2t,8+2t,4-t)$ 為飛機 Y 在 t 秒後的位置，$\overline{PQ}$ 的長度為何？以 t 表示你的答案。

學生：$\overline{PQ}$

$=\sqrt{(1+2t-2t)^2+(2+3t-8-2t)^2+(-1-t-4+t)^2}$

$=\sqrt{1+t^2-12t+36+25}$

$=\sqrt{t^2-12t+62}$

老師：要找到 P 到 Q 的最短距離，我們用配方法來求根號內二次多項式的最小值：

$t^2-12t+62$

$=t^2-12t+36+62-36$

$=(t-6)^2+26$

所以當 $t=6$，P 最靠近 Q。

例題二

說明：這題考如何判斷空間中兩直線互為歪斜線

（英文）Which of the following lines is skew to z-axis?

(1) $L_1:\begin{cases}x=0\\z=0\end{cases}$　(2) $L_2:\begin{cases}y=0\\x+z=0\end{cases}$　(3) $L_3:\begin{cases}z=0\\x+y=1\end{cases}$

(4) $L_4:\begin{cases}x=1\\y=1\end{cases}$　(5) $L_5:\begin{cases}y=1\\z=1\end{cases}$

（中文）下列各直線中，請選出和 z 軸互為歪斜線的選項。

(1) $L_1:\begin{cases}x=0\\z=0\end{cases}$　(2) $L_2:\begin{cases}y=0\\x+z=0\end{cases}$　(3) $L_3:\begin{cases}z=0\\x+y=1\end{cases}$

(4) $L_4:\begin{cases}x=1\\y=1\end{cases}$　(5) $L_5:\begin{cases}y=1\\z=1\end{cases}$

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Teacher: Let's review the definition of skew lines in space. What can we say about two skew lines in space?

Student: Two lines are skew if they are not parallel and they don't intersect.

Teacher: That also means if two lines are parallel or intersect, then they are not skewed. To check whether two lines are parallel, we can see if one's direction vector is a scalar multiple of the others. To check whether two lines intersect, we can assume a point $P(0,0,t)$ on z-axis and plug P into the lines to see if there is any solution.

Now, we plug $P(0,0,t)$ into L_1.

Student: I get $t=0$, so there is an intersection between L_1 and z-axis and they are not skew.

Teacher: Plug $P(0,0,t)$ into L_2.

Student: I also get $t=0$, so there is an intersection between L_2 and z-axis and they are not skew.

Teacher: plug $P(0,0,t)$ into L_3.

Student: Since $0+0\neq 1$, so there is no intersection between L_3 and z-axis. The direction vector of L_3 is $(1,-1,0)$ which is not a scalar multiple of the direction vector of the z-axis, $(0,0,1)$, so they are not parallel. We can conclude that L_3 is skewed to the z-axis.

Teacher: Plug $P(0,0,t)$ into L_4.

Student: Since $0\neq 1$, so there is no intersection between L_4 and z-axis. The direction vector of L_4 is $(0,0,1)$ which is a scalar multiple of the direction vector of z-axis, $(0,0,1)$, so they are parallel. We can conclude that L_4 is not skewed to the z-axis.

Teacher: Plug $P(0,0,t)$ into L_5.

Student: Since $0+0\neq 1$, so there is no intersection between L_5 and z-axis. The direction vector of L_5 is $(1,0,0)$ which is not a scalar multiple of the direction vector of the z-axis, $(0,0,1)$, so they are not parallel. We can conclude that L_5 is skewed to the z-axis.

Teacher: In summary, L_3 and L_5 are skewed to the z-axis.

老師： 我們先複習歪斜線的定義，空間中兩直線怎樣算是歪斜？

學生： 兩歪線不相交也不平行。

老師： 也就是說兩直線若相交或是平行就不算是歪斜線了。要看說兩直線是否平行，我們可以比較兩直線的方向向量是否成比例。如果要知道說兩直線是否相交，我們假設 z 軸上一點 $P(0,0,t)$ ，並將 P 代入到下列的直線方程式去求解. 將 $P(0,0,t)$ 代入 L_1.

學生： 我得到 $t=0$, 所以 L_1 和 z 軸有相交，它們不是互為歪斜線。

老師： 將 $P(0,0,t)$ 代入 L_2.

學生： 我也得到 $t=0$, 所以 L_2 和 z 軸有相交，它們不是互為歪斜線。

老師： 將 $P(0,0,t)$ 代入 L_3.

學生： 因為 $0+0\neq 1$,所以 L_3 和 z 軸不相交。 L_3 的方向向量為 $(1,-1,0)$ 也不是 z 軸方向向量 $(0,0,1)$ 的倍數，所以它們沒有平行。由以上我們可以得知 L_3 與 z 軸互為歪斜.

老師： 將 $P(0,0,t)$ 代入 L_4.

學生： 因為 $0\neq 1$,所以 L_4 和 z 軸不相交。 L_4 的方向向量為 $(0,0,1)$ 是 z 軸方向向量 $(0,0,1)$ 的倍數，所以它們平行。由以上我們可以得知 L_4 與 z 軸為平行.

老師： 將 $P(0,0,t)$ 代入 L_5.

學生： 因為 $0+0\neq 1$,所以 L_5 和 z 軸不相交。L_5 的方向向量為 $(1,0,0)$ 也不是 z 軸方向向量 $(0,0,1)$ 的倍數，所以它們沒有平行。由以上我們可以得知 L_5 與 z 軸互為歪斜.

老師： 總結 L_3 與 L_5 與 z 軸互為歪斜。.

單元七　條件機率與獨立事件
Conditional Probability and Independent Events

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■　前言 Introduction

為了要準備畢業典禮，班上同學分成場佈組及表演組。如果我們想要隨機指派一名學生致詞，選到住宿的同學的機率是多少？但如果我們知道致詞的學生來自於場佈組時，選到住宿的同學去致詞的機率變成多少呢？是否跟前一個問題的答案一樣呢？在第二個問題中「來自於場佈組」是我們新得到的一個條件，計算已知另一個條件的情況下該事件會發生的機率稱作條件機率。若一事件 A 的發生不影響另一事件 B 發生的機率，我們稱事件 A 及 B 是獨立事件，我們在此節中也會學習判斷兩事件是否獨立。

■　詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
conditional probability	**條件機率**	draw without replacement	取後不放回
sample space	樣本空間	**independent events**	**獨立事件**
event	事件	dependent events	相依事件
intersection	交集	complement	餘集
union	聯集	**disjoint (mutually exclusive)**	**互斥**

multiplication rule	機率的乘法原理	coin	硬幣
general addition rule	加法原理	die (dice)	骰子(多個骰子)

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ **What is the probability that _____________ given that ___________?**

例句：**What is the probability that** the card is a diamond **given that** it is red**?**
在抽到紅卡的條件下，抽到的卡是方塊的機率為何？

❷ **What is the probability that ______________ when rolling a die?**

例句：**What is the probability that** you get a 3 **when rolling a die?**
擲一公正骰子時，擲出 3 點的機率為何？

❸ **What is the probability that ___________ and ___________?**

例句：**What is the probability that t**he card is a queen **and** it is red**?**
抽到的卡片為皇后且為紅色的機率為何？

❹ **What is the probability that the _______ you draw is _______ if you draw the _______ from the _____ without replacement?**

例句：**What is the probability that you the** 2nd marble **you draw is** red **if you draw the** marbles **from the** bag **without replacement?**
當你從袋中取球且不放回時，第二次抽到紅球的機率為何？

❺ **Show that _____ and ______ are independent.**

例句：**Show that** event A **and** B **are independent.**

試證 A、B 兩事件獨立。

❻ **Given A and B are independent, and $P(A)$ = ___, $P(B)$ = ___, find $P(A \cap B)$.**

例句：**Given** A **and** B **are independent events, and** $P(A) = 0.2$ **,** $P(B) = 0.3$**, find** $P(A \cap B)$**.**

若 A、B 兩事件獨立，且 $P(A) = 0.2$、$P(B) = 0.3$，試求 $P(A \cap B)$。

❼ **Flip a coin _____________.**

例句：**Flip a coin** three times.

擲一個公正的硬幣 3 次。

■ 問題講解 Explanation of Problems

說明

In this lesson, we will explore, conditional probability and independent events.

Assume we randomly select a student from our class to answer a question, what is the probability the selected student is a girl? From the theoretical probability, we know the answer to this question is the number of female students divided by the number of total students in class. But if we separate our class into two groups, one with odd seat numbers and the other with even seat numbers, what is the probability that a female student will be selected given that her seat number is even? From the example, this additional condition “with even seat numbers” will affect the result when we calculate the probability. A probability that considers a given condition such as this is called a conditional probability. To find the probability that event A happens given that event B happens, we use the notation $P(A|B)$ (read as the probability of A given B) to calculate the conditional probability. The formula $P(A|B)$ is $\frac{P(A \cap B)}{P(B)}$.

What if $P(A|B) = P(A)$? That means the outcome of event B will not affect the probability of event A. We also call that event A and event B are independent events. Furthermore, If A and B are independent, to find the probability that both A and B happens, we can find the product of $P(A)$ and $P(B)$. That is $P(A \cap B) = P(A)P(B)$.

Alternatively, we can use $P(A \cap B) = P(A)P(B)$ to show A and B are independent.

運算問題的講解

例題一

說明：本題是在玩撲克牌的情境下導入條件機率的問題。

（英文）You draw a card at random from a standard deck of 52 cards. The probability that each card is drawn is the same. Find each of the following conditional probabilities:

a) The card is a diamond, given that it is red.

b) The card is a king, given that it is a face card.

（中文）一副撲克牌中有 52 張卡，假設每張卡抽到的機會相等。試回答下列問題。

a) 已知抽到紅牌的條件下，求抽到方塊的機率。

b) 已知抽到花牌的條件下，求抽到國王的機率。

Teacher: Let's have a brief introduction about a deck of cards. There are 52 cards, including four suits: spade, heart, diamond and club. Spades and clubs are black. Hearts and diamonds are red. Each suit contains 13 cards, labeled as Ace, 2, 3, …, 10, J, Q, and K. Face cards are J(Jack), Q(Queen) and K(King). Assume S is the sample space, and $n(S) = 52$. What does part(a) ask for?

Student: What is the probability that the card you draw is a diamond given that it is red?

Teacher: Assume A represents the event you draw a diamond, B represents the event you draw a red card and $A \cap B$ represents the event you draw a red and diamond card. What is $P(B)$ and $P(A \cap B)$?

Student: Since there are 26 red cards, $P(B) = \frac{26}{52} = \frac{1}{2}$.

Teacher: Way to go. What is $P(A \cap B)$?

Student: Since there are 13 red diamond cards, $P(A \cap B) = \frac{13}{52} = \frac{1}{4}$.

Teacher: To find a conditional probability of A given B, we calculate $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$. For part(b), Assume C represents the event you draw a king, D represents the event you draw a face card and $C \cap D$ represents the event you draw a king and face card, what is $P(D)$and $P(C \cap D)$?

Student: Since there are 12 face cards, $P(D) = \frac{12}{52}$.

Teacher: Great. What is $P(C \cap D)$?

Student: Since there are 4 face cards which are king, $P(C \cap D) = \frac{4}{52}$.

Teacher: To find a conditional probability of C given D, we calculate $P(C|D) = \frac{P(C \cap D)}{P(D)} = \frac{\frac{4}{52}}{\frac{12}{52}} = \frac{1}{3}$.

老師：我們先來簡單介紹一下一副撲克牌裡面有哪些卡。一副撲克卡裡有 52 張卡，分成 4 種花色分別為黑桃，紅心，方塊及梅花。黑桃和梅花是黑色的卡，而紅心及方塊是紅色的卡。每種花色皆有 13 張，從 Ace, 2, 3, … 10, J, Q, 一直到 K。花牌則是 J、Q、和 K。假設 S 為樣本空間，則 $n(S) = 52$，題目(a)在問什麼？

學生：已知抽到紅牌的條件下，求抽到方塊的機率。

老師：假設事件 A 為抽到方塊的事件，事件 B 是抽到紅牌的事件 $A \cap B$ 則代表同時抽到紅牌跟方塊的事件。試求出 $P(B)$及 $P(A \cap B)$？

學生：因為有 26 張紅牌，$P(B) = \frac{26}{52} = \frac{1}{2}$。

老師：不錯哦！試找到 $P(A \cap B)$？

學生：因為有 13 張紅色且為方塊的卡，所以 $P(A \cap B) = \frac{13}{52} = \frac{1}{4}$。

老師：為了求出在 B 事件發生的條件下，A 事件發生的機率，我們計算條件機率如下：

$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$。同樣的，在問題(b)中，假設 C 是抽到國王的事件，D 代表抽到花牌的事件，而 $C \cap D$ 同時抽到國王和花牌的事件。試求出 $P(D)$ 及 $P(C \cap D)$？

學生：因為有 12 張花牌，所以 $P(D) = \frac{12}{52}$。

老師：　很好。試求出 $P(C \cap D)$？

學生：　因為有 4 張國王的花牌，所以 $P(C \cap D) = \frac{4}{52}$。

老師：　我們來計算在 D 發生的條件下，C 發生的機率，

$$P(C|D) = \frac{P(C \cap D)}{P(D)} = \frac{\frac{4}{52}}{\frac{12}{52}} = \frac{1}{3}。$$

例題二

說明： 本題是判斷兩事件是否獨立或互斥。

（英文）Roll a fair die. Let $A = \{1, 2\}$ be the event of getting a 1 or 2, $B = \{2, 3, 4\}$ be the event of getting a 2, 3, or 4, and $C = \{4, 5, 6\}$ be the event of getting a 4, 5, or 6. Answer the following questions:

(a) Are A and B independent? Disjoint?

(b) Are A and C independent? Disjoint?

（中文）擲一公正骰子，令 $A = \{1, 2\}$ 出現 1 點或 2 點的事件，$B = \{2, 3, 4\}$ 出現 2 點、3 點、或 4 點的事件，而 $C = \{4, 5, 6\}$ 是出現 4 點、5 點、或 6 點的事件，判斷下列結果：

(1) 事件 A、B 是否獨立？是否互斥？

(2) 事件 A、C 是否獨立？是否互斥？

Teacher:　Since $A = \{1, 2\}$ and $B = \{2, 3, 4\}$, can you tell me what $A \cap B$ is?

Student:　$A \cap B = \{2\}$.

Teacher:　Good! Are A and B disjoint?

Student:　No, they are not disjoint. Because 2 is in both A and B.

Teacher:　Great! If the intersection of two events is not an empty set, then they are not disjoint. Next, find $P(A)$, $P(B)$ and $P(A \cap B)$.

Student:　$P(A) = \frac{2}{6} = \frac{1}{3}$, $P(B) = \frac{3}{6} = \frac{1}{2}$, $P(A \cap B) = \frac{1}{6}$.

Teacher:　To show whether A and B are independent, we can check whether $P(A \cap B) = P(A)P(B)$.

Student:　I found out that $P(A \cap B) = \frac{1}{6} = \frac{1}{3} \times \frac{1}{2} = P(A)P(B)$, so A and B are independent.

Teacher:　Way to go. Now, find $P(C)$ and $A \cap C$.

Student: $P(C)=\frac{3}{6}=\frac{1}{2}$. $A\cap C=\phi$, So A and C are disjoint.

And $P(A\cap C)=0\neq\frac{1}{3}\times\frac{1}{2}=P(A)P(C)$, so A and C are not independent.

Teacher: Excellent!

老師： 因為 $A=\{1,2\}$，$B=\{2,3,4\}$。試找出 $A\cap B$？

學生： $A\cap B=\{2\}$。

老師： 很好！A 和 B 是否互斥？

學生： 不，他們不互斥。因為 2 同時在 A 和 B 這兩個事件中。

老師： 很棒哦！如果兩集合的交集非空集合，則他們不互斥。接下來，算出 $P(A)$、$P(B)$及 $P(A\cap B)$。

學生： $P(A)=\frac{2}{6}=\frac{1}{3}$、$P(B)=\frac{3}{6}=\frac{1}{2}$、$P(A\cap B)=\frac{1}{6}$。

老師： 要判斷 A，B 兩集合是否獨立，我們可以算算看 $P(A\cap B)=P(A)P(B)$ 是否成立。若成立的話，則他們為獨立。

學生： 我發現 $P(A\cap B)=\frac{1}{6}=\frac{1}{3}\times\frac{1}{2}=P(A)P(B)$，所以 A 和 B 為獨立事件。

老師： 厲害哦！那我們來算$P(C)$ 及 $A\cap C$。

學生： $P(C)=\frac{3}{6}=\frac{1}{2}$，$A\cap C=\phi$，所以 A 和 C 互斥。

而$P(A\cap C)=0\neq\frac{1}{3}\times\frac{1}{2}=P(A)P(C)$，所以 A 和 C 相依。

老師： 你答對了！

例題三

說明：判斷兩事件為獨立事件。

（英文）A survey found that 80% of Taiwanese people have a home phone, 70% have a cell phone, and 56% have both.

(1) Are having a home phone and having a cell phone independent events?

(2) What is the probability that a Taiwanese has a home phone given that she/he has a cell phone?

（中文）一項調查指出 80%的台灣民眾有市話，而 70%的民眾有手機門號，56%的民眾則同時兩者都有。

(1) 有市話及有手機門號是否為獨立事件？

(2) 在有手機門號的條件下，台灣民眾有市話的機率為何？

Teacher: Let H represent that a Taiwanese person has a home phone, and C represent that a Taiwanese person has a cell phone. What is $P(H)$, $P(C)$ and $P(H \cap C)$?

Student: Given the mention of "80% of Taiwanese have a home phone" and "70% have a cell phone", I can get $P(H)=0.8$ and $P(C)=0.7$. Since it also mentions "56% have both", so $P(H \cap C)=0.56$.

Teacher: Excellent! To determine whether H and C are independent, we check whether $P(H) \cdot P(C) = P(H \cap C)$. Here, we have

$P(H) \cdot P(C)$

$= 0.8 \cdot 0.7$

$=0.56$

$= P(H \cap C)$

Teacher: Thus, having a home phone and having a cell phone are independent events.

Teacher: Next, let's find the probability, $P(H|C)$, that a Taiwanese has a home phone given that she/he has a cell phone. How do we calculate $P(H|C)$?

Student: $P(H|C)=\dfrac{P(H\cap C)}{P(C)}=\dfrac{0.56}{0.7}=0.8$.

Teacher: Good job! Additionally, we notice that $P(H|C)=P(H)$, indicating that the occurrence of C does not impact the probability of H occurring.

老師： 假設 H 為台灣民眾有市話的事件，C 為有手機門號的事件。

請問 $P(H)$、$P(C)$ 及 $P(H\cap C)$ 分別為多少。

學生： 因為題目有提到「80%的台灣民眾有市話」，所以 $P(H)=0.8$。題目也提到說「70%的民眾有手機門號」，所以 $P(C)=0.7$。最後它也有提到「56%的民眾則同時兩者都有」，所以 $P(H\cap C)=0.56$。

老師： 很好。要判斷 H 和 C 為獨立事件，我們檢查一下 $P(H)\cdot P(C)=P(H\cap C)$ 這個式子是否成立。

$P(H)\cdot P(C)$

$=0.8\cdot 0.7$

$=0.56$

$=P(H\cap C)$

老師： 所以，有市話及有手機門號為獨立事件。

老師： 接下來，我們要找出有手機門號的條件下，台灣民眾有市話的機率 $P(H|C)$ 為何。如何計算 $P(H|C)$?

