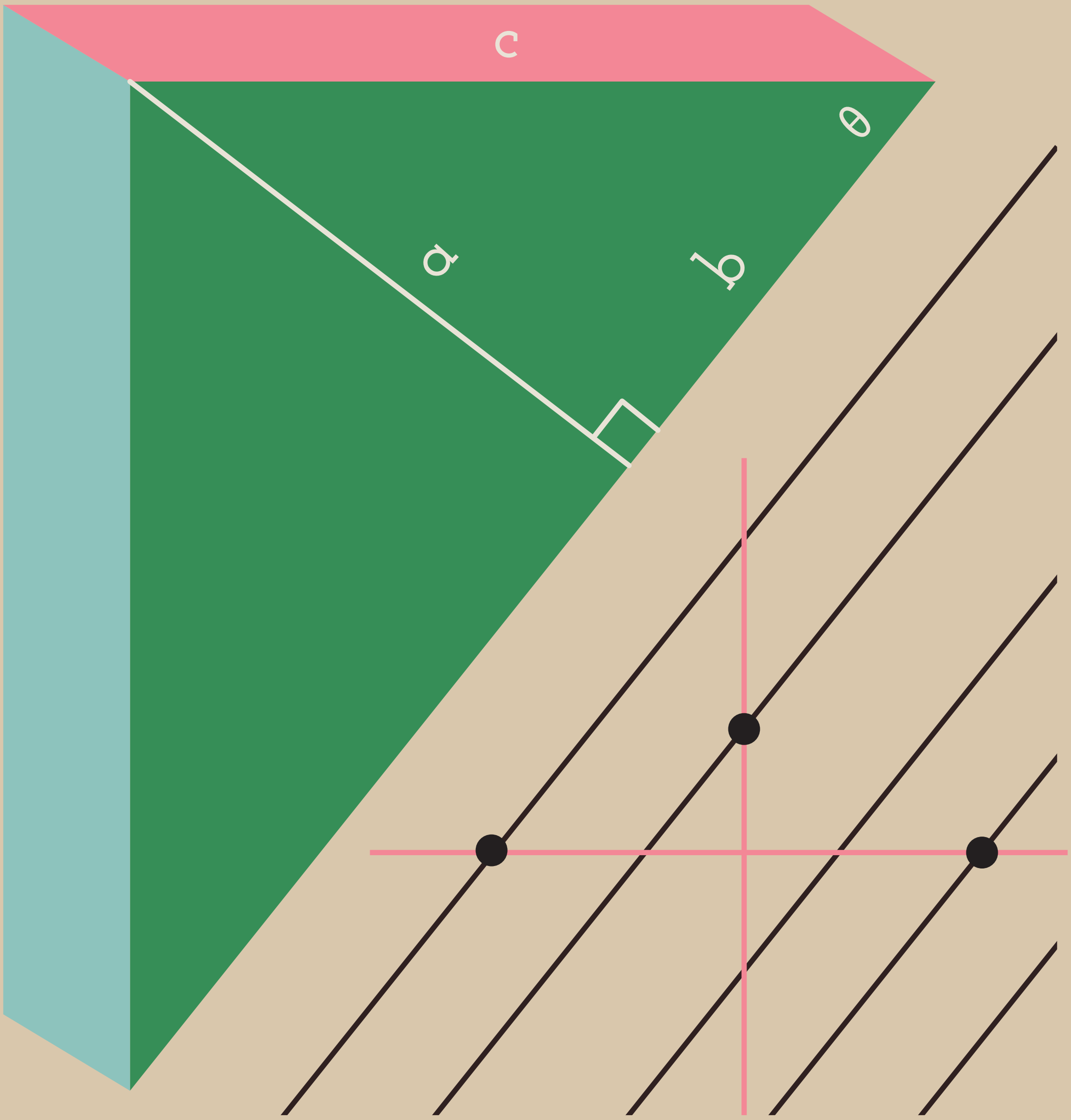
高中數學領域

雙語教學資源手冊 英語授課用語

A Reference Handbook for Senior High School Bilingual Teachers in the Domain of Mathematics: Instructional Language in English

〔高二上學期〕



臺師大雙語教學研究中心 發行

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單元一　三角函數的圖形
Graphs of the Trigonometric Functions

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■ 前言 Introduction

正弦和餘弦函數的波浪狀特性與周期性讓它們成為描述許多自然現象的理想選擇。這些現象，如季節性的氣溫變化、海洋的潮起潮落，以及月球的各個階段，都可以透過這些三角函數來表現或模擬。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
radian	**弧度；弳（度）**	**arc length**	**弧長**
sector	**扇形**	angular speed	角速度
periodic	**週期性的**	**amplitude**	**振幅**
symmetry	**對稱性**	**frequency**	**頻率**
clockwise	順時針的	counterclockwise	逆時針的
domain	定義域	range	值域
oscillate	震盪	shift	平移
stretch	延伸	shrink	收縮

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ If ___________, then ________.