學生： $P(H|C)=\dfrac{P(H\cap C)}{P(C)}=\dfrac{0.56}{0.7}=0.8$.

老師： 算對了！我們也觀察到 $P(H|C)=P(H)$，代表有沒有手機門號不會影響有市話這個事件發生的機率。

꧁ 應用問題 / 學測指考題 ꧂

例題一

說明：考機率的概念及條件機率的算法。

（英文）Assume A and B are two events of the sample space S such that $P(A) = P(B) = 0.6$. Which of the following is true?

(1) $P(A \cup B) = 1$

(2) $P(A \cap B) = 0.2$

(3) $P(A|B) = 1$

(4) $P(A|B) = P(B|A)$

(5) A and B are independent.

（中文）今設 A 和 B 為樣本空間中的兩個事件，已知$P(A) = P(B) = 0.6$，請選出正選的選項。

(1) $P(A \cup B) = 1$　　(2) $P(A \cap B) = 0.2$　　(3) $P(A|B) = 1$

(4) $P(A|B) = P(B|A)$　　(5) A 和 B 為獨立事件。

（100 年指考數乙單選第 1 題）

Teacher: Let's check whether item 1 is correct. What does the general addition rule say about $P(A \cup B)$?

Student: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Teacher: That's correct. Since $P(A \cap B)$ is not given here, we can't conclude the value of $P(A \cup B)$. Therefore we can't choose item 1. Let's check item 2. We can use the general addition rule again. Since $P(A \cup B)$is not given here, we can't conclude the value of $P(A \cap B)$. Hence, we can't choose item 2.

Student: That makes sense to me.

Teacher: Let's check item 3. How do we find $P(A|B)$?

Student: $P(A|B) = \frac{P(A \cap B)}{P(B)}$. Wow, since I don't know $P(A \cup B)$, I can't decide what $P(A|B)$ is.

Teacher: Great! So, we can't choose item (3). Let's check item 4. Can we compare $P(A|B)$ and $P(B|A)$ here?

Student: Yes, since $P(A|B)=\frac{P(A\cap B)}{P(B)}=\frac{P(A\cap B)}{0.6}=\frac{P(B\cap A)}{P(A)}=P(B|A)$, so item 4 is correct.

Teacher: Awesome! Let's finish this one by checking item (5). To check A and B are independent, we can check whether $P(A\cap B)=P(A)P(B)$. But $P(A\cap B)$ is not given here, we can't conclude whether $P(A\cap B)=P(A)P(B)$. As a result, we can't choose item (5).

老師： 我們來看看選項 1 是否正確。機率中的加法原理告訴我們怎麼計算$P(A\cup B)$?

學生： $P(A\cup B)=P(A)+P(B)-P(A\cap B)$.

老師： 正確。因為$P(A\cap B)$未知，我們無法確定 $P(A\cup B)$的答案，所以選項 1 不是答案。我們可以用加法原理來回答選項 2。同理 $P(A\cup B)$也不知道，我們無法確定 $P(A\cap B)$的值，所以選項 2 也不正確。

學生： 嗯，我懂了。

老師： 來看選項 3，我們如何算$P(A|B)$？

學生： $P(A|B)=\frac{P(A\cap B)}{P(B)}$。同樣的我不知道 $P(A\cup B)$，無法確定 $P(A|B)$ 的值。

老師： 很好。來看選項 4，我們能比較 $P(A|B)$ 和 $P(B|A)$的值嗎？

學生： 是的，因為$P(A|B)=\frac{P(A\cap B)}{P(B)}=\frac{P(A\cap B)}{0.6}=\frac{P(B\cap A)}{P(A)}=P(B|A)$，所以選項 4 為真。

老師： 太棒了！讓我們看完選項 5 來完成這題！要判斷事件 A 和 B 是否獨立，我們可以算算看 $P(A\cap B)=P(A)P(B)$。但 $P(A\cap B)$ 並不知道，我們不能確定 $P(A\cap B)=P(A)P(B)$。於是，選項 5 不能選。

例題二

說明：利用已知的列聯表來計算機率的題目。

（英文）The probabilities that an adult Taiwanese man has high blood pressure and/or high cholesterol are shown in the contingency table

		Blood pressure	
		High	OK
Cholesterol	High	0.14	0.18
	OK	0.19	0.49

a. What's the probability that a man with high blood pressure has high cholesterol?

b. Are getting high blood pressure and getting high cholesterol independent?

c. Are getting high blood pressure and getting high cholesterol mutually exclusive?

（中文）台灣成年男性得到高血壓或高膽固醇的機率如下表，

		血壓	
		高血壓	正常血壓
膽固醇	高膽固醇	0.14	0.18
	正常膽固醇	0.19	0.49

a. 在有高血壓的情況下，一名男性有高膽固醇的機率為何？

b. 得到高血壓和得到高膽固醇是否獨立？

c. 得到高血壓和得到高膽固醇是否互斥？

Teacher: To begin with, let's add one more row below the table to find the marginal distribution of a man with "high blood" pressure and "OK" pressure. See the table below.

		Blood pressure	
		High	OK
Cholesterol	High	0.14	0.18
	OK	0.19	0.49
	total		

Now add each cell in the column of "High" Blood pressure, then add each cell in the column of "OK" blood pressure.

Student: Like that?

		Blood pressure	
		High	OK
Cholesterol	High	0.14	0.18
	OK	0.19	0.49
	total	0.33	0.67

Teacher: Correct. So we know the probability of people getting high pressure is 0.33 and for people with OK blood pressure is 0.67. Next, add one more column to the right to find the marginal distribution of a man with "high" cholesterol and "OK" cholesterol. See the table below.

		Blood pressure		
		High	OK	total
Cholesterol	High	0.14	0.18	
	OK	0.19	0.49	
	total	0.33	0.67	

Add each cell in a row of "High" cholesterol, then add each cell in a row of "OK" cholesterol.

Student: Done.

		Blood pressure		
		High	OK	total
Cholesterol	High	0.14	0.18	0.32
	OK	0.19	0.49	0.68
	total	0.33	0.67	

Teacher: Great! Let's do part a. What's the probability that a man with high blood pressure has high cholesterol? We should treat "a man with high blood pressure" as a given condition and calculate the probability of a man having high cholesterol given that a man with high blood pressure. Suppose B represents a man with high blood pressure and C represents a man with high cholesterol, what is $P(C|B)$?

Student: $P(C|B) = \frac{P(C \cap B)}{P(B)} = \frac{0.14}{0.33} = \frac{14}{33}$

Teacher: Good job! For part b, to check whether B and C are independent, we check that if $P(B \cap C)$ is the same as $P(B) \times P(C)$. What is $P(B \cap C)$?

Student: $P(B \cap C)$=0.14

Teacher: What is $P(B) \times P(C)$?

Student: $P(B) = 0.33$, $P(C) = 0.32$, and $P(B) \times P(C) = 0.33 \times 0.32 \approx 0.0348 \neq 0.14$. So, B and C are not independent.

Teacher: Fantastic! For part c, $P(B \cap C) = 0.14$ because some people who are both high blood pressure and high cholesterol, $B \cap C \neq \phi$. B and C are not disjoint.

老師：首先我們先在列聯表的下方加一列來計算高血壓和正常血壓的分布。如下表所示：

膽固醇		血壓	
		高血壓	正常血壓
	高膽固醇	0.14	0.18
	正常膽固醇	0.19	0.49
	總計		

現在將高血壓那一欄的兩個資料加起來，再將正常血壓那一欄的兩個資料加起來，把結果放在最下面的那一列。

學生：像這樣？

膽固醇		血壓	
		高血壓	正常血壓
	高膽固醇	0.14	0.18
	正常膽固醇	0.19	0.49
	總計	0.33	0.67

老師： 你答對了。現在我們知道高血壓及正常血壓的機率分別為 0.33 和 0.67。
下一步，在列聯表的右方加入一欄來觀察高膽固醇和正常膽固醇的機率分布，如下表：

膽固醇		血壓		
		高血壓	正常血壓	總計
	高膽固醇	0.14	0.18	
	正常膽固醇	0.19	0.49	
	總計	0.33	0.67	

在高膽固醇那一列中，把兩個資料加起來，也把正常膽固醇那一列中的兩個資料加起來，將結果放在右邊那一欄中。

學生： 算好了。

膽固醇		血壓		
		高血壓	正常血壓	總計
	高膽固醇	0.14	0.18	0.32
	正常膽固醇	0.19	0.49	0.68
	總計	0.33	0.67	

老師： 很好！我們可以來做題目 a.了。當一個男性有高血壓的情況下，他有高膽固醇的機率為何？我們要將“ a man with high blood pressure” 解讀成已知的條件下，去計算他有高膽固醇的機率。假設事件 B 為男性有高血壓的事件，C 為男性有高膽固醇的事件，試算出$P(C|B)$?

學生： $P(C|B)=\dfrac{P(C\cap B)}{P(B)}=\dfrac{0.14}{0.33}=\dfrac{14}{33}$

老師： 做對了！題目 b.中，要判斷 B 和 C 是否獨立，我們可以檢查一下 $P(B\cap C)$和$P(B)\times P(C)$是否相同。$P(B\cap C)$是多少呢？

學生： $P(B\cap C)$=0.14。

老師： 試算出$P(B)\times P(C)$?

學生： $P(B)=0.33$、$P(C)=0.32$，而$P(B)\times P(C)=0.33\times 0.32\approx 0.0348\neq 0.14$，所以，$B$ 和 C 並非獨立事件。

老師： 太好了！題目 c 中，因為$P(B\cap C)=0.14\neq 0$，代表$B\cap C\neq\phi$, 所以 B 和 C 並非互斥。

單元八　貝氏定理與主觀、客觀機率
Bayes' Theorem, Objective Probability and Subjective Probability

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■ 前言 Introduction

假設一公司生產的產品一部分由甲工廠製造，而另一部分由乙兩工廠製造，而若我們知道甲、乙兩工廠生產產品分別的良率，但我們要如何計算當我們知道是抽出不良品的時候，該不良品來自於甲工廠呢？在抽取之前，根據古典機率理論我們知道隨機抽到甲工廠產品的機率，在此情境中稱為「事前機率」，但在知道抽到的是不良品時，我們利用新的資訊去重新評估了該產品是來自甲工廠的機率稱為事後機率，在本節我們利用貝氏定理來計算事後機率。

這此節中，我們也跟學生介紹主觀機率及客觀機率。利用或解讀這個新的資訊可能來自於多次重複試驗後一事件出現的頻率，我們稱它叫客觀機率；而根據你對一事件的相信程度去判斷該事件發生的機率，我們稱它為主觀機率。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
conditional probability	**條件機率**	**objective probability**	**客觀機率**
Bayes' formula	**貝氏定理**	**subject probability**	**主觀機率**
prior probability	事前機率	posterior probability	事後機率
tree diagram	樹狀圖	complement rule	餘集法則

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ **What is the probability that_____________ given that __________?**

例句：**What is the probability that** people have hepatitis **given that** the test result is positive?

在檢驗結果為陽性的條件下，此人有肝炎的機率為何？

❷ **If _____________, what's the probability ____________?**

例句：**If** one of those products turns out to be defective, **what's the probability** it's from factory A?

若在該產品是有瑕疵的情況下，該產品來自工廠 A 的機率是？

❸ **Let *A* be the event of ____________________.**

例句：**Let *A* be the event of** people who have type A blood.

假設 A 為民眾為 A 血型的事件。

❹ **We can make a tree diagram to discuss ______________.**

例句：**We can make a tree diagram to discuss** this problem.

我們可以繪製樹狀圖來討論這個問題。

❺ **What's the probability that ___________________?**

例句：**What's the probability that** a patient testing negative is carrying HIV?

當患者被檢驗為陰性時，該患者確實患有 HIV 的機率為何？

❻ **Assume $P(A)$ represents the probability that ______ given that ________.**

例句：**Assume $P(A)$ represents the probability that** a patient has diabetes **given that** his/her test result is negative.

設 A 代表在一病患檢驗為陰性的情況下，該病患確實有糖尿病的機率。

■ 問題講解 Explanation of Problems

☙ 說明 ❧

Assume the probability of Taiwanese male adults getting diabetes is 12.4% (https://www.mohw.gov.tw/fp-5020-63343-1.html). If the results of a large study suggested that among people with diabetes, 99% of the test conducted were positive, while for people without diabetes 97% of the test conducted were negative. To answer the probability of people who have diabetes given the test was positive, we can use Bayes' formula to calculate as follows,

Let A be the event of people who have diabetes and B be the event of people whose test results are positive,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) \times P(B|A)}{P(B)}.$$

Here, we call $P(A)$ the prior probability which represents the probability we know people who have diabetes. Given the condition that the test results are positive, combined with the information of knowing the probability that people's test results are positive given that they have diabetes (that is we know $P(B|A)$), we are able to revise $P(A)$ by calculating $P(A|B)$. The method is called Bayes' formula and we call $P(A|B)$ the posterior probability.

Objective probability is the frequency based on the occurrence of similar events of frequent trials and Subjective probability is the level of your belief toward a certain event.

運算問題的講解

例題一

說明：本題是利用列聯表來幫助學生了解條件機率及貝氏定理的應用。

（英文）A car factory produced 25% of electric cars and the rest of the fuel cars. A small number of its cars had defects: 2% of the electric cars are defective and only 1% of the fuel ones are defective. If one of those cars turns out to be defective, what's the probability it's the electric type?

（中文）一家汽車工廠製造 25%的電動車及 75%的燃油車。在他們製造的車中發現 2%的電動車中有瑕疵，而 1%的燃油車有瑕疵。若在發現有瑕疵的車款的情況下，該車款是電動車的機率為何？

- **Solve by making a contingency table.**

Teacher: We can make a contingency table to discuss this question. Let E = the electric cars, F = the fuel cars, D = cars with defects, and D' = cars without defects, the contingency table of car types and defective or not is constructed as below,

	E	F	Total
D			
D'			
Total			

Let's begin with the marginal distribution. Given that 25% of cars are electric and 75 % of cars are fuel, we can fill in the values in the last row as follows,

	E	F	Total
D			
D'			
Total	0.25	0.75	1

It also states that 2% of electric cars are defective, while only 1% of fuel cars are defective. Consequently, we can fill in the first row as follows,

	E	F	Total
D	$0.25 \times 0.02 = 0.005$	$0.75 \times 0.01 = 0.0075$	$0.005 + 0.0075 = 0.0125$
D'			
Total	0.25	0.75	1

The second row (representing non-defective cars) can be obtained by subtracting the first row (representing defective cars) from the last row (marginal distribution of car types). The result is as follows,

	E	F	Total
D	$0.25 \times 0.02 = 0.005$	$0.75 \times 0.01 = 0.0075$	$0.005 + 0.0075 = 0.0125$
D'	0.245	0.7425	0.9875
Total	0.25	0.75	1

The question asks for the probability that a car is electric given that it is defective, $P(E|D)$.

How do we find $P(E|D)$?

Student: From the previous lesson, $P(E|D) = \frac{P(E \cap D)}{P(D)}$.

Teacher: Find what $P(E \cap D)$ and $P(D)$ are from the contingency table.

Student: $P(E \cap D) = 0.005$ and $P(D) = 0.0125$.

Therefore, we can find $P(E|D) = \frac{P(E \cap D)}{P(D)} = \frac{0.005}{0.125} = \frac{1}{25}$

Teacher: Correct!

- 用列聯表來解題

老師： 我們可以用列聯表來討論這一題。令 E = 電動車、F = 燃油車、D = 有瑕疵的車，而 D' = 無瑕疵的車。繪製車款(燃油車或是電動車)及是否有瑕疵的列聯表如下：

	E	F	總計
D			
D'			
總計			

我們先完成車款的邊際分布如下：

	E	F	總計
D			
D'			
總計	0.25	0.75	1

因為 2%的電動車有瑕疵，而 1% 的燃油車有瑕疵，我們可以完成第一列如下：

	E	F	總計
D	$0.25\times0.02=0.005$	$0.75\times0.01=0.0075$	$0.005+0.0075=0.0125$
D'			
總計	0.25	0.75	1

第二列可由第三列減去第一列來得到：

	E	F	總計
D	$0.25\times0.02=0.005$	$0.75\times0.01=0.0075$	$0.005+0.0075=0.0125$
D'	0.245	0.7425	0.9875
總計	0.25	0.75	1

在選到一台有瑕疵的車款的條件下，選到電動車的機率為 $P(E|D)$，如何計算 $P(E|D)$?

學生： 由前一節我們學到 $P(E|D)=\frac{P(E\cap D)}{P(D)}$.

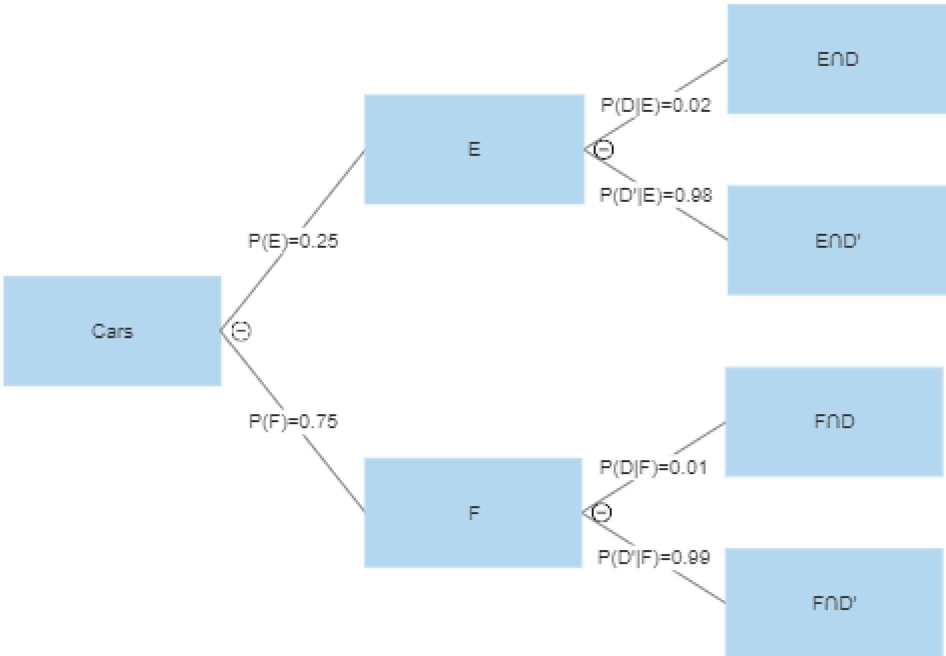
老師： 從列聯表來找到 $P(E\cap D)$和 $P(D)$的值。

學生： $P(E\cap D)=0.005$ 而 $P(D)=0.0125$ 。所以我們可以得到 $P(E|D)=\frac{P(E\cap D)}{P(D)}=\frac{0.005}{0.125}=\frac{1}{25}$

老師： 答對了！

- **Solve by making a tree diagram.**

Teacher: We can make a tree diagram to discuss this question. Let E=the electric cars, F= the fuel cars, D=cars with defect, and D'= cars without defect, the probability the car is electric given that it has defect is $P(E|D)$. According to this question, we can make a tree diagram below.



How do we find $P(E|D)$?

Student: From the previous lesson, $P(E|D)=\frac{P(E\cap D)}{P(D)}$.

Teacher: Good! By the addition rule, we can rewrite $P(D)=P(E\cap D)+P(F\cap D)$.

Student: Like this? $P(E|D)=\frac{P(E\cap D)}{P(D)}=\frac{P(E\cap D)}{P(E\cap D)+P(F\cap D)}$.

Teacher: Correct. Next, by the multiplication rule, we can rewrite $P(E\cap D)=P(E)P(D|E)$ and $P(F\cap D)=P(F)P(D|F)$

Student: Like this?

$$P(E|D)=\frac{P(E\cap D)}{P(D)}=\frac{P(E\cap D)}{P(E\cap D)+P(F\cap D)}$$

$$=\frac{P(E)P(D|E)}{P(E\cap D)+P(F\cap D)}$$

$$=\frac{P(E)P(D|E)}{P(E)P(D|E)+P(F)P(D|F)}$$

Teacher: We can plug in our data from the tree diagram to simplify it now.

$$= \frac{P(E)P(D|E)}{P(E)P(D|E) + P(F)P(D|F)}$$

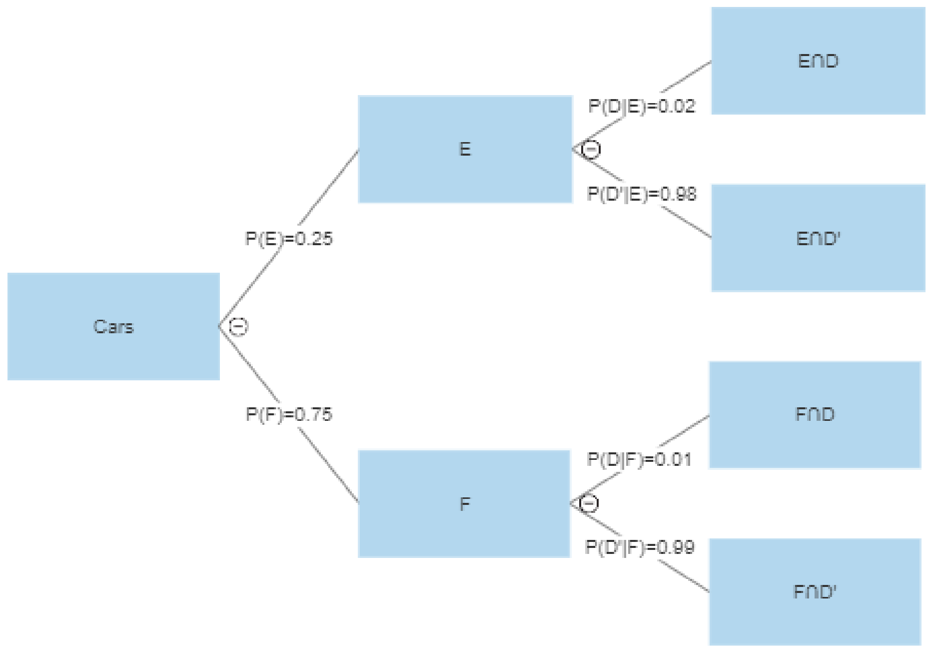
$$= \frac{0.25 \times 0.01}{0.25 \times 0.01 + 0.75 \times 0.02}$$

$$= \frac{25}{175} = \frac{1}{7} \approx 0.14286$$

Here we call the probability of selecting an electric car, $P(E)$, the prior probability. After knowing the probability of defective cars among different cars, we revise our probability of selecting an electric car given it is defective, $P(E|D)$, the posterior probability. The process is called the Bayes' theorem.

- **用樹狀圖來解題**

老師： 我們可以製作樹狀圖來討論這個問題，令 E=電動車、F= 燃油車、D=有瑕疵的車，而 D'= 無瑕疵的車。在選到一台有瑕疵的車款的條件下，選到電動車的機率為 $P(E|D)$。依照題意製作樹狀圖如下：



如何計算$P(E|D)$？

學生： 從前一節我學了 $P(E|D) = \frac{P(E \cap D)}{P(D)}$。

老師： 很好！我們可以由機率的加法原理得到 $P(D) = P(E \cap D) + P(F \cap D)$，請改寫該式的分母。

學生： 像這樣？$P(E|D) = \frac{P(E \cap D)}{P(D)} = \frac{P(E \cap D)}{P(E \cap D) + P(F \cap D)}$。

老師： 答對了。接下來我們可以改寫 $P(E\cap D)=P(E)P(D|E)$ 及 $P(F\cap D)=P(F)P(D|F)$。

學生： 像這樣？

$$P(E|D)=\frac{P(E\cap D)}{P(D)}=\frac{P(E\cap D)}{P(E\cap D)+P(F\cap D)}$$

$$=\frac{P(E)P(D|E)}{P(E\cap D)+P(F\cap D)}$$

$$=\frac{P(E)P(D|E)}{P(E)P(D|E)+P(F)P(D|F)}$$

老師： 沒錯，接下來我們可以將樹狀圖的資料帶入此式中

$$P(E|D)=$$

$$=\frac{P(E)P(D|E)}{P(E)P(D|E)+P(F)P(D|F)}$$

$$=\frac{0.25\times 0.01}{0.25\times 0.01+0.75\times 0.02}$$

$$=\frac{25}{175}=\frac{1}{7}\approx 0.14286$$

這裡我們稱選到電動車的機率$P(E)$為事前機率。在知道瑕疵車款於 D 兩種車發生的機率後，我們重新評估在選到瑕疵車款的情況下，選到電動車的機率 $P(E|D)$為事後機率。這樣的方法我們稱它為貝氏定理。

例題二

說明：本題是判斷是用哪一種機率來判斷一個事件發生的機率。

（英文）Determine what kind of probability is applied to explain the following statements:

(A) Subjective probability (B) Objective probability (C) Classical probability

(1) Roll a die once. The probability of getting a “2” is $\frac{1}{6}$.

(2) Sherry has rolled a die 10 times and recorded the frequency of each number. She found that “2” appeared 3 times so she concluded that the probability of getting a “2” next time is $\frac{3}{10}$.

(3) When David went to school today, he observed many things related to “two”, such as meeting two red lights in a row, bumping into two friends, and spotting two magpies perching in the tree. So, he felt the probability he would get a “2” when rolling a die was more than 50%.

（中文）試判斷以下情境分別應用了哪一種機率觀念？

(A) 主觀機率 (B) 客觀機率 (C) 古典機率

(1) 丟一個骰子 1 次，出現兩點的機率為 $\frac{1}{6}$。

(2) 雪莉丟一個骰子 10 次且記錄各點數出現的次數。她發現「2 點」出現了三次，所以她覺得下一次丟出「2 點」的機率為 $\frac{3}{10}$。

(3) 當大衛上學的時候，他發現途中有很多跟”2”相關的事物，譬如接連遇到兩個紅燈，遇到了兩位朋友，看到了兩隻喜鵲停在同一棵樹上。所以他覺得他接下來要丟骰子時，出現 2 點的機率大於 50%。

Teacher: What do you think of question (1)?

Student: I think it belongs to classical probability. Assume this is a fair die, the probability of getting each number is the same which is $\frac{1}{6}$.

Teacher: Great! What do you think of question (2)?

Student: I think it belongs to objective probability because the probability is derived from previous experience and data.

Teacher: Good! What do you think of question (3)?

Student:　It belongs to subjective probability because he estimates the probability based on his hunch.

Teacher:　Fantastic! You got all three correct.

老師：　你覺得問題 1 是屬於何種機率？

學生：　我覺得是古典機率。因為丟一公正的骰子，出現每一點的機率都是 $\frac{1}{6}$.

老師：　很好。那你覺得問題 2 是屬於何種機率？

學生：　我覺得是客觀機率，因為題目有提到 2 點在之前發生的比例，根據過往經驗來做出判斷是客觀機率。

老師：　沒錯。那問題 3？

學生：　是主觀機率，因為那是大衛自己用直覺來對這一個事件的做出的判斷.

老師：　很厲害，你三題都答對了。

應用問題 / 學測指考題

例題一

說明：利用貝氏定理來計算檢驗結果是”偽陰性”的機率。

（英文）In July 2005 the journal Annals of Internal Medicine published a report on the reliability of HIV testing. Results of a large study suggested that among people with HIV, 99.7% of tests conducted were (correctly) positive, while for people without HIV, 98.5% of the tests were (correctly) negative. A clinic serving an at-risk population offers free HIV testing, believing that 15% of the patients may actually carry HIV. What’s the probability that a patient testing negative is carrying HIV?

（中文）在 2005 年 7 月時，Annals of Internal Medicine 發表了一個報告是關於 HIV 檢驗的可靠性。在一個大型的研究結果顯示若民眾確實有感染 HIV 的情況下，99.7% 的檢驗結果是陽性；而當民眾無感染 HIV 的情況下，有 98.5%的檢驗結果是陰性。有一個診所在一個高風險群眾中，提供 HIV 檢驗。該診所相信該高風險群眾中有 15%的民眾確實有感染 HIV。試算出在該患者確實有感染 HIV 的情況下，驗出陰性的機率為何？

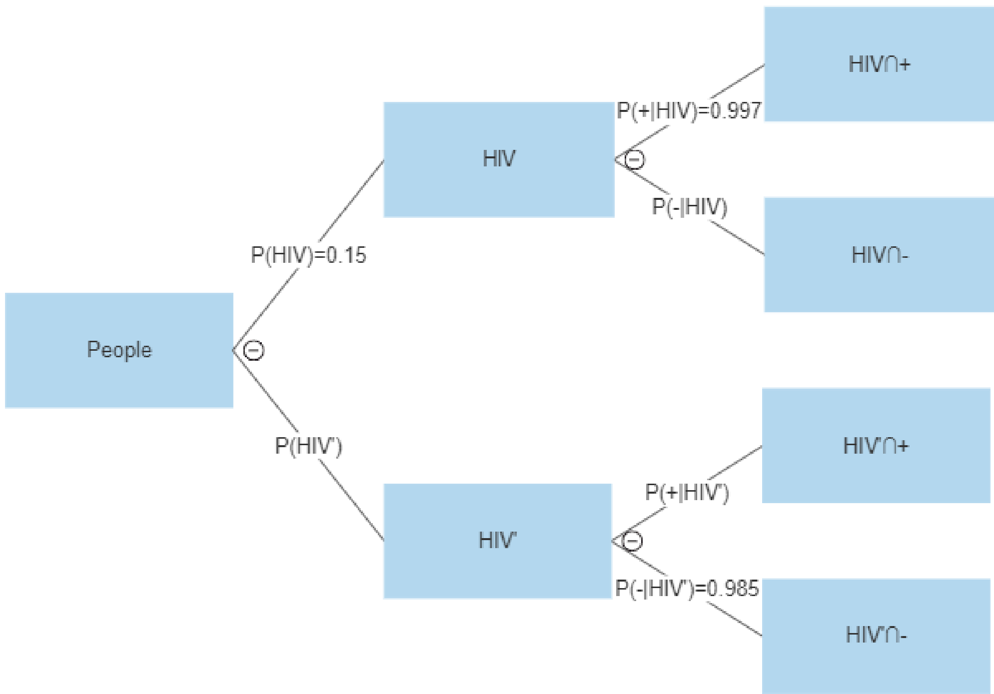
Teacher: A negative test result for a sick person is called a false negative. A false negative is considered to be more serious because sick people will assume they are healthy and can't get treatment in time. So, we hope the false-negative rate is as low as possible.

Student: Oh! I see the point of finding the false-negative rate.

Teacher: Let's begin. Assume HIV= people with HIV, HIV'= people without HIV, += people with positive testing results, and - = people with negative testing results, what do you know $P(+|HIV)$, $P(-|HIV')$, and $P(HIV)$?

Student: Because of people with HIV, 99.7% of tests conducted were (correctly) positive, $P(+|HIV) = 0.997$. Because people without HIV 98.5% of the tests were (correctly) negative, $P(-|HIV') = 0.985$. The question mentions that 15% of the patients may carry HIV, so $P(HIV) = 0.15$.

Teacher: Draw the tree diagram and put the data into it.



To complete the tree, we need to find $P(HIV')$, $P(-|HIV)$, and $P(+|HIV')$. Can you find them?

Student: Yes, I can. By using the complement rule, I will get the following results:

$P(HIV') = 1 - P(HIV) = 0.85$

$P(-|HIV) = 1 - P(+|HIV) = 0.003$

$P(+|HIV') = 1 - P(-|HIV') = 0.015$

Teacher: Terrific! Now, use the general multiplication rule to find $P(HIV \cap +)$, $P(HIV \cap -), P(HIV' \cap +)$, and $P(HIV' \cap -)$.

Student: I'll calculate those,

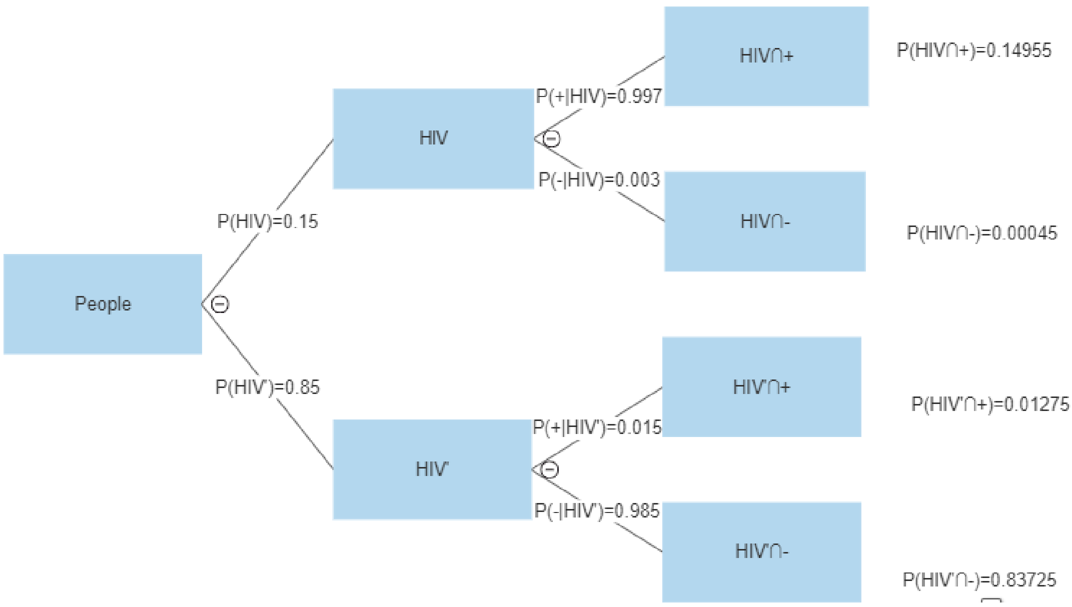
$$P(HIV \cap +) = P(HIV) \times P(+|HIV) = 0.15 \times 0.997 = 0.14955$$

$$P(HIV \cap -) = P(HIV) \times P(-|HIV) = 0.15 \times 0.003 = 0.00045$$

$$P(HIV' \cap +) = P(HIV') \times P(+|HIV') = 0.85 \times 0.015 = 0.01275$$

$$P(HIV' \cap -) = P(HIV') \times P(-|HIV') = 0.85 \times 0.985 = 0.83725$$

Teacher: That is hard work. We can complete our tree diagram now.



So, the probability that a patient testing negative is carrying HIV is

$$P(HIV|-)$$

$$= \frac{P(HIV \cap -)}{P(-)}$$

$$= \frac{P(HIV \cap -)}{P(HIV \cap -) + P(HIV' \cap -)}$$

$$= \frac{P(HIV) \times P(-|HIV)}{P(HIV) \times P(-|HIV) + P(HIV') \times P(-|HIV')}$$

$$= \frac{0.00045}{0.00045 + 0.83725}$$

$$= 0.000537$$

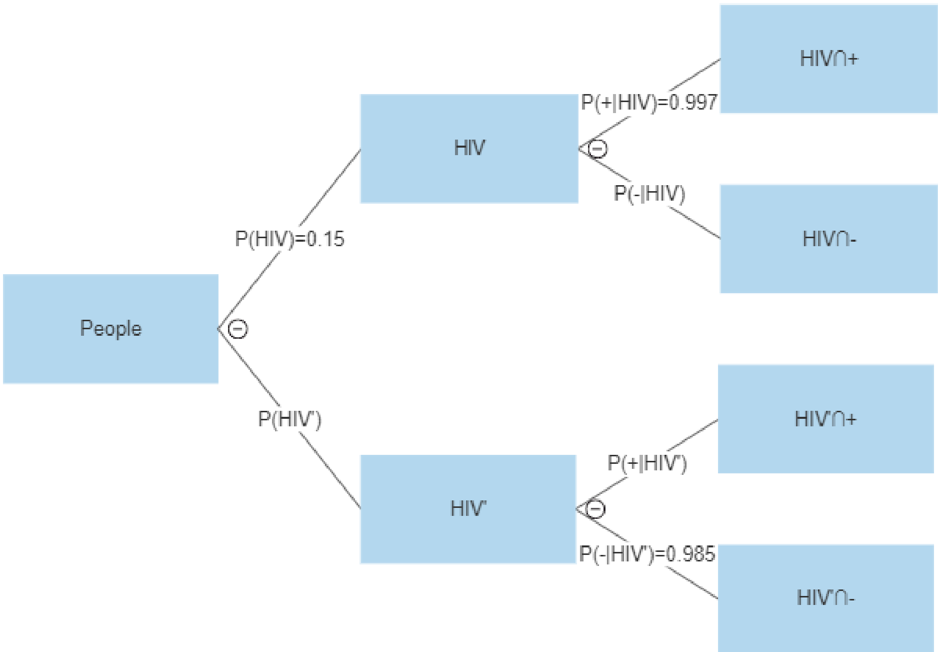
老師：　當真正有生病的民眾的檢驗結果為陰性時，我們稱該檢驗結果為偽陰性。偽陰的結果通常是被視為比較嚴重的，因為這樣的話病人會誤以為他們是健康而無法即時接受治療。所以我們希望一個檢驗結果的偽陰性能愈低愈好。

學生：　嗯！我看出計算偽陰比例的重要了。

老師： 那我們開始吧！假設 HIV= 感染 HIV 的事件, HIV'= 無感染 HIV 的事件，+= 檢驗結果為陽性的事件, and -= 檢驗結果為陰性的事件，試找出$P(+|HIV)$、$P(-|HIV')$、及$P(HIV)$?

學生： 當感染 HIV 的患者中，有 99.7% 的檢驗結果為陽性，所以 $P(+|HIV) = 0.997$。在無感染 HIV 民眾中，98.5% 的檢驗結果為陰性，所以$P(-|HIV') = 0.985$。題目有提到 15% of 的民眾確實有感染 HIV，所以$P(HIV) = 0.15$。

老師： 繪製樹狀圖並把剛找到的數據填入。



要完成這個樹狀圖，還需找到 $P(HIV')$、$P(-|HIV)$、和$P(+|HIV')$。試找出 $P(HIV')$、$P(-|HIV)$，和 $P(+|HIV')$?