例句(1)：**If** a central angle of a circle intercepts an arc equal in length to the radius**, then** the measure of that angle is 1 radian.

如果圓的中心角所截取的弧長等於半徑的長度，那麼該角度的量值為 1 弧度。

例句(2)：**If** a point on the unit circle corresponds to an angle in the standard position**, then** its *y*-coordinate gives the value of the sine of that angle.

如果單位圓上的一點對應於標準位置的角度，那麼它的 *y* 坐標給出了該角度的正弦值。

❷ A periodic function ______.

例句(1)：**A periodic function** repeats its values in regular intervals or periods.

一個週期性函數在固定的間隔或週期內重複其值。

例句(2)：**A periodic function** displays a repetitive pattern at equal intervals of its domain.

一個週期性函數在其定義域的等間隔中顯示重複的模式。

❸ A function *f*(*x*) is said to be _______.

例句(1)：**A function *f*(*x*) is said to be** periodic if there exists a non-zero number *P* such that $f(x+P)=f(x)$ for all values of *x*.

對於一個函數 $f(x)$，如果存在一個非零數 P 使得對所有的$f(x+\mathrm{P})=f(x)$，則稱 $f(x)$ 為週期性函數。

例句(2)：**A function *f*(*x*) is said to be** sinusoidal if it can be written as $f(x)=a\sin(bx+c)+d$.

若一函數 *f*(*x*) 可被表示為 $f(x)=a\sin(bx+c)+d$.，則該函數被認為是正弦波型。

❹ The amplitude of a sinusoidal function is ______.

例句(1)：**The amplitude of a sinusoidal function is** half the height of the wave.
正弦波型函數的振幅是波高的一半。

例句(2)：**The amplitude of a sinusoidal function** $f(x) = a\sin(bx + c) + d$ **is** $|a|$.
對於正弦波型函數 $f(x) = a\sin(bx + c) + d$ 的圖形，它的振幅是$|a|$。

❺ The period of a sinusoidal function is ______.

例句(1)：**The period of a sinusoidal function** is the length of its domain required to complete one full cycle.
對於正弦波型函數，其週期是完成一個完整週期所需的定義域的長度。

例句(2)：**The period of a sinusoidal function** $f(x) = a\sin(bx + c) + d$ is $\frac{2\pi}{|b|}$.
對於正弦波型函數 $f(\mathrm{x}) = a\sin(bx + c) + d$ 的圖形，它的週期是 $\frac{2\pi}{|b|}$ 。

❻ The graph of $g(x)$ is a vertical/ horizontal stretch/shrink of the graph of $f(x)$ by a factor of ___.

例句(1)：**The graph of** $g(x) = 2\cos x$ **is a vertical stretch of the graph of** $f(x) = \cos x$ **by a factor of** 2.
$g(x) = 2\cos x$ 的圖形是 $f(x) = \cos x$ 的圖形經過垂直拉伸 2 倍後的結果。

例句(2)：The graph of $g(x) = \cos 2x$ **is a horizontal shrink of the graph of** $f(x) = \cos x$ **by a factor of** $\frac{1}{2}$.
$g(x) = \cos 2x$ 的圖形是 $f(x) = \cos x$ 的圖形經過水平收縮 $\frac{1}{2}$ 倍後的結果。

❼ The graph of $g(x)$ is a vertical/ horizontal shift of the graph of $f(x)$ by ______

例句(1)：**The graph of** $g(x) = \cos x + 2$ **is a vertical shift of** $f(x) = \cos x$ **upward by 2 units.**
$g(x) = \cos x + 2$ 的圖形是 $f(x) = \cos x$ 的圖形經過垂直平移往上 2 個單位後的結果。

例句(2)：**The graph of** $g(x) = \cos(x + 2)$ **is a horizontal shift of the graph of** $f(x) = \cos x$ to the left **by 2 units**.
$g(x) = \cos(x + 2)$ 的圖形是 $f(x) = \cos x$ 的圖形經過水平向左平移 2 個單位後的結果。

■ 問題講解 Explanation of Problems

☙ 說明 ❧

In this section, we will explore the real-life applications of the periodic nature of trigonometric functions. To understand these applications, it's essential to grasp the concepts of radians and graph transformations.