學生： 好哦！由餘集定理，我可以算得以下的機率：

$P(HIV') = 1 - P(HIV) = 0.85$

$P(-|HIV) = 1 - P(+|HIV) = 0.003$

$P(+|HIV') = 1 - P(-|HIV') = 0.015$

老師： 太棒了！接下來我們可以用機率的乘法原理來找到 $P(HIV \cap +)$、$P(HIV \cap -)$、$P(HIV' \cap +)$，和 $P(HIV' \cap -)$。

學生：　計算如下：

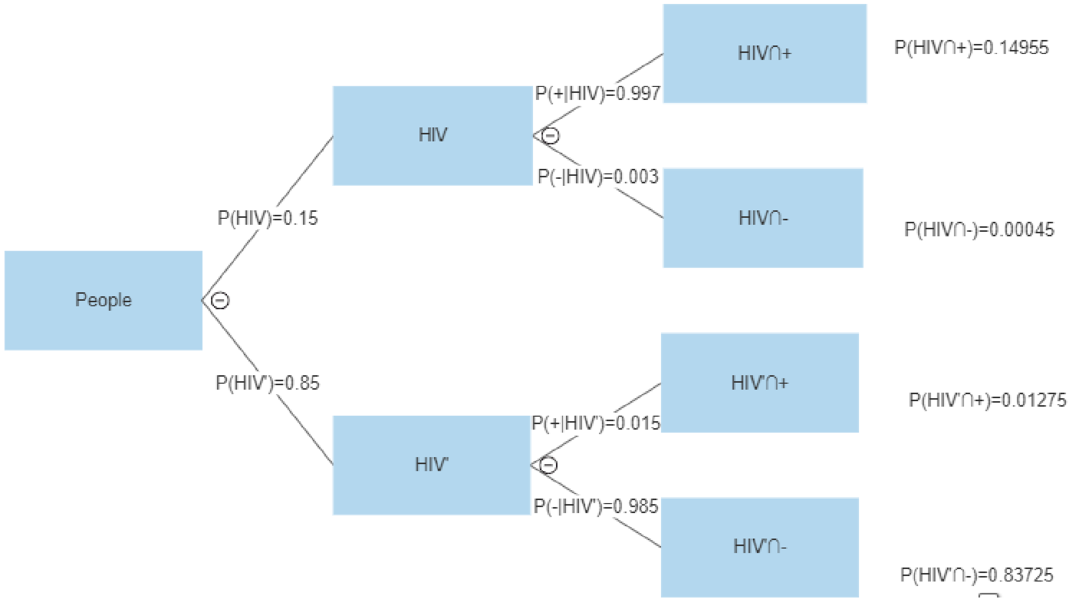
$P(HIV \cap +) = P(HIV) \times P(+|HIV) = 0.15 \times 0.997 = 0.14955$

$P(HIV \cap -) = P(HIV) \times P(-|HIV) = 0.15 \times 0.003 = 0.00045$

$P(HIV' \cap +) = P(HIV') \times P(+|HIV') = 0.85 \times 0.015 = 0.01275$

$P(HIV' \cap -) = P(HIV') \times P(-|HIV') = 0.85 \times 0.985 = 0.83725$

老師：　這真的是個大工程！算完後我們可以完成樹狀圖了：



所以，當一個民眾的檢驗結果為陰性，他真的有感染 HIV 的機率為：

$$P(HIV|-)$$
$$= \frac{P(HIV \cap -)}{P(-)}$$
$$= \frac{P(HIV \cap -)}{P(HIV \cap -) + P(HIV' \cap -)}$$
$$= \frac{P(HIV) \times P(-|HIV)}{P(HIV) \times P(-|HIV) + P(HIV') \times P(-|HIV')}$$
$$= \frac{0.00045}{0.00045 + 0.83725}$$
$$= 0.000537$$

例題二

說明：考貝氏定理的應用。

（英文）It is known that 30% of the population was infected with certain pandemics in some regions. There are two test results, positive and negative, for this pandemic. If we know that among the people infected by this pandemic, 80% of the test results are positive. We also know that among people without infection, 60% of the test results are negative. To reduce the rate of false negatives, some experts suggest that we should test a patient three times consecutively. Assume P represents the probability that a patient is infected given that his/her test result is negative and P' represents the probability that a patient is infected given that his/her results are tested negative for three consecutive times. Which of the following items is $\frac{P}{P'}$ closest to?

(1) 7 (2) 8 (3) 9 (4) 10 (5) 11

（中文）已知某地區有 30%的人口感染某傳染病。針對該傳染病的快篩試劑檢驗，有陽性或陰性 兩結果。已知該試劑將染病者判為陽性的機率為 80%，將未染病者判為陰性的機率則為 60%。為降低該試劑將染病者誤判為陰性的情況，專家建議連續採檢三次。若單次採檢判為陰性者中，染病者的機率為 P；而連續採檢三次皆判為陰性者中，染病者的機率為 P'。試問 $\frac{P}{P'}$ 最接近哪一選項？

(1) 7 (2) 8 (3) 9 (4) 10 (5) 11

（111 年學測數 A 單選第 5 題）

Teacher: For a question like this, we also make a tree diagram to help us analyze. Let D= people with infection, D'=People without infection, "+"= people tested positive and "-"=people tested negative. Find $P(D)$, $P(D')$, $P(+|D)$and $P(-|D')$.

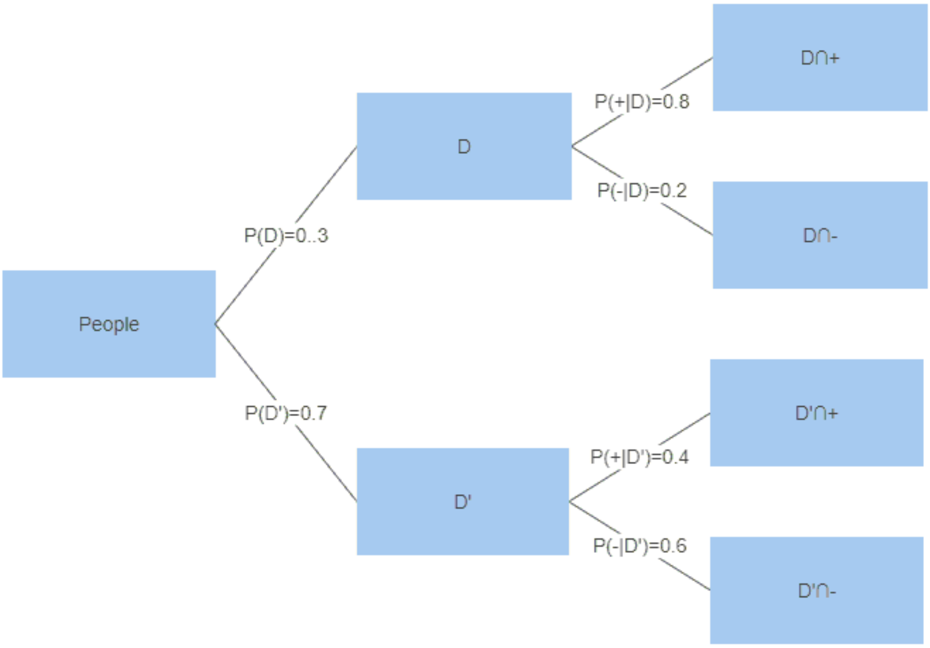
Student: Because 30% of the population was infected, I get$P(D) = 0.3$. $P(D') = 1 - 0.3 = 0.7$ by applying the complement rule.

This question mentions 80% of the test results are positive given people with infection, so $P(+|D) = 0.8$. The sentence "60% of the test results are negative given people without infection" tells us $P(-|D') = 0.6$

Teacher: Great! You read it correctly. Next, find $P(-|D)$ and $P(+|D')$.

Student: By applying the complement rule, $P(-|D) = 1 - P(+|D) = 0.2$ and $P(+|D') = 1 - P(-|D') = 0.4$.

Teacher: Good job. Now, we can complete the tree diagram as follows,



To find the probability of people with infection given that their test results are negative, we need to find $P(D|-)$. From Bayes' theorem,

$$P(D|-)$$
$$=\frac{P(D\cap -)}{P(-)}$$
$$=\frac{P(D\cap -)}{P(D\cap -)+P(D'\cap -)}$$
$$=\frac{P(D|-)\times P(D)}{P(D|-)\times P(D)+P(D'|-)\times P(D')}$$
$$=\frac{0.3\times 0.2}{0.3\times 0.2+0.7\times 0.6}=\frac{1}{8}$$

Next, to find the probability of people with infection given that their test results are negative three times in a row, let "$-^3$"=people are tested negative 3 times in a row. Assuming that the tests are independent of each other, the probability of getting negative results three times in a row given that people are infected is $0.2^3=0.008$. The probability of getting negative results three times in a row given that people are not infected is $0.6^3=0.216$.

Apply Bayes' theorem to find $P(D|-)$,

$P(D|-^3)$

$$= \frac{P(D \cap -^3)}{P(-^3)}$$

$$= \frac{P(D \cap -^3)}{P(D \cap -^3) + P(D' \cap -^3)}$$

$$= \frac{P(D|-^3) \times P(D)}{P(D|-^3) \times P(D) + P(D'|-^3) \times P(D')}$$

$$= \frac{0.3 \times 0.2^3}{0.3 \times 0.2^3 + 0.7 \times 0.6^3} = \frac{24}{1536} = \frac{1}{64}$$

Now, find the ratio of $\frac{P(D|-)}{P(D|-^3)}$.

Student: $\frac{P(D|-)}{P(D|-^3)} = \frac{\frac{1}{8}}{\frac{1}{64}} = 8$, so I choose item 2

Teacher: Well done. This question also suggests a method to help reduce the rate of false negatives. If a patient is tested negative once, we can't be sure he/she is free of the disease. We have to test a few more times to be more certain.

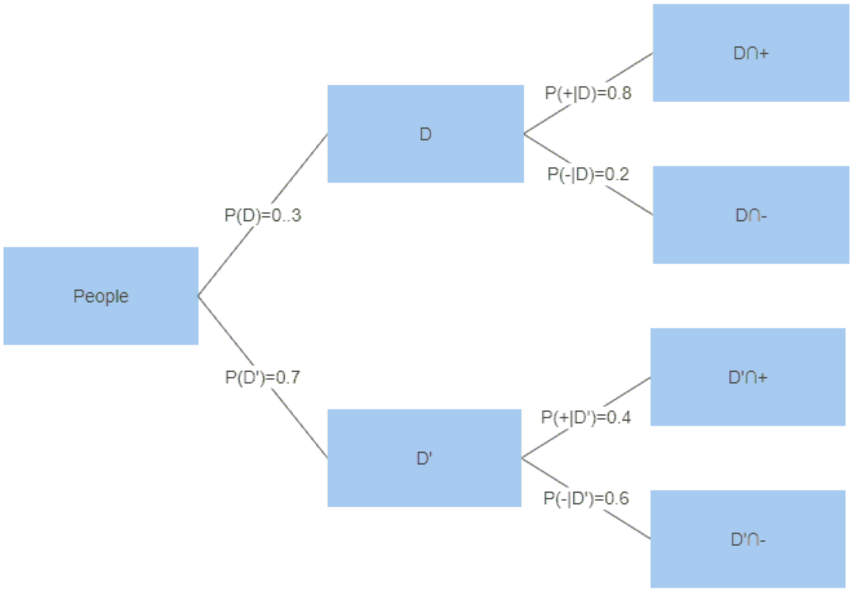
老師： 我們可以用樹狀圖來幫我們分析這道題目。
假設 D= 感染傳染病的民眾，D'=沒有感染傳染病的民眾，“+” = 檢驗結果為陽性，及 “-” = 檢驗結果為陰性。
試找出 $P(D)$, $P(D')$, $P(+|D)$和 $P(-|D')$。

學生： 因為 30%的民眾有被感染，所以$P(D)=0.3$。用餘集法則，我可以找到 $P(D') = 1 - 0.3 = 0.7$。
題目提到說若民眾有感染些傳染病的情況下 80%的檢驗結果為陽性，所以我可以知道 $P(+|D) = 0.8$。另外題目也提到“將未染病者判為陰性的機率則為60%”，可得到 $P(-|D') = 0.6$。

老師： 很好！你有將題目意思讀懂。接下來，找出 $P(-|D)$及$P(+|D')$。

學生： 用餘集法則，可以算出$P(-|D) = 1 - P(+|D) = 0.2$ 和 $P(+|D') = 1 - P(-|D') = 0.4$。

老師： 很好！接下來我們可以完成下圖這個樹狀圖了：



如果要找到在檢驗結果為陰性的情況下，該民眾確實有染病的機率為 $P(D|-)$，我們可以用貝氏定理計算$P(D|-)$如下：

$$P(D|-)$$
$$=\frac{P(D\cap -)}{P(-)}$$
$$=\frac{P(D\cap -)}{P(D\cap -)+P(D'\cap -)}$$
$$=\frac{P(D|-)\times P(D)}{P(D|-)\times P(D)+P(D'|-)\times P(D')}$$
$$=\frac{0.3\times 0.2}{0.3\times 0.2+0.7\times 0.6}=\frac{1}{8}$$

最後，要算出連續採檢三次皆判為陰性者中，是染病者的機率，假設"$-^3$"代表連續三次採檢為陰的事件。在檢驗結果互不影響的狀況下，若該民眾有感染該傳染病，連續三次採檢為陰的機率為$0.2^3=0.008$. 若該民眾無感染該傳染病，連續三次採檢為陰的機率為$0.6^3=0.216$.

我們用貝氏定理來計算$P(D|-^3)$如下：我們可以來計算$\frac{P(D|-)}{P(D|-^3)}$了。

$$P(D|-^3)$$
$$=\frac{P(D\cap -^3)}{P(-^3)}$$
$$=\frac{P(D\cap -^3)}{P(D\cap -^3)+P(D'\cap -^3)}$$
$$=\frac{P(D|-^3)\times P(D)}{P(D|-^3)\times P(D)+P(D'|-^3)\times P(D')}$$
$$=\frac{0.3\times 0.2^3}{0.3\times 0.2^3+0.7\times 0.6^3}=\frac{24}{1536}=\frac{1}{64}$$

學生： $\frac{P(D|-)}{P(D|-^3)}=\frac{\frac{1}{8}}{\frac{1}{64}}=8$，所以我選擇選項(2)。

老師： 太好了！這個題目也告訴我們一個可以降低偽陰性的比例的辦法。如果有人檢驗結果為陰性，我們還是無法很確認該民眾是否真的沒有染病。多檢驗幾次可以更加確定哦！

單元九　三元一次方程組
System of Linear Equations with Three Unknowns

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■ 前言 Introduction

我們在國中時學過如何解二元一次方程組。高中的這個單元，我們將進階到解決有三個未知數的一次方程式，稱為三元一次方程組。除了常用的加減消去法，我們也會學習高斯消去法來做擴增矩陣的列運算。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
system	系統	apply	應用
elimination	**消除**	**eliminate**	**消去**
Gaussian Elimination	高斯消去法	**substitute**	**取代**
substitution	**取代**	yield	產生
equivalent	**等價的**	**interchange**	**互換**
original	原來的	transform	轉換
coefficient	**係數**	reduce	減少
entry	**輸入**	align	對齊

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ The key step in this method is ________.

例句(1)：**The key step in this method is to** eliminate the variable.
這個方法的關鍵步驟是消除變數。

例句(2)：**The key step in Gaussian Elimination is** row reduction.
高斯消去法的關鍵步驟是列的簡化。

❷ We use _____ for _____.

例句(1)：**We use** 0 **for** the missing coefficient of the *y*-variable in the equation.
我們用零來代表方程式裡缺少的那個變數。

例句(2)：**We use** parentheses **for** grouping terms in the equation.
我們用括號來將方程式裡的各項分組。

❸ Align ____ vertically in the equations.

例句(1)：Align the variables vertically in the equations.
將方程式中的變數垂直對齊。

例句(2)：Align the decimal points vertically in the columns of numbers.
將數字欄位中的小數點垂直對齊。

❹ Multiply ____ by _____.

例句(1)：**Multiply** one of the equations **by** a nonzero constant.
將其中一個方程式乘上一個非零常數。

例句(2)：**Multiply** a row **by** a nonzero constant.
將其中一行乘上一個非零常數。

❺ Add __________ to ________.

例句(1)：**Add** a multiple of one row **to** another row.
將某一行的倍數加到另一行上。

例句(2)：**Add** a nonzero constant **to** both sides of the equation.
將一個非零常數加到等式的兩邊。

❻ Substitute ____ for ____.

例句(1)：We **substitute** 1 **for** z in the beginning.
我們首先將 z 替換成 1。

例句(2)：**Substitute** the obtained value of x **for** the x-variable in the original equation to find the corresponding y-value
將求得的 x 值代入原方程式的 x 變數，以找到對應的 y 值。

❼ Replace ____ with ____

例句(1)：**Replace** z **with** 1 in the equation of $x-2y+z=4$.
將方程式 $x-2y+z=4$ 的 z 替換為 1。

例句(2)：**Replace** the fractional coefficients **with** whole numbers by multiplying the entire equation by the least common denominator.
將整個方程式乘以最小公倍數便可以將分數係數替換為整數。

■ 問題講解 Explanation of Problems

☙ 說明 ❧

In this section, we solve the system of equations based on the elimination method. The elimination method will be extended to Gaussian Elimination, where we have to conduct the row operations. Real-life problems regarding the system of equations will be mentioned as well.

The method for solving a system of three linear equations is similar to that for solving a system of two linear equations. First, eliminate one of the variables to obtain a system of two linear equations. Subsequently, we use the method for solving a system of two linear equations to find its solution.

In a system of linear equations, write the constant terms of each equation on the right side of the equals sign, while the terms containing variables are written on the left side in a fixed order, as in the system of three linear equations shown in Figure (a).

Solve the system of equations	Augmented matrix
$\begin{cases} 2x+3y-z=10 \\ x-2y+5z=-15 \\ -x+y+2z=-7 \end{cases}$	$\left[\begin{array}{ccc\|c} 2 & 3 & -1 & 10 \\ 1 & -2 & 5 & -15 \\ -1 & 1 & 2 & -7 \end{array}\right]$ ← first row, ← second row, ← third row; ↑ 1st column, ↑ 2nd column, ↑ 3rd column, ↑ 4th column
Figure(a)	Figure(b)

Extract the coefficients of the variables and the constant terms from this system of equations in order, and represent them as shown in Figure (b), which is called the augmented matrix of the system of equations. The vertical line in the augmented matrix is used to distinguish between the "coefficients" and the "constant terms," but it can also be omitted.

Transforming the augmented matrix to triangular form ($\begin{bmatrix} 1 & a & b & d \\ 0 & 1 & c & e \\ 0 & 0 & 1 & f \end{bmatrix}$ is Gaussian elimination.

Below are the operations needed to transform a system of linear equations into triangular form. These operations result in an equivalent system.

- Interchange any two equations of the system.
- Multiply (or divide) one of the equations by any nonzero real number.
- Add a multiple of one equation to any other equation in the system.

For example,

Augmented matrix

$$\left[\begin{array}{ccc|c} \mathbf{2} & \mathbf{3} & \mathbf{-1} & \mathbf{10} \\ \mathbf{1} & \mathbf{-2} & \mathbf{5} & \mathbf{-15} \\ \mathbf{-1} & \mathbf{1} & \mathbf{2} & \mathbf{-7} \end{array}\right]$$

Interchange the 1st and the 2nd row.

$$\left[\begin{array}{ccc|c} \mathbf{1} & \mathbf{-2} & \mathbf{5} & \mathbf{-15} \\ \mathbf{2} & \mathbf{3} & \mathbf{-1} & \mathbf{10} \\ \mathbf{-1} & \mathbf{1} & \mathbf{2} & \mathbf{-7} \end{array}\right]$$

$\times(-2)$

a.) Multiply the 1st row by (-2) and add it to the 2nd row

b.) Add the 1st row to the 3rd row

$$\left[\begin{array}{ccc|c} \mathbf{1} & \mathbf{-2} & \mathbf{5} & \mathbf{-15} \\ \mathbf{0} & \mathbf{7} & \mathbf{-11} & \mathbf{40} \\ \mathbf{0} & \mathbf{-1} & \mathbf{7} & \mathbf{-22} \end{array}\right] \leftarrow \times(-1)$$

Multiply (-1) to the third row.

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & -15 \\ 0 & 7 & -11 & 40 \\ 0 & 1 & -7 & 22 \end{array}\right] \times(-7)$$

Multiply (-7) to the 3rd row and add it to the 2nd row.

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & -15 \\ 0 & 0 & 38 & -114 \\ 0 & 1 & -7 & 22 \end{array}\right] \leftarrow \times\left(\frac{1}{38}\right)$$

Multiply $\frac{1}{38}$ to the 2nd row.

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & -15 \\ 0 & 0 & 1 & -3 \\ 0 & 1 & -7 & 22 \end{array}\right]$$

Interchange the 2nd and 3rd row.

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & -15 \\ 0 & 1 & -7 & 22 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

Gaussian Elimination
1. Write the augmented matrix of the system of linear equations.
2. Use row operations to rewrite the augmented matrix so that the first nonzero entry of the first row is 1.
3. Use row operations to rewrite the previous augmented matrix so that the first entry of the second row is 0.
4. Use row operations to rewrite the previous augmented matrix so that the first two entries of the third row is 0.

運算問題的講解

例題一

說明：本題是三元一次方程組的基本題型，可用以與高斯消去法做連結。

（英文）Solve the system of equations $\begin{cases} x-2y+z=4 \\ 7y+4z=-17 \\ z=1 \end{cases}$

（中文）試求三元一次方程組 $\begin{cases} x-2y+z=4 \\ 7y+4z=-17 \\ z=1 \end{cases}$ 的解。

Teacher: In this problem, we substitute 1 for z in the beginning.

The system then becomes a system of two linear equations. $\begin{cases} x-2y=3 \\ 7y=-21 \end{cases}$

I believe you are familiar with the way to solve a system of two linear equations.

Who can tell me what the next step is?

Student: Divide both sides of the second equation by 7.

It yields $y=-3$.

We then substitute -3 for y in the first equation.

It leads to $x+6=3$.

$x=-3$

Teacher: Very clear explanation.

老師：在這個問題中，我們一開始先將 1 代入 z。

然後，可以得到一個由兩個線性方程式組成的方程組。$\begin{cases} x - 2y = 3 \\ 7y = -21 \end{cases}$

我相信大家很熟悉求解兩個線性方程式的方法。

誰能告訴我下一步是什麼？

學生：將第二個方程的兩邊都除以 7。

得到 $y = -3$。

然後，我們將 -3 代入第一個方程的 y。

得到 $x + 6 = 3$。

也就是 $x = -3$。

老師：解釋得很清楚。

例題二

說明：本題仍是三元一次方程組的基本題型，但計算過程較前一題複雜。

（英文）Solve the system of equations $\begin{cases} x - 2y + z = 4 \\ 2x + 3y + 6z = -9 \\ 3x - 2y + 4z = 1 \end{cases}$

（中文）試求三元一次方程組 $\begin{cases} x - 2y + z = 4 \\ 2x + 3y + 6z = -9 \\ 3x - 2y + 4z = 1 \end{cases}$ 的解。

Teacher: This system of equations is slightly more difficult than the previous one.

The key step in solving this system is to eliminate the variables one by one.

If we want to eliminate the x-variable in the second equation, we multiply the first equation by -2 and add the result to the second equation.

The new second equation becomes $7y + 4z = -17$.

How to eliminate the x-variable in the third equation?

Student: We multiply the first equation by -3 and add the result to the third equation.

The new third equation becomes $4y + z = -11$.

Teacher: Very good.

Now, we have a system of two linear equations $\begin{cases} 7y+4z=-17 \\ 4y+z=-11 \end{cases}$.

We can solve it by elimination or substitution.

If we still want to apply the elimination method, which variable shall we eliminate first?

Student: I think it is easier to eliminate the z-variable.

We multiply the new third equation $(4y+z=-11)$ by -4 and add the result to the new second equation $(7y+4z=-17)$.

We get $-9y=27$.

$y=-3$

Teacher: After we obtain the value of y, we substitute -3 for y in the equation of $4y+z=-11$.

We have $-12+z=-11$

$z=1$

How about the value of x?

Student: Replace y with -3 and replace z with 1 in the equation of $x-2y+z=4$.

We have $x+6+1=4$.

$x=-3$.

Teacher: Well done.

The key step in solving the system of multivariable equations is to eliminate the variables one by one.

The elimination becomes easy if the coefficient of a variable is 1.

老師： 這個方程組比前一題略為困難。

解這個方程組的關鍵步驟是逐一消除變數。

如果我們想消除第二個方程式中的 x，我們必須先將第一個方程式乘以 -2，然後將結果加到第二個方程式上。

新的第二個方程式變成 $7y+4z=-17$。

如何消除第三個方程式中的 x 變數呢？

學生： 我們必須先將第一個方程式乘以 -3，然後將結果加到第三個方程式上。

新的第三個方程式變成 $4y+z=-11$。

老師： 很好。現在我們有一個由兩個線性方程式組成的方程組。

我們可以採用消去法或代入法求解。

如果我們仍然想應用消去法，我們應該先消除哪個變數？

學生： 我認為消除 z 變數會比較容易。

我們可以將新的第三個方程式 $(4y + z = -11)$ 乘以 -4，然後將結果加到新的第二個方程式 $(7y + 4z = -17)$ 上。

得到 $-9y = 27$。相當於 $y = -3$。

老師： 在我們得到 y 值後，我們將 -3 代入方程式 $4y + z = -11$。

得到 $-12 + z = -11$，相當於 $z = 1$。

那麼 x 值呢？

學生： 我們可以將 y 改成 -3，將 z 改成 1，代入方程 $x - 2y + z = 4$。

得到 $x + 6 + 1 = 4$。相當於 $x = -3$。

老師： 做得很好。

從這一題的計算過程我們可以發現，解多個變數方程組的關鍵步驟是逐一消除變數。如果變數的係數為 1，消除變數就變得容易多了。

例題三

說明：本題仍是三元一次方程組的基本題型，但以擴增矩陣的列運算作為解題方法。

（英文）(1) Write the augmented matrix corresponding to the system $\begin{cases} x-2y+z=4 \\ 2x+3y+6z=-9 \\ 3x-2y+4z=1 \end{cases}$

(2) Solve the system of equations with Gaussian Elimination.

（中文）寫出三元一次方程組的擴增矩陣並以高斯消去法求解。

$$\begin{cases} x-2y+z=4 \\ 2x+3y+6z=-9 \\ 3x-2y+4z=1 \end{cases}$$

Teacher: The matrix derived from the system of equations is called the "augmented matrix". We align the variables of each equation vertically and then list the coefficients and constants as the entries of the matrix.

The dimension of the augmented matrix is three by four (3×4).

It is $\begin{bmatrix} 1 & -2 & 1 & 4 \\ 2 & 3 & 6 & -9 \\ 3 & -2 & 4 & 1 \end{bmatrix}$.

Any questions about the augmented matrix?

Student: No.

Teacher: The key step of Gaussian Elimination is row reduction.

We expect the augmented matrix to be in a triangular form.

In other words, all the entries below the main diagonal are zero.

To make a_{21} zero, we multiply the first row by -2 and add the result to the second row.

The new augmented matrix becomes $\begin{bmatrix} 1 & -2 & 1 & 4 \\ 0 & 7 & 4 & -17 \\ 3 & -2 & 4 & 1 \end{bmatrix}$.

How to make a_{31} zero?

Student: We multiply the first row by -3 and add the result to the third row.

The new augmented matrix becomes $\begin{bmatrix} 1 & -2 & 1 & 4 \\ 0 & 7 & 4 & -17 \\ 0 & 4 & 1 & -11 \end{bmatrix}$.

Teacher: Very good.

If you compare this augmented matrix with the calculation process in the previous question, they actually represent the same equations.

There are several ways to make the entries in the lower triangle zero.

For example, we can multiply the second row by $\frac{-4}{7}$ and add the result to the third row.

The new augmented matrix becomes $\begin{bmatrix} 1 & -2 & 1 & 4 \\ 0 & 7 & 4 & -17 \\ 0 & 0 & \frac{-9}{7} & \frac{-9}{7} \end{bmatrix}$.

Now, the entries in the lower triangle are zeros.

Check the third row, what is the value of z?

Student: It is 1.

Teacher: Correct.

We can multiply the third row by $\frac{-7}{9}$.

The augmented matrix becomes $\begin{bmatrix} 1 & -2 & 1 & 4 \\ 0 & 7 & 4 & -17 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

The equivalent system to this augmented matrix is $\begin{cases} x - 2y + z = 4 \\ 7y + 4z = -17 \\ z = 1 \end{cases}$

This system is the same as the first example.

I would like to skip other steps in solving this problem.

Any questions?

Student: I'm not good at the calculation of fractions.

Any other methods that I can use to operate the augmented matrix?

Teacher: This is a good question.

In the augmented matrix $\begin{bmatrix} 1 & -2 & 1 & 4 \\ 0 & 7 & 4 & -17 \\ 0 & 4 & 1 & -11 \end{bmatrix}$, we can multiply the third row by -2 and add the result to the second row.

We will get $\begin{bmatrix} 1 & -2 & 1 & 4 \\ 0 & -1 & 2 & 5 \\ 0 & 4 & 1 & -11 \end{bmatrix}$,

What will be your next step?

Student: I will multiply the second row by 4 and add the result to the third row.

The augmented matrix becomes $\begin{bmatrix} 1 & -2 & 1 & 4 \\ 0 & -1 & 2 & 5 \\ 0 & 0 & 9 & 9 \end{bmatrix}$.

Teacher: It works.

The result also tells you that $z = 1$.

You can solve the remaining variables with substitution.

老師： 將係數和常數按照對應位置列出的矩陣稱為擴增矩陣。

我們先將每個方程式的變數垂直對齊，然後將係數和常數列為矩陣的元素。

擴增矩陣的維度是三行四列 (3×4)。

它是$\begin{bmatrix} 1 & -2 & 1 & 4 \\ 2 & 3 & 6 & -9 \\ 3 & -2 & 4 & 1 \end{bmatrix}$。同學們，對擴增矩陣有任何問題嗎？

學生： 沒有。

老師： 高斯消去法的關鍵步驟是行的消除。

我們期望擴增矩陣呈現三角形式。

換句話說，主對角線以下的所有元素都為零。

為了使元素 a_{21} 為零，我們將第一行乘以 -2，並將結果加到第二行。

新的擴增矩陣變為 $\begin{bmatrix} 1 & -2 & 1 & 4 \\ 0 & 7 & 4 & -17 \\ 3 & -2 & 4 & 1 \end{bmatrix}$。

如何使元素 a_{31} 為零呢？

學生： 我們將第一行乘以 -3，並將結果加到第三行。

新的擴增矩陣變為 $\begin{bmatrix} 1 & -2 & 1 & 4 \\ 0 & 7 & 4 & -17 \\ 0 & 4 & 1 & -11 \end{bmatrix}$。

老師： 非常好。

如果你比較這個擴增矩陣和前一個問題的計算過程，你應該可以發現他們實際上代表了相同的方程式。

有好幾種方法可以讓下三角的元素為零。

例如，我們可以將第二行乘以 $\frac{-4}{7}$，並將結果加到第三行。

新的擴增矩陣變為$\begin{bmatrix} 1 & -2 & 1 & 4 \\ 0 & 7 & 4 & -17 \\ 0 & 0 & \frac{-9}{7} & \frac{-9}{7} \end{bmatrix}$。現在，下三角的元素都是零。

請檢查第三行，z 的值是多少？

學生： z 值是 1。

老師： 正確。

我們可以將第三行乘以 $\frac{-7}{9}$。

擴增矩陣變為$\begin{bmatrix} 1 & -2 & 1 & 4 \\ 0 & 7 & 4 & -17 \\ 0 & 0 & 1 & 1 \end{bmatrix}$。

這個擴增矩陣對應的方程組是. $\begin{cases} x - 2y + z = 4 \\ 7y + 4z = -17 \\ z = 1 \end{cases}$。

這個方程組與第一個例子相同，我想跳過解決這個問題的其他步驟。

關於這題的解法，有其他問題嗎？

學生： 我不擅長計算分數。

有其他方法可以操作擴增矩陣嗎？

老師： 這是一個好問題。

在擴增矩陣 $\begin{bmatrix} 1 & -2 & 1 & 4 \\ 0 & 7 & 4 & -17 \\ 0 & 4 & 1 & -11 \end{bmatrix}$. 中，我們可以將第三行乘以 -2，並將結果加到第二行。

我們可以得到 $\begin{bmatrix} 1 & -2 & 1 & 4 \\ 0 & -1 & 2 & 5 \\ 0 & 4 & 1 & -11 \end{bmatrix}$。

那麼，你覺得下一步是什麼？

學生： 我們可以將第二行乘以 4，再將結果加到第三行。

擴增矩陣變為 $\begin{bmatrix} 1 & -2 & 1 & 4 \\ 0 & -1 & 2 & 5 \\ 0 & 0 & 9 & 9 \end{bmatrix}$。

老師： 這是個可行的方法。

結果還是 $z=1$。

我們接著可以透過代換法來求解其餘的變數。

≈ 應用問題 / 學測指考題 ≈

例題一

說明：這題是三元一次方程式無解的指考題。

（英文）Let c be a real number such that the system of three linear equations has no solution. Find the value of c.

$$\begin{cases} x - y + z = 0 \\ 2x + cy + 3z = 1 \\ 3x - 3y + cz = 0 \end{cases}$$

（中文）設 c 為實數使得三元一次方程組 $\begin{cases} x - y + z = 0 \\ 2x + cy + 3z = 1 \\ 3x - 3y + cz = 0 \end{cases}$ 無解。試選出 c 之值。

(1) 3　(2) 2　(3) 0　(4) 2　(5) 3

（111 年數甲指考第 2 題)

Teacher: First of all, we write the augmented matrix for this system.

It is $\begin{bmatrix} 1 & -1 & 1 & 0 \\ 2 & c & 3 & 1 \\ 3 & -3 & c & 0 \end{bmatrix}$,

We can apply the process similar to the previous problem to make a_{21} and a_{31} zero.

Who can explain the process?

Student: To make a_{21} zero, we multiply the first row by -2 and add the result to the second row.

To make a_{31} zero, we multiply the first row by -3 and add the result to the third row.

The new augmented matrix is $\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 2+c & 1 & 1 \\ 0 & 0 & c-3 & 0 \end{bmatrix}$,

Teacher: Check the third row, you can find out that there will be infinitely many solutions to the systems when $c - 3 = 0$.

We can divide the third row by $c - 3$.

The equivalent matrix is $\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 2+c & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

We then multiply the third row by -1 and add the result to the second row.

What is the result?

Student: It is $\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 2+c & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$,

Teacher: Check the second row, what value of c will yield no solution?

Student: If $2+c=0$, the corresponding equation is $0x+0y+0z=1$.

There is no solution when $c=-2$.

Teacher: Very clear explanation.

Finding the determinant of the coefficient matrix also helps figure out what value of c will yield no solution.

We will leave this method to the next section.

老師： 首先，我們寫出這個方程組的擴增矩陣。

他是$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 2 & c & 3 & 1 \\ 3 & -3 & c & 0 \end{bmatrix}$。

誰可以用前一題的方法把這個矩陣的 a_{21} 和 a_{31}變成零？

學生： 我們可以將第一行乘以 -2，並將結果加到第二行。

我們可以將第一行乘以 -3，並將結果加到第三行。

新的擴增矩陣是$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 2+c & 1 & 1 \\ 0 & 0 & c-3 & 0 \end{bmatrix}$。

老師： 檢查一下矩陣的第三行，應該很容易可以發現 $c=3$ 的時候有無限多解。

我們可以把第三行除以$c-3$。

等價的擴增矩陣是$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 2+c & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$。如果我把矩陣的第三行乘以$-1$，加到矩陣的第二行，結果是什麼？

學生： 是 $\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 2+c & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$。

老師：　現在讓我們檢查這個矩陣的第二行。c 值為多少的時候為變成無限多解？

學生：　如果 $2+c=0$，第二行代表的方程式是 $0x+0y+0z=1$。

　　　　也就是說，這個方程式在 $c=-2$ 時無解。

老師：　解釋得很清楚。計算係數矩陣的行列式也可以幫助我們找到這一題的答案。

　　　　但是我們把這種解法留到下一個章節講解。

例題二

說明：本題需要先依題意假設三元一次方程組，再進一步求解。

（英文）Given the quadratic function $f(x)$ satisfying $f(-1)=k, f(1)=9k$, $f(3)=-15k$ and $k>0$. If a is the x-coordinate for the vertex of $f(x)$, which of the following is true?

(A) $a \leq -1$

(B) $-1<a<1$

(C) $a=1$

(D) $1<a<3$

(E) $3 \leq a$

（中文）已知實係數二次多項式函數 $f(x)$ 滿足$f(-1)=k$，$f(1)=9k$，$f(3)=-15k$，其中$k>0$。設函數 $y=f(x)$ 圖形頂點的 x 坐標為 a，試選出正確的選項。

(1) $a \leq -1$　(2) $-1<a<1$　(3) $a=1$　(4) $1<a<3$　(5) $3 \leq a$

（110 年數乙指考第 2 題）

Teacher:　We can assume $f(x)=\alpha x^2+\beta x+\gamma$.

To make the problem easier to solve, we further assume $k=1$.

Who can list the corresponding system of equations based on the given information?

Student:　It is $\begin{cases} \alpha-\beta+\gamma=1 \\ \alpha+\beta+\gamma=9 \\ 9\alpha+3\beta+\gamma=-15 \end{cases}$.

Teacher:　How to solve the system of equations?

Student: We can subtract the first equation from the second one. It yields $2\beta = 8$.

Replace β with 4 in each equation.

The equivalent system is $\begin{cases} \alpha + \gamma = 5 \\ 9\alpha + \gamma = -27 \end{cases}$.

Teacher: What is the next step?

Student: Subtraction again. The second equation minus the first equation.

It yields $8\alpha = -32$.

Alpha (α) is -4.

Gamma (γ)should be 9.

Teacher: What is the x-coordinate for the vertex of $f(x) = -4x^2 + 4x + 9$.

Student: We can apply the formula $x = \frac{-b}{2a}$.

In this case, $x = \frac{-4}{-8} = \frac{1}{2}$.

The correct choice is (B).