By using an online graphing calculator like Desmos, we can easily generate the graphs of sine and cosine, as shown below in Figure (a) and Figure (b) respectively.

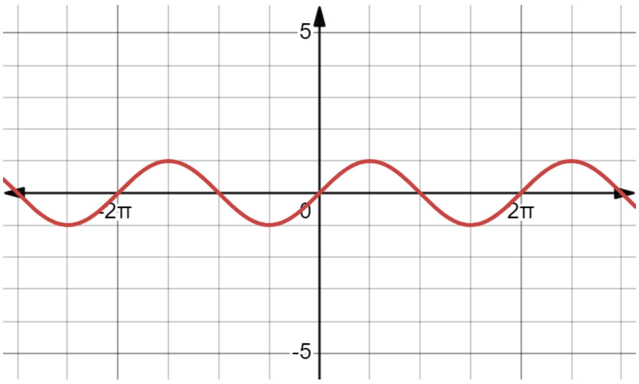


Figure (a)

According to the graph of $y = \sin x$, we can conclude the following characteristics.

- Domain: $(-\infty, \infty)$
- Range: $[-1,1\}$
- Amplitude: 1
- Period: 2π
- Symmetry: Origin symmetry

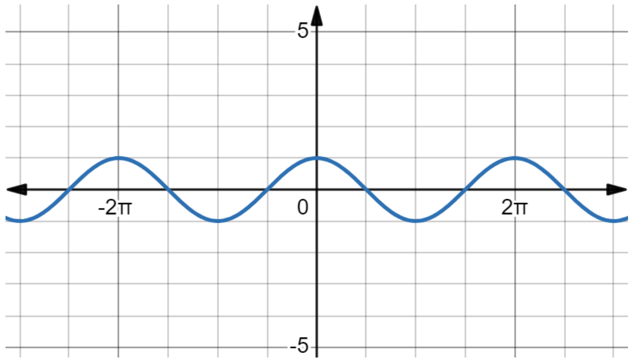
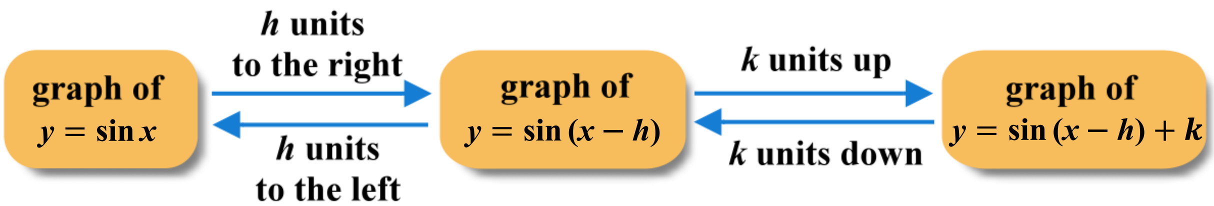


Figure (b)

According to the graph of $y = \cos x$, we can conclude the following characteristics.

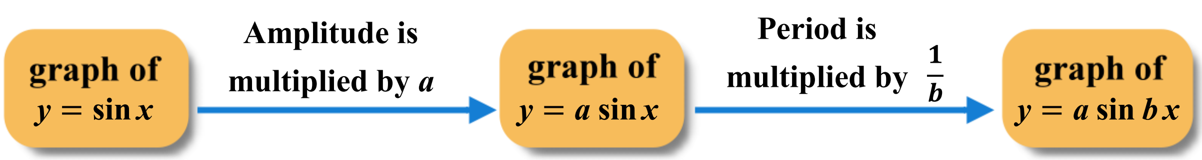
- Domain: $(-\infty, \infty)$
- Range: $[-1,1\}$
- Amplitude: 1
- Period: 2π
- Symmetry: y-axis symmetry

Regarding the concept of translating the graph of the sine function, we represent it with the following flowchart, where h and k are positive.



For example, the graph of the function $y = \sin(x - 2) + 3$ can be obtained by translating the graph of $y = \sin x$ 2 units to the right and then 3 units up.

Regarding the concept of stretching or compressing the sine function graph, we represent it with the following flowchart, where a and b are positive.



For example, the graph of the function $y = 2\sin(3x)$ can be obtained by vertically stretching by a factor of 2 and horizontally shrinking by a factor of $\frac{1}{3}$.

Similar logic applies to the transformation of the graphs of $y = \cos x$.

運算問題的講解

例題一

說明：本題是弧度與圓心角的基本關係運算。

（英文）Given that the radius of the circle is 6 and the arc length is 7, based on the relationship "ratio of the arc length to the circle's radius equals the radian,"
(a) express the central angle in radians.
(b) express the central angle in degrees.
(c) find the area of the sector.