Teacher: It is easier for us to solve this problem when assuming $k = 1$.

If k is another positive value, the graph of this function has a vertical translation.

It actually does not affect the value of the x-coordinate.

老師： 我們可以假設 $f(x) = \alpha x^2 + \beta x + \gamma$。

為了讓問題更容易解決，我們進一步假設$k = 1$。

誰能列出根據所給信息得出的相應方程組？

學生： 應該是$\begin{cases} \alpha - \beta + \gamma = 1 \\ \alpha + \beta + \gamma = 9 \\ 9\alpha + 3\beta + \gamma = -15 \end{cases}$。

老師： 如何解這個方程組呢？

學生： 我們可以從第二個方程式中減去第一個方程式，得到 $2\beta = 8$。

將 β 替換為 4，代入每個方程式中。

等價方程組是$\begin{cases} \alpha + \gamma = 5 \\ 9\alpha + \gamma = -27 \end{cases}$。

老師： 下一個解題步驟是？

學生： 再次相減。

第二個方程式減去第一個方程式。得到 $8\alpha = -32$。

α 為−4。γ 應該是 9。

老師： 一元二次函數 $f(x) = -4x^2 + 4x + 9$ 的頂點 x 坐標為何？

學生： 我們可以套用公式 $x = \frac{-b}{2a}$，所以答案應該是 $x = \frac{-4}{-8} = \frac{1}{2}$。

老師： 當我們假設 $k = 1$ 時，這道問題很容易解決。

如果 k 是另一個正值，這個函數的圖形會垂直平移。

但這實際上並不會影響 x 坐標的值。

單元十　矩陣的運算
Operations with Matrices

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■ 前言 Introduction

將表格中的數據依照其原來的位置排成一個矩形的陣列，即可形成矩陣。矩陣在人工智慧中的數據處理和通訊網路的訊號處理等等領域中已被廣泛應用。本單元將介紹矩陣的運算與性質。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
matrix	**矩陣**	**dimension**	**（矩陣的）階；維度**
array	陣列	subscript	下標
row	**列**	**column**	**行**
diagonal	對角線	**element**	**（矩陣的）元；元素**
horizontal	水平的	**vertical**	**垂直的**
corresponding	對應的	undefined	未定義的
augmented	擴增的	coefficient	係數
identity	**本身；身份**	**inverse**	**相反的**

determinant	行列式值	**distributive**	分配的
associative	結合的	**communicative**	交換的

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ An $m \times n$ matrix is ___________.

例句(1)：**An $m \times n$ matrix is** a matrix having *m* horizontal rows and *n* vertical columns.
一個 $m \times n$階矩陣是一個具有 *m* 個水平列和 *n* 個垂直行的矩陣。

例句(2)：**An $m \times n$ matrix is** a rectangular array of numbers with *m* rows and *n* columns.
一個 $m \times n$ 階矩陣是一個具有 *m* 列和 *n* 行的矩形數字陣列。

❷ The element a_{ij} represents ___________.

例句(1)：**The element a_{ij} represents** the number in the *i*th row and *j*th column of the matrix.
元素 a_{ij} 代表矩陣中第 *i* 列與第 *j* 行的數字。

例句(2)：**The element a_{ij} represents** the number located at the intersection of the *i*th row and the *j*th column within the matrix.
元素 a_{ij} 代表在矩陣中第 *i* 列與第 *j* 行交叉點的數字。

❸ Two matrices are equal matrices if ___________.

例句(1)：**Two matrices are equal matrices if** they have the same dimension and corresponding elements.
如果兩個矩陣具有相同的維度和對應元素，則它們是相等的矩陣。

例句(2)：**Two matrices (of the same dimension) are equal matrices if** their corresponding elements at each position are equal.
若兩個具有相同維度的矩陣在每個位置上對應的元素都相等，則被視為相等。

❹ We add/subtract two matrices of the same dimension by _______.

例句(1)：**We add two matrices of the same dimension by** adding their corresponding elements.
我們將相同維度的兩個矩陣相加，是透過將它們對應位置的元素進行加法運算。

例句(2)：**We subtract two matrices of the same dimension by** performing subtraction operations on their corresponding elements.
我們將相同維度的兩個矩陣相減，是透過將它們對應位置的元素進行減法運算。

❺ The product of the real number k and the $m \times n$ matrix _______.

例句(1)：**The product of the real number k and the $m \times n$ matrix** still has the dimension of $m \times n$.
實數 k 與 $m \times n$ 矩陣的乘積仍然具有 $m \times n$ 的維度。

例句(2)：**The product of the real number k and the $m \times n$ matrix** $A = [a_{ij}]$ is called a scalar multiple of A.
實數 k 和 $m \times n$ 矩陣 $A = [a_{ij}]$ 的乘積被稱為矩陣 A 的係數積。

❻ To calculate the product AB of two matrices, _____.

例句(1)：**To calculate the product AB of two matrices,** the number of columns in matrix A must be the same as the number of rows in matrix B.
要計算兩個矩陣 AB 的乘積，矩陣 A 的列數必須與矩陣 B 的行數相同。

例句(2)：**To calculate the product AB of two matrices,** we sum up the products of the entries of a row of A and the corresponding entries of a column of B.
要計算兩個矩陣 AB 的乘積，我們將矩陣 A 的一行中的元素與矩陣 B 的相應列中的元素進行相乘，然後將這些乘積相加。

❼ A square matrix has ________.

例句(1)：**A square matrix has** the same number of columns and rows.

一個方陣具有相同的列數和行數。

例句(2)：**A square matrix has** entries a_{11}, a_{22}, a_{33} and so on in its main diagonal.

一個方陣在其主對角線上的元素依次為 a_{11}、a_{22}、a_{33} 等等。

■ 問題講解 Explanation of Problems

說明

In this section, the fundamental operations of matrices, which include addition, subtraction, scalar multiplication, and multiplication are covered. Matrices share several properties with real numbers, but it is important to note that matrix multiplication is not commutative.

Arranging a group of "numbers" or "texts" into a rectangular format of m rows and n columns, and enclosing them with the square bracket to be considered as one entity, is called an $m \times n$ order matrix or a matrix of order $m \times n$, as shown below.

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{matrix} \text{1st row} \\ \text{2nd row} \\ \vdots \\ m\text{th row} \end{matrix}$$

1st column

2nd column

$\cdots$

***n*th column**

This can be abbreviated as $A = [a_{ij}]_{m\times n}$ where the element a_{ij} located in the ith row and jth column is called the (i, j) element of A. Furthermore, matrices are generally represented by uppercase letters such as A, B, C, etc.

(1) Addition or Subtraction

Let $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$ be matrices of the same order, defined as follows:

- $A+B=[a_{ij}+b_{ij}]_{m\times n}$ (adding elements at corresponding positions);
- $A-B=[a_{ij}-b_{ij}]_{m\times n}$ (subtracting elements at corresponding positions).

(2) Scalar Product

Let $A=[a_{ij}]_{m\times n}$ and k be a real number, kA is defined as

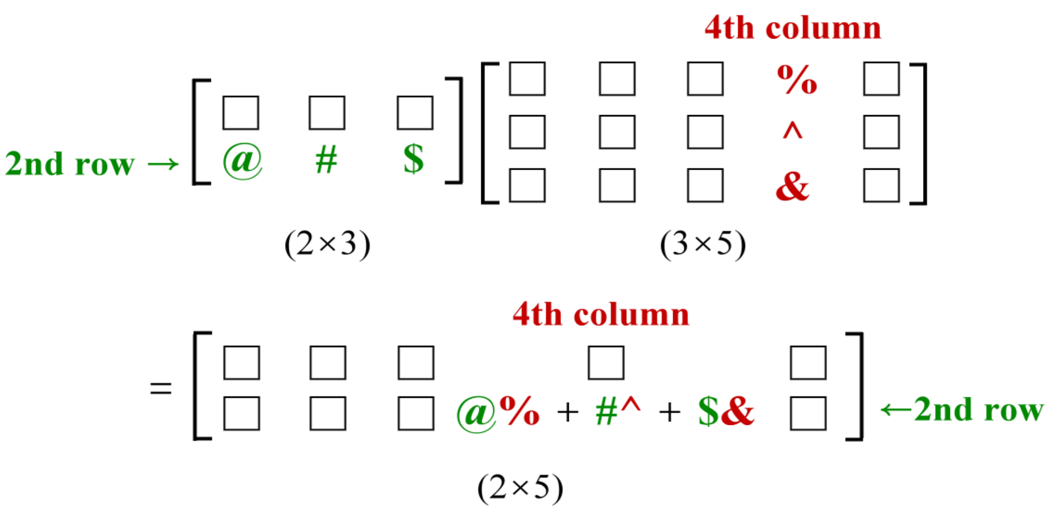
$kA=k[a_{ij}]_{m\times n}=[ka_{ij}]_{m\times n}$ (every element a_{ij} is multiplied by k).

(3) Multiplication

If $A=[a_{ij}]_{m\times n}$ is an $m\times n$ matrix and $B=[b_{ij}]_{n\times p}$ is an $n\times p$ matrix, then the product of A and B, $C=[c_{ij}]_{m\times p}$ is defined as an $m\times p$ matrix satisfying

$$c_{ij}=(a_{i1}b_{1j})+(a_{i2}b_{2j})+\ldots+(a_{in}b_{nj})$$

This can be seen as the operation similar to the dot product of the i^{th} row of A with the j^{th} column of B.



運算問題的講解

例題一

說明：本題是矩陣的加法，教師可向學生比較矩陣的加法和實數的加法。

（英文）For the following matrices, evaluate $A + B$ and $C + D$ if possible.

a.) $A=\begin{bmatrix}2 & -1\\1 & 0\end{bmatrix}$, $B=\begin{bmatrix}4 & 2\\2 & 1\end{bmatrix}$

b.) $C=\begin{bmatrix}2 & -1 & 5\\1 & 0 & 6\end{bmatrix}$, $D=\begin{bmatrix}4 & 3\\2 & 1\end{bmatrix}$

（中文）對於以下的矩陣，如果可能的話，計算 $A + B$ 和 $C + D$ 的值。

a.) $A=\begin{bmatrix}2 & -1\\1 & 0\end{bmatrix}$, $B=\begin{bmatrix}4 & 2\\2 & 1\end{bmatrix}$

b.) $C=\begin{bmatrix}2 & -1 & 5\\1 & 0 & 6\end{bmatrix}$, $D=\begin{bmatrix}4 & 3\\2 & 1\end{bmatrix}$

Teacher: Before we calculate the sum of two matrices, we need to ensure that both matrices have the same dimension.
If two matrices have different dimensions, their sum is undefined.
In the two problems mentioned here, can any of you tell me which one is undefined?

Student: The answer in part b.) is undefined because the dimension of matrix C is 2 by 3, while the dimension of matrix D is 2 by 2. Matrices of different dimensions cannot be added.

Teacher: Your explanation is very clear.
Now, let's solve the problem in part a).
To add two matrices of the same dimensions, we add their corresponding entries.
Can any of you provide the answer to part a)?

Student: This is an easy question.
The answer is $\begin{bmatrix}6 & 1\\3 & 1\end{bmatrix}$.

Teacher: Well done

老師：　在我們計算兩個矩陣的和之前，我們需要確保這兩個矩陣具有相同的維度。
如果兩個矩陣具有不同的維度，它們的和是未定義的。
在這裡提到的這兩個問題中，有沒有人可以告訴我哪一個是未定義的呢？

學生：　b）的答案是未定義的，因為矩陣 C 的維度是 2×3，而矩陣 D 的維度是 2×2。
不同維度的矩陣無法相加。

老師：　你的解釋非常清楚。現在，讓我們解決 a）的問題。
要將兩個相同維度的矩陣相加，我們將它們對應位置的元素相加。有沒有人能夠提供 a）的答案呢？

學生：　這是個很簡單的問題。
我的答案是 $\begin{bmatrix} 6 & 1 \\ 3 & 1 \end{bmatrix}$。

老師：　很好。

例題二

說明：本題是矩陣係數積的基本題型。

（英文）Solve for X in the equation $3X + A = B$, where $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$.

（中文）在方程式 $3X + A = B$ 中求解 X，其中 $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ 且 $B = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$。

Teacher:　Matrices share several properties with real numbers.
In this problem, we aim to solve for X in terms of A and B, treating A, B, and X as variables.
Can anyone tell me the answer?

Student:　To simplify the equation, we subtract B from both sides: $3X = B - A$
Next, we divide both sides by three:
Therefore, the final result is:

$$X = \frac{1}{3}(B - A).$$

Teacher:　Very clear explanation.
We then substitute the matrix for the variable. What is the answer?

Student: $B-A=\begin{bmatrix}2 & 3\\ 1 & 1\end{bmatrix}$

$$X=\begin{bmatrix}\frac{2}{3} & 1\\ \frac{1}{3} & \frac{1}{3}\end{bmatrix}$$

Teacher: Well done.

老師： 矩陣與實數有幾個共同的特性。在這個問題中，我們的目標是根據變數 A、B 和 X 來解 X。

有沒有人能告訴我答案呢？

學生： 為了簡化方程，我們從兩邊減去 B：$3X = B - A$

接下來，我們將兩邊都除以 3：

因此，最終的結果是：

$X = \frac{1}{3}(B - A)$。

老師： 非常清晰的解釋。

然後我們將矩陣代入變數中。答案是什麼呢？

學生： $B-A=\begin{bmatrix}2 & 3\\ 1 & 1\end{bmatrix}$

$$X=\begin{bmatrix}\frac{2}{3} & 1\\ \frac{1}{3} & \frac{1}{3}\end{bmatrix}$$

老師： 很棒。

例題三

說明：本題是矩陣的乘法。

（英文）Find the product of AB and BA where $A = \begin{bmatrix} 2 & -1 & 5 \\ 1 & 0 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$.

（中文）計算 AB 和 BA 的乘積，其中 $A = \begin{bmatrix} 2 & -1 & 5 \\ 1 & 0 & 6 \end{bmatrix}$ 且 $B = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$。

Teacher: To calculate the product AB of two matrices, the number of columns in matrix A must be the same as the number of rows in matrix B.
What are the dimensions of the two matrices?

Student: The dimension of matrix A is 2 by 3, and the dimension of matrix B is 2 by 2.

Teacher: By checking the dimension, can you tell which product is undefined?

Student: The product AB is undefined because the number of columns in matrix A is different from the number of rows in matrix B.

Teacher: The product AB is undefined, while the product BA is defined.
This explains that matrix multiplication is not commutative.
What is the dimension of the product BA?

Student: 2 by 3.

Teacher: Now, we have to sum up the products of the entries of a row of B and the corresponding entries of a column of A.
We need to calculate the sum of the products of the entries in a row of matrix B with the corresponding entries in a column of matrix A.
Can any of you tell me the answer?
Please come up to the dais and calculate the product of matrix BA.

Student:

$$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & -1 & 5 \\ 1 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 4 \times 2 + 2 \times 1 & 4 \times (-1) + 2 \times 0 & 4 \times 5 + 2 \times 6 \\ 2 \times 2 + 1 \times 1 & 2 \times (-1) + 1 \times 0 & 2 \times 5 + 1 \times 6 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & -4 & 32 \\ 5 & -2 & 16 \end{bmatrix}$$

Teacher: Mary's calculation process is very clear.
It's recommended that students, when calculating the product of matrices, first write down the individual element calculations and then carefully evaluate them.

老師： 要計算兩個矩陣的乘積 AB，矩陣 A 的列數必須與矩陣 B 的行數相同。
這兩個矩陣的維度是什麼呢？

學生： 矩陣 A 的維度是 2×3，而矩陣 B 的維度是 2×2。

老師： 通過檢查維度，你能夠告訴我哪個乘積是未定義的嗎？

學生： 矩陣 AB 的乘積是未定義的，因為矩陣 A 的列數與矩陣 B 的行數不同。

老師： 矩陣 AB 的乘積是未定義的，但是矩陣 BA 的乘積是有定義的。這解釋了矩陣乘法不具有交換律。
矩陣 BA 的維度是多少？

學生： 是 2×3。

老師： 現在，我們需要將矩陣 B 的一行中的項目與矩陣 A 的相應列的項目的乘積相加。我們需要計算矩陣 B 的一行中的項目與矩陣 A 的相應列中的項目的乘積之和。
有沒有人可以告訴我答案呢？請上台計算矩陣 BA 的乘積。

學生：
$$\begin{bmatrix}4 & 2\\2 & 1\end{bmatrix}\times\begin{bmatrix}2 & -1 & 5\\1 & 0 & 6\end{bmatrix}=\begin{bmatrix}4\times2+2\times1 & 4\times(-1)+2\times0 & 4\times5+2\times6\\2\times2+1\times1 & 2\times(-1)+1\times0 & 2\times5+1\times6\end{bmatrix}$$
$$=\begin{bmatrix}10 & -4 & 32\\5 & -2 & 16\end{bmatrix}$$

老師： 瑪莉的計算過程很清楚。
建議同學們計算矩陣的乘積時先寫下個別元素的計算式，再細心求值。

應用問題 / 學測指考題

例題一

說明：這題是矩陣係數積的運算。教師可直接引導學生將每個元素乘以係數。時間充足的情況下，可仔細解釋題意，甚至介紹菜餚名稱。

（英文）A group of three restaurants serves various dishes, each specializing in a different cuisine. The number of dishes prepared at restaurant j for a particular cuisine i in one day is represented by a_{ij} in matrix A.

$$A = \begin{bmatrix} 100 & 40 & 320 \\ 50 & 20 & 160 \end{bmatrix}$$

Determine the updated production levels when the overall production of dishes is increased by 10%.

（中文）有三家餐廳組成的一個團隊提供多種菜餚，每家餐廳專注於不同的料理風格。在一天內，餐廳 j 為特定料理風格 i 準備的菜餚數量由矩陣 A 中的 a_{ij} 表示。

$$A = \begin{bmatrix} 100 & 40 & 320 \\ 50 & 20 & 160 \end{bmatrix}$$

當菜餚的整體生產量增加 10%時，請確定更新後的生產水平。

Teacher: According to the description, the number of dishes prepared at restaurant j for a particular cuisine i in one day is represented by a_{ij} in matrix A.
Can any of you tell me the number of restaurants and cuisines mentioned in the question?

Student: The dimension of matrix A is 2 by 3.
There are 2 rows and 3 columns. It means there are 2 cuisines and 3 restaurants.

Teacher: Assuming these two dishes are named "Xiaolongbao" and "Stinky Tofu," these three restaurant names are "Luhe," "Haihe," and "Konghe."
Could any of you explain the meaning of entry 320 with some context?

Student: Three hundred twenty is in the first row and third column.
It means that the Konghe restaurant sold 320 " Xiaolongbao " on that day.

Teacher: Very clear explanation.
If the production of these cuisines is increased by 10%, what is the new production level?

Student: Multiply 320 by 10%. It is 32.

Teacher: Your answer should be the sum of 320 and 32 because the new production level takes the original production into account. Therefore, we can calculate 320 multiplied by 110%, which is 352.

Now, let's determine the solution to this problem using the scalar product of the matrix.

Who would like to show the work on the board?

Student: Let me try.

$$1.1 \times \begin{bmatrix} 100 & 40 & 320 \\ 50 & 20 & 160 \end{bmatrix} = \begin{bmatrix} 110 & 44 & 352 \\ 55 & 22 & 176 \end{bmatrix}$$

Teacher: Great.