（中文）假設圓的半徑為 6，且弧長為 7，根據「弧長和圓半徑的比例是弳度」的原理，
(a) 請以弳度表示圓心角。
(b) 請以度數表示圓心角。
(c) 找出扇形面積。

Teacher: What is the ratio of the arc length to the radius in this question?

Student: It is 7 to 6.

Teacher: Great!

Based on the relationship "ratio of arc length to the circle's radius equals the radian," the central angle is $\frac{7}{6}$ radians.

How do you convert radians to degrees?

Student: One radian equals $\frac{180}{\pi}$ degrees. Multiply $\frac{7}{6}$ by $\frac{180}{\pi}$, we will have $\frac{210}{\pi}$ degrees.

Teacher: Well done.

How do you find the area of this sector?

Student: The area of the whole circle is πr^2, which is 36π.

Two hundred and ten degrees divided by pi is $\frac{7}{12\pi}$ of the circle.

(Seven-sixths of a radian is $\frac{7}{12\pi}$ of the circle.)

Therefore, the area of the circle is $\frac{7}{12\pi} \times 36\pi$, which is 21.

Teacher: Your explanation is very clear.

We can also find the area of the sector using the formula：$A = \frac{1}{2}r^2\theta$, where θ is measured in radians.

Can any of you find the area using this formula?

Student: $\frac{1}{2} \times 6^2 \times \frac{7}{6} = 21$. (One-half times six squared times seven-sixths equals twenty-one.)

Teacher: As you can see, both methods yield the same results.

老師： 這道題目中弧長與半徑的比例是多少？

學生： 是七比六。

老師： 很好！

根據「弧長與圓半徑的比等於弧度」的關係，圓心角度為 $\frac{7}{6}$ 弳。

如何將弧度轉換成角度？

學生： 1 弳等於 $\frac{180}{\pi}$ 度。$\frac{7}{6}$ 乘以 $\frac{180}{\pi}$，會得到 $\frac{210}{\pi}$ 度。

老師： 做得很好。

如何找到這個扇形的面積呢？

學生： 整個圓的面積是 πr^2，等於 36π。$\frac{210}{\pi}$ 度是 $\frac{7}{12\pi}$ 個圓。

(或者，$\frac{7}{6}$ 弳度是$\frac{7}{12\pi}$ 個圓。)

因此，圓的面積為 $\frac{7}{12\pi} \times 36\pi$，等於 21。

老師： 解釋得非常清楚！

我們也可以使用 $A = \frac{1}{2}r^2\theta$ 的公式找到扇形的面積，其中 θ 是弧度。

你們有誰可以用這個公式找到面積嗎？

學生： $\frac{1}{2} \times 6^2 \times \frac{7}{6} = 21$。

老師： 正如你們所看到的，兩種方法都會得到相同的結果。

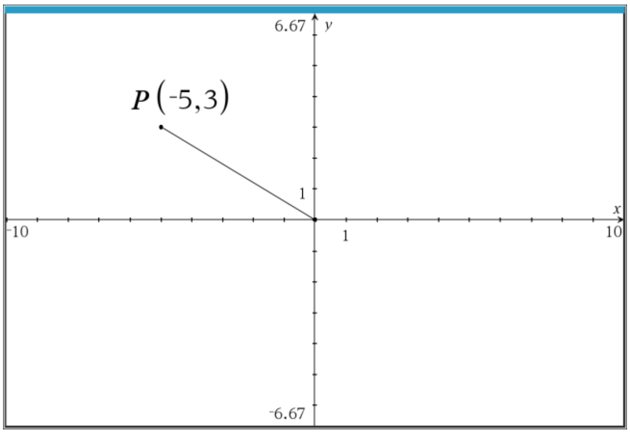
例題二

說明：本題是廣義的三角函數的基本運算。

（英文）Let theta be an angle in the standard position with its terminal side passing through the point $P(-5, 3)$. Determine the sine, cosine and tangent values for theta.

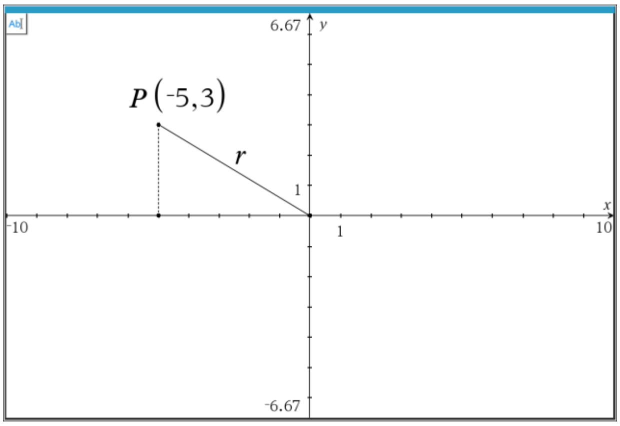
（中文）假設 θ 是一個在標準位置的角，其終邊經過點 $P(-5, 3)$。計算 θ 的正弦、餘弦和正切值。

Teacher: First of all, let's draw this angle in the coordinate plane.



Draw a perpendicular segment from the point P to the x-axis, and then we can see a reference triangle here.

What is the hypotenuse of this reference triangle?



Student: Applying the Pythagorean Theorem, it would be $\sqrt{3^2+5^2}$ (the square root of the sum of 3 squared and 5 squared), which is $\sqrt{34}$.

Teacher: The hypotenuse, represented by r, is always positive.

Sine theta is y over r. It is $\frac{3}{\sqrt{34}}$. How about cosine theta?

Student: Cosine theta is x over r. It is $\frac{-5}{\sqrt{34}}$.

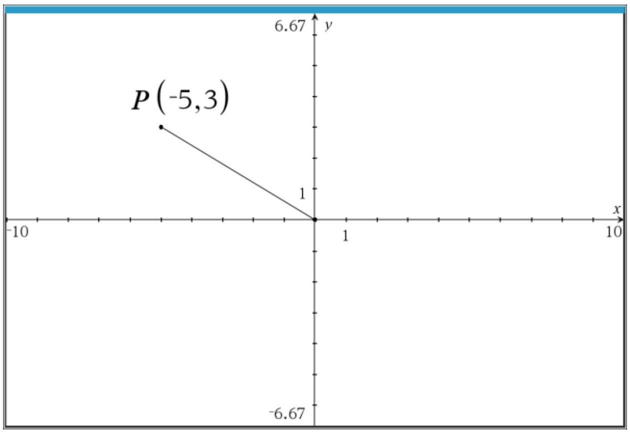
Teacher: Similarly, who knows the answer to tangent theta?

Student: Tangent theta is y over x. It is $\frac{3}{-5}$.

Teacher: Great.

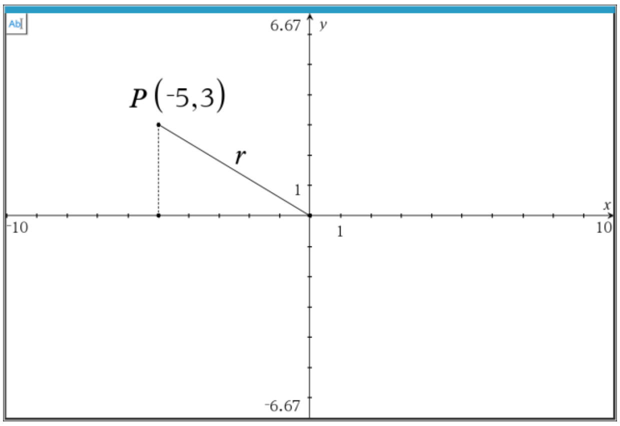
When we set an angle in the standard position on the coordinate plane, the coordinates of any point on the angle's terminal side can be used to determine the values of trigonometric functions

老師： 首先，我們在坐標平面上畫出這個角度。



從點 P 畫一條垂直線段到 x 軸，我們可以得到一個參考用的三角形。

這個三角形的斜邊長是多少？



學生： 應用畢氏定理，$\sqrt{3^2+5^2}$ 即 $\sqrt{34}$。

老師： 斜邊用 r 表示，始終為正值。

$\sin\theta$ 是 y 除以 r，$\frac{3}{\sqrt{34}}$。那 $\cos\theta$ 呢？

學生：　$\cos\theta$ 是 x 除以 r，$\frac{-5}{\sqrt{34}}$。

老師：　同樣地，誰知道 $\tan\theta$ 的答案呢？

學生：　$\tan\theta$ 是 y 除以 x，$\frac{3}{-5}$。

老師：　太棒了。
當我們在坐標平面上將一個角置於標準位置時，角終邊上的任何點的坐標都可以用來決定三角函數的值。

例題三

說明：本題是正弦函數圖形的變換。

（英文）Using the graph of $y = \sin x$, sketch the graph of $y = 3\sin 2x - 1$ and determine the corresponding period, maximum, and minimum values.