老師： 根據描述，餐廳 j 在一天中為特定的菜式 i 準備的菜品數量由矩陣 A 中的a_{ij}表示。

有沒有人可以告訴我這個問題中提到的餐廳和菜式的數量？

學生： 矩陣 A 的維度是 2 × 3。

有 2 行和 3 列。這表示有 2 種菜式和 3 家餐廳。

老師： 假設這兩道菜品分別被稱為「小籠包」和「臭豆腐」，而這三家餐廳則分別是「陸和」、「海和」和「空和」。

有沒有人能夠解釋一下 320 的含義並加上一些背景說明呢？

320 位於第一行第三列。

學生： 這表示「空和」餐廳當天售出了 320 道「小籠包」。

老師： 非常清楚的解釋。

如果這些菜式的生產量增加了 10%，新的生產水平是多少呢？

學生： 將 320 乘以 10%，答案應該是 32。

老師： 你的答案應該是 320 加 32 的總和，因為新的生產水平考慮了原始生產水平。

因此，我們可以計算 320 乘以 110%，得到 352。

現在，讓我們使用矩陣的係數積來解決這個問題。

誰願意在黑板上展示解題過程呢？

學生： 讓我試試看。

$$1.1 \times \begin{bmatrix} 100 & 40 & 320 \\ 50 & 20 & 160 \end{bmatrix} = \begin{bmatrix} 110 & 44 & 352 \\ 55 & 22 & 176 \end{bmatrix}$$

老師： 做得很好。

例題二

說明：這題是兩個矩陣的乘積，教師教學上可與前一題矩陣的係數積做比較。

（英文）A restaurant offers two dishes, pasta, and sushi, and serves them at three different tables. The number of units of dish i served at table j is represented by a_{ij} in the matrix A.

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 10 & 0 & 6 \end{bmatrix}$$

The profit per unit for each dish is represented by the matrix B.

$B = [50 \quad 10]$

Calculate the product BA and explain its significance in terms of the restaurant's earnings.

（中文）一家餐廳提供兩道菜，義大利麵和壽司，並在三張不同的桌子上提供服務。菜品 i 在桌子 j 上提供的數量由矩陣 A 中的 a_{ij} 表示。

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 10 & 0 & 6 \end{bmatrix}$$

每道菜的每單位利潤由矩陣 B 表示。

$B = [50 \quad 10]$

計算乘積 BA 並解釋其在餐廳收益方面的意義。

Teacher: This problem requires you to calculate the product of BA and interpret its meaning.
We can begin by listing the calculation process.
Would any of you like to demonstrate the calculation process on the board?

Student: I think I can do it.

$$[50 \quad 10] \times \begin{bmatrix} 2 & 1 & 3 \\ 10 & 0 & 6 \end{bmatrix}$$

$$= [50 \times 2 + 10 \times 10 \quad 50 \times 1 + 10 \times 0 \quad 50 \times 3 + 10 \times 6]$$

$$= [200 \quad 50 \quad 210]$$

Teacher: For the element in the first row and first column, we are considering the revenue of the first restaurant.
50×2 represents 50 dollars per pasta multiplied by 2 dishes of pasta.
10×10 represents 10 dollars per sushi multiplied by 10 pieces of sushi.

The sum of these amounts gives us the total revenue received by the first restaurant.

Can any of you explain the meaning of the other elements?

Student: For the element in the first row and second column, we are considering the revenue of the second restaurant.

50×1 represents 50 dollars per pasta multiplied by 1 dish of pasta.

10×0 represents that no sushi was sold.

50 is the total revenue received by the second restaurant.

Teacher: Your explanation is very clear.

How about the remaining element?

Student: 210 is the total revenue received by the third restaurant.

Teacher: When you think the interpretation is more difficult than the calculation, you can start from the interpretation of a specific element.

老師： 這個問題要求你計算 BA 的乘積並解釋其意義。

我們可以從列出計算過程開始。

你們中是否有人願意在黑板上展示計算過程呢？

學生： 我覺得我可以試試。

$$\begin{bmatrix}50 & 10\end{bmatrix} \times \begin{bmatrix}2 & 1 & 3\\ 10 & 0 & 6\end{bmatrix}$$

$$= \begin{bmatrix}50 \times 2 + 10 \times 10 & 50 \times 1 + 10 \times 0 & 50 \times 3 + 10 \times 6\end{bmatrix}$$

$$= \begin{bmatrix}200 & 50 & 210\end{bmatrix}$$

老師： 對於第一行和第一列的元素，我們正在考慮第一家餐廳的收入。

50×2 代表每份義大利麵 50 美元，乘以 2 份義大利麵。

10×10 代表每份壽司 10 美元，乘以 10 片壽司。

這些金額的總和給出了第一家餐廳的總收入。

你們當中有人可以解釋其他元素的意義嗎？

學生： 對於第一行和第二列的元素，我們正在考慮第二家餐廳的收入。

50×1 代表每份義大利麵 50 美元，乘以 1 份義大利麵。

10×0 表示沒有售出壽司。

50 是第二家餐廳的總收入。

老師： 你的解釋非常清楚。那麼最後一個元素呢？

學生： 210 是第三家餐廳的總收入。

老師： 當你覺得解釋比計算更困難時，你可以從解釋特定元素的意義開始。

單元十一　矩陣的應用
Matrix Applications

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■ 前言 Introduction

矩陣可以用來描述和解決複雜問題。舉例而言，圖片的伸縮或平移是在平面上的線性轉換，可以透過矩陣的係數積或乘積實現。除此之外，轉移矩陣也是一個很重要的生活應用；在市場研究中，轉移矩陣能夠分析產品或服務的消費者轉換模式，有助於制定更有效的銷售策略；生態學研究中，轉移矩陣可用來研究物種在不同環境中的遷移和變化。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
transformation	**變換**	**translation**	**平移**
combination	**組合**	**transition**	**轉移**
rotation	**旋轉**	stability	穩定性
reflection	**鏡射；反射**	remaining	剩餘的
symmetry	**對稱**	approach	趨近
vertical	**垂直的**	**horizontal**	**水平的**
map	**映射**	**image**	**對應點**

yield	產生	**stretch**	**延伸**
scale sth up/down	放大/縮小	shrink	收縮

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ Linear transformations in the plane ________.

例句(1)：**A "linear transformation in the plane"** refers to a mathematical operation that takes points from the plane and maps them to new points while preserving certain properties.

「平面上的線性轉換」是指一種數學運算，它從平面中取點，並將它們映射到新的點並同時保留某些特性。

例句(2)：**Linear transformations in the plane** include operations like scaling, rotation, and reflection.

平面上的線性轉換包括縮放、旋轉和鏡射等。

❷ ______ stretched by a factor of ________.

例句(1)：On the coordinate plane, if point $P(x, y)$ is **stretched by a factor of h** along the x-axis (where $h > 0$) and a factor of k along the y-axis (where k > 0), with the origin O as the center, then the resulting point is $P'(x', y')$ is (hx, ky).

在坐標平面上，若以原點 O 為中心，將點 $P(x, y)$ 沿著 x 軸方向伸展 h 倍（其中 $h > 0$），並沿著 y 軸方向伸展 k 倍（其中 $k > 0$），則所得的新點為 $P'(x', y')$，其坐標為 (hx, ky)。

例句(2)：The image was vertically **stretched by a factor of 3**, resulting in a taller and slimmer appearance.

這個影像在垂直方向被拉伸了 3 倍，使其呈現更高且更修長的外觀。

❸ Transform _____ according to ________.

例句(1)：**Transform** the point $P(x, y)$ **according to** the linear relationship equations $x' = 3x - y$ and $y' = x + 2y$.

根據線性關係式 $x' = 3x - y$ 和 $y' = x + 2y$，將點 $P(x, y)$ 進行變換。

例句(2)：By applying the appropriate matrix operations, we can **transform the original coordinates according to** the given linear transformation equations.

透過適當的矩陣運算，我們可以根據所給的線性變換方程式將原始坐標進行變換。

❹ ______ perform a linear transformation _______.

例句(1)：Using the 2 by 2 matrix to **perform a linear transformation** from point P to point P'.

使用這個 2×2 矩陣進行從點 P 到點 P' 的線性變換。

例句(2)：Using the given matrix, we can **perform a linear transformation** on the original coordinates to map them onto a new set of points in the plane.

利用給定的矩陣，我們可以對原始坐標進行線性變換，將它們映射到平面上的一組新點。

❺ ______ after (undergoing) a linear transformation by _______.

例句(1)：If the determinant of a 2×2 matrix A is non-zero, then a line segment **after a linear transformation by** A will remain a line segment.

若一個二階矩陣 A 的行列式不為零，則線段則經過 A 的線性變換後仍為一條線段。

例句(2)：**After undergoing a linear transformation by the matrix** A**,** the vector $\vec{u}$ exhibits a change in both magnitude and direction.

在經過矩陣 A 的線性變換後，向量 $\vec{u}$ 的大小和方向都會有所改變。

❻ ______ be seen as a linear combination of ______.

例句(1)：The vector $\vec{u}$ can be seen **as a linear combination of** the vectors $\vec{a}$ and $\vec{b}$.

向量 $\vec{u}$ 可視為向量 $\vec{a}$ 和向量 $\vec{b}$ 的線性組合。

例句(2)：The solution to the system of equations can be represented **as a linear combination of** the basis vectors.

這個方程組的解可以表示為基底向量的線性組合。

❼ rotate ______ by an angle ______.

例句(1)：Taking the origin O as the center, **rotating point** $P(x, y)$ **by an angle** θ counterclockwise yields the point $P'(x', y')$.

以原點 O 為中心，將點 $P(x, y)$逆時針方向旋轉 θ 角度後得到點 $P'(x', y')$。

例句(2)：Taking the origin O as the center, **rotating point** $Q(x, -y)$ **by an angle** 2θ in the counterclockwise direction results in the point $P'(x', y')$.

以原點 O 為中心，將點$Q(x, -y)$逆時針方向旋轉 2θ 角度後，得到點$P'(x', y')$。

❽ reflect ______ across ______.

例句(1)：**Reflecting the point** $P(x, y)$ **across** the x-axis results in the symmetric point $Q(x, -y)$.

將點 $P(x, y)$ 對 x 軸進行對稱，得到對稱點 $(x, -y)$。

例句(2)：When you **reflect the point** $P(x, y)$ **across** the line $y = x$, it yields the point $P'(y, x)$.

當你將點 $P(x, y)$ 對 $y = x$的線進行對稱，得到的點為 $P'(y, x)$。

❾ the relationship between ______ and ______.

例句(1)：What is **the relationship between** the area of the new shape after a linear transformation **and** the area of the original shape?

在經過線性變換後的新圖形面積與原圖形的面積之間存在什麼關係呢？

例句(2)：Exploring **the relationship between** the elements in the matrix **and** the resulting image helps us understand how each element contributes to the transformation.
探索矩陣中元素與對應點影像之間的關係，有助於我們理解每個元素針對變換產生的效果。

⑩ ______ while the remaining ______.

例句(1)：In the urban area, 80% of people will remain in the city in the following year, **while the remaining** 20% migrate to the suburbs.
在都市地區，隔年有 80% 的人仍留在城市，而剩下的 20% 遷移到郊區。

例句(2)：Those who repay on time, 70% of them will continue to do so in the following month, **while the remaining** 30% will have delayed repayments.
那些準時還款的人，隔月有 70% 仍會繼續準時還款，而其餘 30% 則會延遲還款。

■ 問題講解 Explanation of Problems

說明

In this section, we will apply the fundamental operations of matrices to perform linear transformations in the plane.

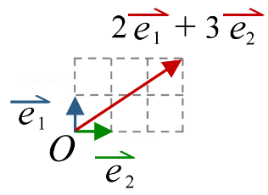
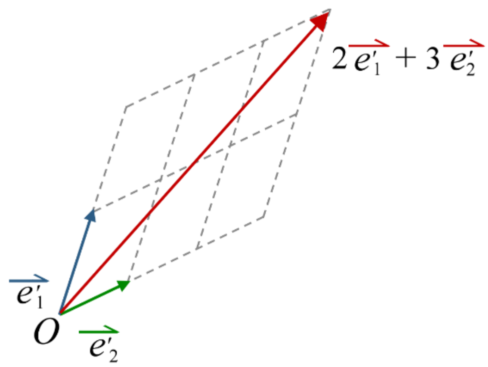
Take the 2×2 matrix A, which is $\begin{bmatrix}1 & 2\\ 3 & 1\end{bmatrix}$, for example.

If $A\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\3\end{bmatrix}$ and $A\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$, then it can be interpreted that point $P(1,0)$ and $Q(0,1)$ are transformed by the 2×2 matrix A to correspond to point $P'(1,3)$ and $Q'(2,1)$.

Additionally, from $A\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x+2y\\3x+1y\end{bmatrix}$, we can also know that a general point $R(x,y)$ can be transformed by the 2×2 matrix A to correspond to $R'(x+2y,3x+y)$.

Given a 2×2 matrix, $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$, $\overrightarrow{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\overrightarrow{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

If $A(2\overrightarrow{e_1} + 3\overrightarrow{e_2}) = 2A\overrightarrow{e_1} + 3A\overrightarrow{e_2} = 2\overrightarrow{e_1'} + 3\overrightarrow{e_2'}$, then $\overrightarrow{e_1'} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\overrightarrow{e_2'} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

	after the transformation	

After the transformation by the 2×2 matrix A, the coefficients of the vector's linear combination remain unchanged. Therefore, we refer to it as a "linear transformation in the plane".

運算問題的講解

例題一

說明：本題是線性轉換的基本題，教師可以圖示說明矩陣於圖形上的應用。

（英文）The transformation matrix $T = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ is given. Find a point Q that, after undergoing a linear transformation by T, corresponds to the point (4, 6).

（中文）給定二階方陣 $T = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$，找到一個點 Q，使其經過 T 的線性變換後的對應點是(4, 6)。

Teacher: First of all, we arrange the coordinate of point Q in a 2 by 1 matrix $\begin{bmatrix} x \\ y \end{bmatrix}$.

According to the descriptions, the matrix equation is $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$.

Do any of you know how to solve the variable matrix $\begin{bmatrix} x \\ y \end{bmatrix}$?

Student: We have to find the inverse of matrix T.

The determinant of matrix T is six minus five. It is one.

T inverse would be $\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$.

We have to multiply the inverse to both sides of the equation.

Teacher: Do we multiply the T inverse in the front of the variable matrix or at the back?

Student: We multiply T inverse in the front of the variable matrix since T inverse is applied to the front of T. Let me show the work on the board as an explanation.

$$\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{Left} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\text{Right} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \times \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \end{bmatrix}$$

Teacher: Well done. The point Q is $(6, -8)$

In this question, we perform the basic linear transformation on points.

Later on, we will extend this transformation to include segments and vectors.

老師： 首先，我們將 Q 點坐標排列成一個 2×1 的矩陣 $\begin{bmatrix} x \\ y \end{bmatrix}$。根據題意，矩陣方程是 $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$。有人知道如何解變數矩陣 $\begin{bmatrix} x \\ y \end{bmatrix}$ 嗎？

學生： 我們必須找到矩陣 T 的反矩陣。

矩陣 T 的行列式是 $6-5=1$。T 的反矩陣是 $\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

接著要將反矩陣乘到方程式的兩邊。

老師： 要將反矩陣 T 乘在變數矩陣的前面還是後面呢？

學生： 乘在變數矩陣的前面，因為反矩陣 T 是乘在矩陣 T 的前面。讓我在黑板上寫下計算過程。

$$\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

左邊：$\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

右邊：$\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \times \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \end{bmatrix}$

老師： 做得很好。Q 點的坐標是 $(6, -8)$。

這題裡，我們練習對點做基本的線性轉換。等一下我們延伸到線段和向量的轉換。

例題二

說明：本題仍是線性轉換的基本題，教師可以補充說明坐標平面上線段經過線性變換後仍為線段等性質。

（英文）Given a linear transformation T that transforms the point (0, 1) to (1, 3) and simultaneously transforms the point (6, −8) to (4, 6), find the corresponding 2×2 matrix T.

（中文）已知某個線性變換 T 將點 (0, 1) 變換為點 (1, 3)，同時將點 (6, −8) 變換為 (4, 6)，求對應的二階方陣 T。

Teacher: This problem is similar to the previous one.

The coordinates of a single point can be organized in a 2 by 1 matrix.

We can assemble the coordinates of two original points into a 2 by 2 matrix, and similarly, we arrange the resulting images in another 2 by 2 matrix.

$A = \begin{bmatrix} 0 & 6 \\ 1 & -8 \end{bmatrix}, B = \begin{bmatrix} 1 & 4 \\ 3 & 6 \end{bmatrix}$

The transformation matrix T is assumed as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

The matrix equation would be $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} 0 & 6 \\ 1 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 3 & 6 \end{bmatrix}$.

How to solve this matrix equation?

Student: We have to find the inverse of matrix A.

The determinant of matrix A is 0−6. It is −6.

A inverse would be $\frac{1}{-6}\begin{bmatrix} -8 & -6 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} & 1 \\ \frac{1}{6} & 0 \end{bmatrix}$.

We have to multiply the inverse to both sides of the equation.

Teacher: Do remember that the matrix multiplication is not commutative.

Do we multiply *A* inverse in the front of the matrix *B* or at the back?

Student: We multiply *A* inverse at the back of the variable matrix since *A* inverse is applied to the back of *A*.

$$\begin{bmatrix}1 & 4\\3 & 6\end{bmatrix}\times\begin{bmatrix}\frac{4}{3} & 1\\\frac{1}{6} & 0\end{bmatrix}=\begin{bmatrix}2 & 1\\5 & 3\end{bmatrix}.$$

Teacher: Great.

Here, the transformation matrix is $\begin{bmatrix}2 & 1\\5 & 3\end{bmatrix}$.

老師：這題題型與前一題類似。

單一點的坐標可以寫成一個 2×1 的矩陣。我們可以將兩個點的坐標組合成一個 2×2 的矩陣，同樣地，將兩個坐標排列至 2×2 的矩陣中

$A=\begin{bmatrix}0 & 6\\1 & -8\end{bmatrix}$、$B=\begin{bmatrix}1 & 4\\3 & 6\end{bmatrix}$

我們假設變換矩陣 *T* 為 $\begin{bmatrix}a & b\\c & d\end{bmatrix}$

矩陣方程式變成 $\begin{bmatrix}a & b\\c & d\end{bmatrix}\times\begin{bmatrix}0 & 6\\1 & -8\end{bmatrix}=\begin{bmatrix}1 & 4\\3 & 6\end{bmatrix}$。

要如何解這個矩陣方程式呢？

學生：我們必須找到矩陣 A 的反矩陣。矩陣 A 的行列式是 0−6，等於−6。

A 的反矩陣是 $\frac{1}{-6}\begin{bmatrix}-8 & -6\\-1 & 0\end{bmatrix}=\begin{bmatrix}\frac{4}{3} & 1\\\frac{1}{6} & 0\end{bmatrix}$

接著要將反矩陣乘到方程式的兩邊。

老師：記住，矩陣乘法不是可交換的。

請問是要將 *A* 的反矩陣乘在變數矩陣的前面還是後面呢？

學生：乘在變數矩陣的後面，因為 *A* 的反矩陣應用在 *A* 的後面。

$$\begin{bmatrix}1 & 4\\3 & 6\end{bmatrix}\times\begin{bmatrix}\frac{4}{3} & 1\\\frac{1}{6} & 0\end{bmatrix}=\begin{bmatrix}2 & 1\\5 & 3\end{bmatrix}$$

老師：在這裡，變換矩陣是 $\begin{bmatrix}2 & 1\\5 & 3\end{bmatrix}$。

例題三

說明：本題是旋轉矩陣在影像變換的用法。由於旋轉矩陣有三角函數，建議教師可適當複習相關內容。

（英文）An equilateral triangle OPQ, with the coordinates of two vertices being $O(0, 0)$ and $P(4, 3)$ is given. Determine the coordinates of the vertex Q.