（中文）利用 $y = \sin x$ 的圖形，畫出 $y = 3\sin 2x - 1$ 的圖形，並求其週期、最大值及最小值。

Teacher: Recall that the range of the function $y = \sin x$ extends from -1 to 1, including both -1 and 1.
What are the maximum and minimum of $y = 3\sin x - 1$

Student: Because $-1 \leq \sin x \leq 1$, $-3 - 1 \leq 3\sin x - 1 \leq 3 - 1$. (The quantity three times the sine of x minus one is bounded by the interval from negative three minus one to three minus one, inclusive.)
Therefore, the maximum is 2 and the minimum is -4.

Teacher: The graph of $y = \sin 2x$ is a horizontal shrink of the graph of $y = \sin x$ by a factor of $\frac{1}{2}$.
The horizontal transformation will not change the maximum and minimum.
Therefore, the maximum is still 2 and the minimum is -4.

Student: How about the period?

Teacher: Because the length of its domain needed to complete one full cycle is halved, the period becomes 2π divided by 2, which equals π.

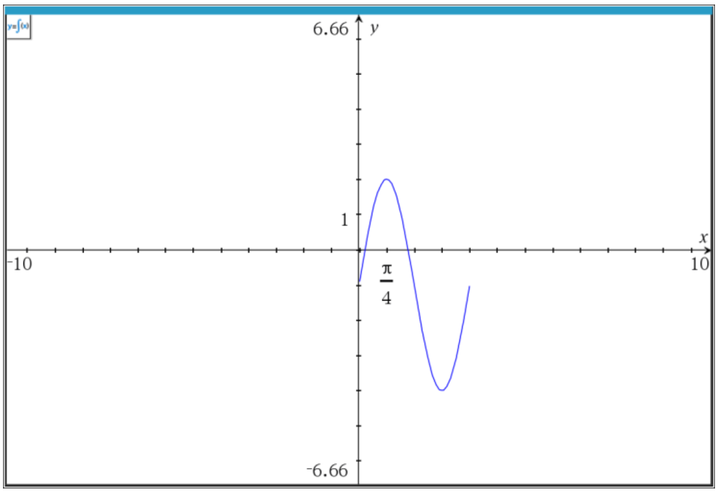
To graph this function, we divide this period into four intervals, with separating points at 0, $\frac{\pi}{4}$, $\frac{\pi}{2}$, $\frac{3\pi}{4}$, and π.

What are the corresponding function values with respect to these separating points?

Student: They are $(0,-1)$, $(\frac{\pi}{4},2)$, $(\frac{\pi}{2},-1)$, $(\frac{3\pi}{4},-4)$ and $(\pi,-1)$.

Teacher: Great.

Now, let's connect these points with a smooth curve.

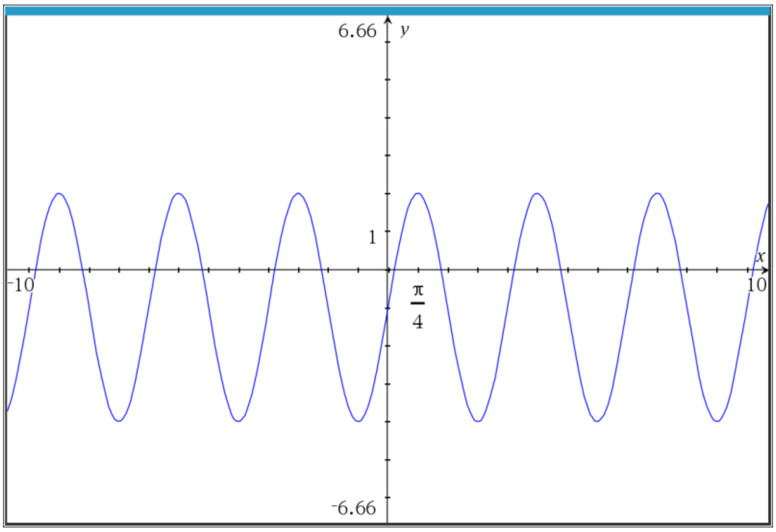


How do we graph $y = 3\sin x - 1$ over all real numbers?

Student: Copy one period of the graph and repeat the pattern.

Teacher: Very good.

The graph of this function is as follows.



老師：回顧一下，函數 $y = \sin x$ 的值域範圍從 -1 到 1，包含 -1 和 1。
$y = 3\sin x - 1$的最大值和最小值是多少？

學生：因為 $-1 \le \sin x \le 1$，$-3 - 1 \le 3\sin x - 1 \le 3 - 1$。因此，最大值為 2，最小值為 -4。

老師：函數 $y = \sin 2x$ 的圖形是函數 $y = \sin x$ 的水平伸縮，收縮量為 $\frac{1}{2}$。水平變化不會影響最大值和最小值。因此，最大值仍然是 2，最小值是 -4。

學生：週期呢？

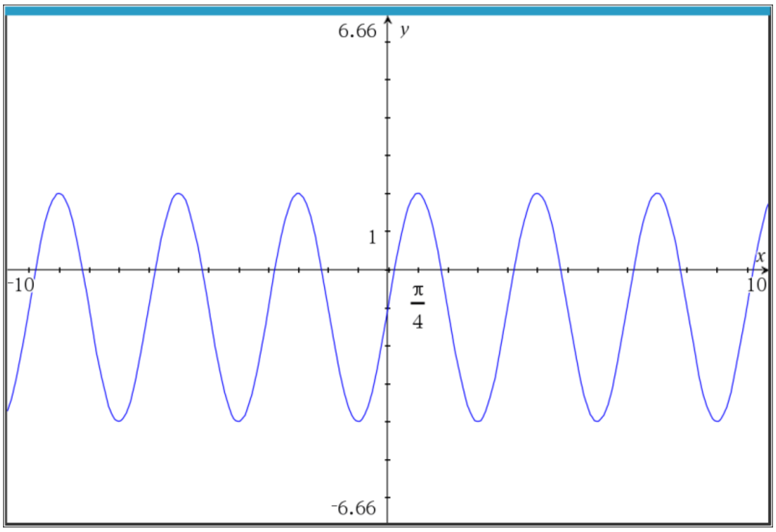
老師：因為需要完成一個完整週期的定義域長度減半，週期變成 2π 的一半，等於 π。為了繪製這個函數，我們將這個週期分為四個區間，分隔點分別是 0、$\frac{\pi}{4}$、$\frac{\pi}{2}$、$\frac{3\pi}{4}$。這些分隔點對應的函數值是什麼？

學生：它們分別是 $(0, -1)$, $(\frac{\pi}{4}, 2)$, $(\frac{\pi}{2}, -1)$, $(\frac{3\pi}{4}, -4)$ 和 $(\pi, -1)$.

老師：很好。現在，我們用平滑曲線連接這些點。
要怎麼畫出函數 $y = 3\sin x - 1$在所有實數上的圖形呢？

學生：複製一個週期的圖形，並重複這個模式。

老師：非常好。函數圖形如下所示。



例題四

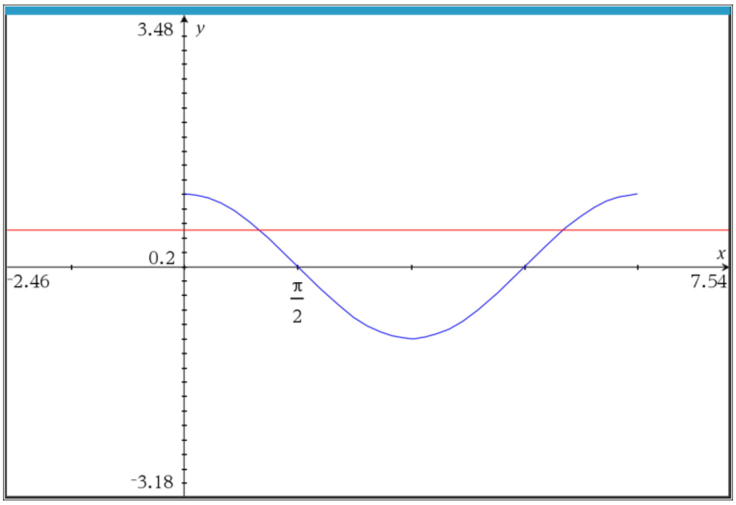
說明：本題是利用三角函數的圖形協助代數運算解題。

（英文）Find the solutions for x such that $\cos x \leq \frac{1}{2}$ over the interval $[0, 2\pi)$.

（中文）在區間 $[0, 2\pi)$ 內，找出滿足 $\cos x \leq \frac{1}{2}$ 的 x 解。

Teacher: We graph $y = \cos x$ and $y = \frac{1}{2}$ over the interval $[0, 2\pi)$.

What are the intersections?



Student: There are two intersections. One is in the first quadrant, and the other is in the 4th quadrant.

$\cos x = \frac{1}{2}$, $x = \frac{\pi}{3}$ and $\frac{5\pi}{3}$

Teacher: Correct.

Now, let's look for the interval so that the graph of $y = \cos x$ is below the graph of $y = \frac{1}{2}$. What is the answer?