（中文）已知正三角形 OPQ 中二頂點坐標為 $O(0 , 0)$, $P(4 , 3)$，求頂點 Q 的坐標。

Teacher: What is the side length of this equilateral triangle?

Student: The square root of 4 squared and 3 squared is 5.

Teacher: $\overline{OQ} = \overline{PQ} = 5$

If we assume the coordinate of the point Q as (x, y), we can apply the distance equation to solve for the coordinate.

$$\begin{cases} x^2 + y^2 = 25 \\ (x-4)^2 + (y-3)^2 = 25 \end{cases}$$

There is another way of solving this question by applying the rotation matrix.

At what angle can we rotate $\overline{OP}$ to obtain $\overline{OQ}$?

Student: Sixty degrees because the interior angle of an equilateral is 60 degrees.

Teacher: Taking the origin O as the center, we can rotate $\overline{OP}$ for 60 degrees clockwise or counterclockwise.

Therefore, there are two possible points. Let's rotate the segment for 60 degrees counterclockwise first.

Could any of you show the work on the board?

Student: We have to apply the rotation matrix.

$$\begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} \times \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \times \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{4-3\sqrt{3}}{2} \\ \frac{4\sqrt{3}+3}{2} \end{bmatrix}.$$

Teacher: Well done. We can apply the same technique to find the other possible point.

If we rotate the segment for 60 degrees clockwise, the angle in the rotation matrix is -60 degrees.

Could any of you show the work to find the other possible point?

Student: I can do it.

$$\begin{bmatrix}\cos(-60°) & -\sin(-60°)\\ \sin(-60°) & \cos(-60°)\end{bmatrix}\times\begin{bmatrix}4\\3\end{bmatrix}=\begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{-\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}\times\begin{bmatrix}4\\3\end{bmatrix}=\begin{bmatrix}\frac{4+3\sqrt{3}}{2}\\ \frac{3-4\sqrt{3}}{2}\end{bmatrix}.$$

Teacher: You did a great job.

This question serves as a reminder of the practicality of using a rotation matrix.

老師： 這個等邊三角形的邊長是多少？

學生： 4 平方加 3 平方，再開平方根是 5。

老師： $\overline{OQ}=\overline{PQ}=5$。如果我們假設點 Q 的坐標為 (x, y)，我們可以應用距離方程式來求出坐標。$\begin{cases} x^2+y^2=25\\ (x-4)^2+(y-3)^2=25\end{cases}$

還有另一種方法：應用旋轉矩陣來解這個問題。

將 $\overline{OP}$ 旋轉到 $\overline{OQ}$ 的角度是多少？

學生： 60 度，因為等邊三角形的內角是 60 度。

老師： 以原點 O 為中心，我們可以將 $\overline{OP}$ 順時針或逆時針旋轉 60 度，因此，有兩個可能的點。首先，我們先逆時針旋轉 60 度。

有人可以在黑板上寫下計算過程嗎？

學生： 我們可以利用旋轉矩陣。

$$\begin{bmatrix}\cos 60° & -\sin 60°\\ \sin 60° & \cos 60°\end{bmatrix}\times\begin{bmatrix}4\\3\end{bmatrix}=\begin{bmatrix}\frac{1}{2} & \frac{-\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}\times\begin{bmatrix}4\\3\end{bmatrix}=\begin{bmatrix}\frac{4-3\sqrt{3}}{2}\\ \frac{4\sqrt{3}+3}{2}\end{bmatrix}$$

老師： 做得好。用相同的技巧來找到另一個可能的點。

如果我們將線段順時針旋轉 60 度，旋轉矩陣中的角度是−60 度。

有人可以來寫出計算過程，求出另一個可能的點嗎？

學生： 我！

$$\begin{bmatrix}\cos(-60°) & -\sin(-60°)\\ \sin(-60°) & \cos(-60°)\end{bmatrix}\times\begin{bmatrix}4\\3\end{bmatrix}=\begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{-\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}\times\begin{bmatrix}4\\3\end{bmatrix}=\begin{bmatrix}\frac{4+3\sqrt{3}}{2}\\ \frac{3-4\sqrt{3}}{2}\end{bmatrix}$$

老師： 做得很好喔！

這個問題提醒了我們旋轉矩陣的實用性。

應用問題／學測指考題

例題一

說明： 這題是利用移轉矩陣計算穩定狀態，涉及矩陣的乘法運算，教師可適當複習之。這些矩陣能夠幫助我們理解系統如何從一個狀態轉移到另一個狀態，以及在轉移過程中可能出現的變化和影響。

（英文）Every Friday, the government distributes two types of lottery tickets, labeled as A and B, to all residents in the county. Each resident can choose to receive either an A-ticket or a B-ticket free of charge by presenting their identification card. Based on long-term statistics, individuals who chose the A-ticket last week will have 85% of them sticking with the A-ticket this week, while the remaining 15% will switch to the B-ticket. On the other hand, individuals who chose the B-ticket last week will have 35% of them switching to the A-ticket this week, while the remaining 65% will stick with the B-ticket. The term "stability" refers to a situation where the proportions of people receiving A and B tickets remain constant every week.

(1) Please write the transition matrix that describes the above phenomenon.

(2) Can you determine the percentage of residents choosing the A-ticket and the B-ticket that would lead to a steady state?

（中文）某縣縣政府每週五對全縣居民發放甲、乙兩種彩券，每位居民均可 憑身分證免費選擇領取甲券一張或乙券一張。根據長期統計，上週選擇甲券的民眾會有 85%在本週維持選擇甲券、15%改選乙券；而選擇乙券的民眾會有 35%在本週改選甲券、65%維持乙券。所謂穩定狀態，係指領取甲券及乙券的民眾比例在每週均保持不變。

(1) 試寫出描述上述現象的轉移矩陣。

(2) 試問領取甲券和乙券民眾各占全縣居民百分比多少時，會形成穩定狀態？

（106 年數乙）

Teacher: A transition matrix can be used to describe the probabilities of transitions between states.

Typically, we record the probability of transitioning to the new state from the initial situation in the first column, and we indicate the probability of transitioning to the new state from the second situation in the second column.

Do any of you know the transition matrix?

Student: It is $\begin{bmatrix} 0.85 & 0.35 \\ 0.15 & 0.65 \end{bmatrix}$.

Teacher: Assume that the percentage of residents choosing the A-ticket and the B-ticket that would lead to stability is $\begin{bmatrix} x \\ y \end{bmatrix}$. Certainly, $x + y = 1$.

What is the equation representing stability?

Student: $\begin{bmatrix} 0.85 & 0.35 \\ 0.15 & 0.65 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$.

Teacher: Multiply the two matrices on the left, what is the result?

Student: $\begin{bmatrix} 0.85x + 0.35y \\ 0.15x + 0.65y \end{bmatrix}$

Teacher: This is actually a system of two linear equations.

$$\begin{cases} 0.85x + 0.35y = x \\ 0.15x + 0.65y = y \end{cases}$$

The first equation indicates the convergence of the percentage of individuals opting for an A-ticket towards stability. The second equation demonstrates the convergence of the percentage of individuals opting for B-ticket towards stability.

Replace y with $1 - x$. We obtain $0.85x + 0.35(1 - x) = x$

What is the value of x?

Student: $x = \dfrac{0.35}{0.5} = 0.7$

Teacher: The percentage of residents choosing the A-ticket and the B-ticket is 0.7 and 0.3 respectively.

老師：　轉移矩陣可用於描述狀態之間轉換的機率。

通常，我們將從初始情況轉移到新狀態的機率記錄在第一行，將從第二種情況轉移到新狀態的機率記錄在第二行。

有人知道轉移矩陣怎麼寫嗎？

學生：　是 $\begin{bmatrix} 0.85 & 0.35 \\ 0.15 & 0.65 \end{bmatrix}$。

老師：　假設形成穩定狀態時，選擇甲券和乙券的民眾所佔的百分比為 $\begin{bmatrix} x \\ y \end{bmatrix}$。總百分比

$x + y = 1$

那麼形成穩定狀態的方程式是什麼？

學生：　$\begin{bmatrix} 0.85 & 0.35 \\ 0.15 & 0.65 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

老師：　將左側的兩個矩陣相乘，會得到？

學生：　$\begin{bmatrix} 0.85x + 0.35y \\ 0.15x + 0.65y \end{bmatrix}$

老師：　這實際上是一個由兩個線性方程式組成的方程組。$\begin{cases} 0.85x + 0.35y = x \\ 0.15x + 0.65y = y \end{cases}$

第一個方程式表示選擇甲券的百分比趨向穩定；第二個方程式表示選擇乙券的百分比趨向穩定。

將 y 替換為 $1 - x$。我們得到 $0.85x + 0.35(1 - x) = x$。x 是多少？

學生：　$x = \dfrac{0.35}{0.5} = 0.7$

老師：　選擇甲券和乙券的民眾百分比分別為 0.7 和 0.3。

例題二

說明： 這題透過矩陣的線性變換判斷圖形面積關係，教師可適當複習面積公式。

（英文）Two coordinates of an equilateral triangle OAB as $O(0, 0)$ and $A(2, 1)$ are given. We scale triangle OAB up by a factor of 3 along the x-axis and 4 along the y-axis, with the origin as the center.

(1) If point $A(2, 1)$ transforms into point A', determine the coordinates of A'.

(2) By what factor does the area of the transformed triangle change compared to the area of the original triangle?

（中文）已知正$\triangle OAB$ 的二頂點坐標為 $O(0, 0)$ 與 $A(2, 1)$，今以原點為中心，將$\triangle OAB$ 沿 x 軸、y 軸方向分別伸縮為 3 倍、4 倍。

(1) 若點 $A(2, 1)$ 變換為點 A'，求 A' 坐標。

(2) 變換後的新三角形面積是變換前三角形面積的多少倍？

Teacher: According to the description, the transformation matrix is $T = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$.

What is the coordinate of point A'?

Student: $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

Teacher: We can apply the rotation matrix to find the third vertex and then find the area of the triangle.

However, we can also assume the coordinate of point B as (p, q) and find the area in terms of p and q.

What is the coordinate of B' given that B is (p, q)?

Student: It would be $(3p, 4q)$

Teacher: Great.

Now, let's find the area by using the determinant.

$\Delta = \frac{1}{2}\left|\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\right|$, where (a_1, a_2) and (b_1, b_2) represent the vectors corresponding to the sides of the formed triangle.

What is the area of the original triangle?

Student: $\frac{1}{2}\left|\begin{vmatrix} 2 & p \\ 1 & q \end{vmatrix}\right| = \frac{1}{2}|2q - p|$.

Teacher: How about the area of the second triangle?

Student: $\frac{1}{2}\left\|\begin{matrix}6 & 3p\\ 4 & 4q\end{matrix}\right\| = \frac{1}{2}|24q-12p| = \frac{12}{2}|2q-p|$.

Teacher: Try to compare the two results, we can find out that the area of the new triangle is 12 times that of the original triangle.

Actually, the ratio is the same as the determinant of the transformation matrix.

老師： 根據題目描述，可列出變換矩陣為 $T=\begin{bmatrix}3 & 0\\ 0 & 4\end{bmatrix}$

點 A' 的坐標是多少？

學生： $\begin{bmatrix}3 & 0\\ 0 & 4\end{bmatrix}\times\begin{bmatrix}2\\ 1\end{bmatrix}=\begin{bmatrix}6\\ 4\end{bmatrix}$

老師： 我們可以用旋轉矩陣來找到第三個頂點，然後找出三角形的面積。

不過，我們也可以假設點 B 的坐標為(p, q)，並以 p 和 q 的形式找出面積。

如果 B 是(p, q)，那麼 B' 的坐標是什麼？

學生： 會是 $B'(3p, 4q)$

老師： 很好。

現在，用行列式來找出面積。

$\Delta=\frac{1}{2}\left\|\begin{matrix}a_1 & b_1\\ a_2 & b_2\end{matrix}\right\|$，其中$(a_1, a_2)$和$(b_1, b_2)$代表所形成三角形邊的向量。

原三角形的面積是多少？

學生： $\frac{1}{2}\left\|\begin{matrix}2 & p\\ 1 & q\end{matrix}\right\| = \frac{1}{2}|2q-p|$。

老師： 第二個三角形的面積呢？

學生： $\frac{1}{2}\left\|\begin{matrix}6 & 3p\\ 4 & 4q\end{matrix}\right\| = \frac{1}{2}|24q-12p| = \frac{12}{2}|2q-p|$。

老師： 比較一下這兩個結果，我們可以發現新三角形的面積是原三角形的 12 倍。

實際上，這個比例與變換矩陣的行列式相同。

例題三

說明：這題利用矩陣解決加密問題。

（英文）Matrices can be used to encrypt messages. Assuming a certain password, *abcd*, satisfies the relationship: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times A = B$, find the corresponding password *abcd*.

（中文）矩陣可以用來加密訊息，假設某組密碼 *abcd* 會滿足關係式：$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times A = B$。已知 $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ 和 $B = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$，試求出對應的密碼 *abcd*。

Teacher: This is an encryption problem, yet its solution is similar to the linear transformation between two sets of points on a plane.
What is the matrix equation according to the descriptions?

Student: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$

Teacher: To solve this matrix equation, we have to multiply A inverse for both sides.
What is A inverse?

Student: The determinant of matrix A is 1.
A inverse is $\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$.

Teacher: Could any of you show the work to solve this matrix equation on the board?

Student: Let me try.
$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 8 \\ -1 & 4 \end{bmatrix}$.

Teacher: Well done.
This question is a simple demonstration of how the matrix multiplication is applied in encryption. We use special matrices to mix up data. To read it, the receiver needs another matrix to unscramble it. This method helps keep information safe when sending it on the internet.

老師：這是一個加密問題，而解題方法類似於平面上兩組點之間的線性變換。
根據題目敘述，矩陣方程式怎麼列？

學生：$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$

老師：　要解這個矩陣方程式，我們必須兩邊乘上 A 的反矩陣。

A 的反矩陣是？

學生：　矩陣 A 的行列式為 1。

A 的反矩陣是 $\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$

老師：　有人可以在黑板上寫出解矩陣方程的過程嗎？

學生：　讓我試試。

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 8 \\ -1 & 4 \end{bmatrix}$。

老師：　做得好。

這題是在演示矩陣乘法如何應用於加密的簡單例題。我們使用特殊的矩陣來打亂數據。若要解讀，接收者必須用另一個矩陣來解密它。這種方法有助於在網路上傳送訊息時保護訊息安全。

國內外參考資源 More to Explore

國家教育研究院樂詞網	
查詢學科詞彙 https://terms.naer.edu.tw/search/	
教育雲：教育媒體影音	
為教育部委辦計畫雙語教學影片 https://video.cloud.edu.tw/video/co_search.php?s=%E9%9B%99%E8%AA%9E	
Oak Teacher Hub	
國外教學及影音資源，除了數學領域還有其他科目 https://teachers.thenational.academy/	
CK-12	
國外教學及影音資源，除了數學領域還有自然領域 https://www.ck12.org/student/	
Twinkl	
國外教學及影音資源，除了數學領域還有其他科目，多為小學及學齡前內容 https://www.twinkl.com.tw/	

Khan Academy	
可汗學院，有分年級數學教學影片及問題的討論 https://www.khanacademy.org/	
Open Textbook (Math)	
國外數學開放式教學資源 http://content.nroc.org/DevelopmentalMath.HTML5/Common/toc/toc_en.html	
MATH is FUN	
國外教學資源，還有數學相關的小遊戲 https://www.mathsisfun.com/index.htm	
PhET: Interactive Simulations	
國外教學資源，互動式電腦模擬。除了數學領域，還有自然科 https://phet.colorado.edu/	
Eddie Woo YouTube Channel	
國外數學教學影音 https://www.youtube.com/c/misterwootube	

國立臺灣師範大學數學系陳界山教授網站	
國高中數學雙語教學相關教材 https://math.ntnu.edu.tw/~jschen/index.php?menu=Teaching_Worksheets	
2024 年第五屆科學與科普專業英文(ESP)能力大賽	
科學專業英文相關教材，除了數學領域，還有其他領域 https://sites.google.com/view/ntseccompetition/%E5%B0%88%E6%A5%AD%E8%8B%B1%E6%96%87%E5%AD%B8%E7%BF%92%E8%B3%87%E6%BA%90/%E7%9B%B8%E9%97%9C%E6%95%99%E6%9D%90?authuser=0	
Desmos Classroom	
國外教學資源，也有免費繪圖功能 https://teacher.desmos.com/?lang=en	

高中數學領域雙語教學資源手冊：英語授課用語

[十一年級下學期]

A Reference Handbook for Senior High School Bilingual Teachers in the Domain of Mathematics: Instructional Language in English

[11th grade 2nd semester]

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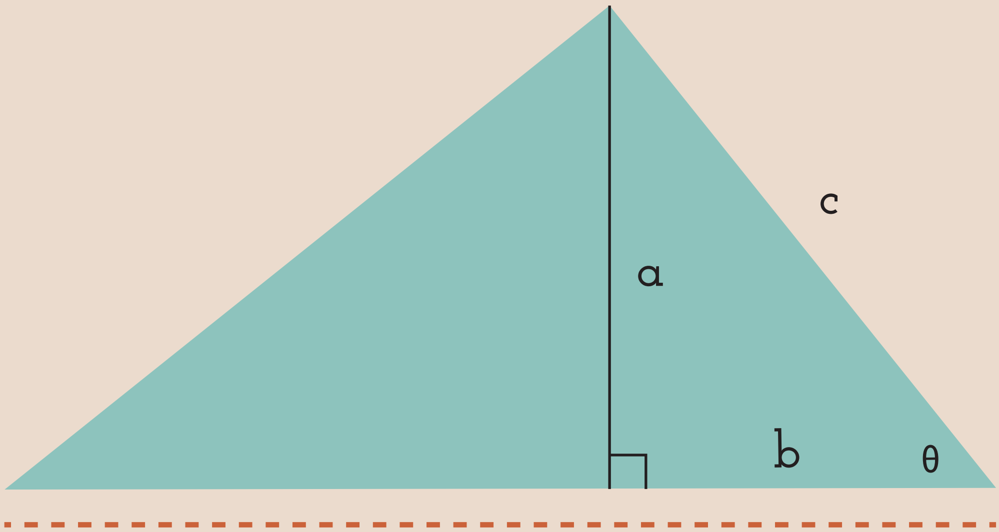
- 語言諮詢：Ramon Mislang
- 綜合規劃：王宏均
- 排版：吳依靜
- 封面封底：JUPE Design



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