Student: The answer would be $\frac{\pi}{3} \leq x \leq \frac{5\pi}{3}$.

Teacher: Graphing is a powerful way to visualize and solve inequalities. The intersection points of the two graphs divide the x-axis into various regions. By examining the relative positions of the graphs, we can determine which regions satisfy the inequality

老師：我們在區間 $[0, 2\pi)$ 上畫出 $y = \cos x$ 和 $y = \frac{1}{2}$ 的圖形。

它們的交點在哪裡？

學生：有兩個交點。一個位於第一象限，另一個位於第四象限。

$\cos x = \frac{1}{2}$. $x = \frac{\pi}{3}$ 和 $\frac{5\pi}{3}$。

老師：答對了。

現在，我們來找一個區間，使 $y = \cos x$ 的圖形位於 $y = \frac{1}{2}$ 的圖形下方。答案是什麼呢？

學生：答案會是 $\frac{\pi}{3} \leq x \leq \frac{5\pi}{3}$。

老師：畫圖是一種很好用的技巧，能夠將題目視覺化並輕鬆解決不等式。這兩個圖形的交點將 x 軸分成不同的區域。透過觀察圖形的相對位置，我們可以確定哪些區域滿足不等式。

應用問題 / 學測指考題

例題一

說明：這題利用角速度解決日常生活的直線速度問題。

（英文）Ming rides his bicycle to school. The radius of his bicycle tire is 35 cm. When Ming rides the bicycle at a constant speed, the angular velocity of the tire is 10 radians/second.

(a) At this speed, how many revolutions does the bicycle tire make in one second?

(b) If the linear distance traveled by the bicycle in one second is d cm, express d using the angular velocity and the radius of the tire.

（中文）小明騎著他的自行車去學校。他的自行車輪胎的半徑是 35 公分。當小明以一定的速度騎自行車時，輪胎的角速度是 10 弧度/秒。

(a) 請問這個速度下，自行車輪胎每秒轉幾圈？

(b) 若小明騎自行車每秒行進的直線距離是 d 公分，請使用角速度與輪胎的半徑來表示 d。

Teacher: What is the circumference of the wheel?

Student: The formula for finding the circumference is $2\pi r$. It is 70π.

Teacher: Great. There are 2π radians in one revolution.

Ten radians correspond to $\frac{10}{2\pi}$ revolutions as given by the proportion $\frac{2\pi}{1} = \frac{10}{x}$.

This means the tire completes $\frac{10}{2\pi}$ revolutions per second, which is approximately 1.59 revolutions per second. Does anyone have any questions about this?

Student: Not really.

Teacher: The linear distance (or arc length) covered by a circle when it rotates through an angle θ radians is given by $d = r\theta = 35 \times 10 = 350$.

Therefore, the distance is 350 cm.

老師： 輪胎的周長是多少？

學生： 求周長的公式是 $2\pi r$。算出來是70π。

老師： 很好。輪胎完整轉一圈是 2π 弳。

根據比例 $\frac{2\pi}{1} = \frac{10}{x}$，10 弧度會對應到 $\frac{10}{2\pi}$ 圈，這表示輪胎每秒完成 $\frac{10}{2\pi}$ 圈大約是每秒 1.59 圈。大家有任何問題嗎？

到這邊有問題嗎？

學生： 沒有。

老師： 由公式$d = r\theta$ 算出，當一個圓以 θ 弳的角度旋轉時，所走過的線性距離（或弧長）是$35 \times 10 = 350$。

因此，行進的直線距離是 350 公分。

例題二

說明：這題利用正餘弦函數圖來模擬乘坐摩天輪時的高度變化。

（英文）The Taichung Lihpao Land Sky Dream Ferris Wheel has a height of 126 meters and a diameter of 112 meters. It takes about 25 minutes to ride a round. Please use trigonometric functions to represent the height variation of passengers as they ride a round.

（中文）台中麗寶樂園天之夢摩天輪高 126 公尺，直徑 112 公尺。搭乘一圈約需 25 分鐘。請使用三角函數來表示乘客在搭乘一圈時的高度變化。

Teacher: What are the lowest and highest points when you take a ride on the Ferris Wheel?

Student: The lowest point would be 14 meters and the highest point is 126 meters.

Teacher: Great. The amplitude is half the height of the wave.
What is the amplitude?

Student: It would be $\frac{126-14}{2}=56$.

Teacher: Great! If we place the center at the midpoint between the lowest and highest points, what is the coordinate?

Student: The y-coordinate of the midpoint is 70, as given by $\frac{126+14}{2}$.

Teacher: Correct. Now, let's figure out the period.
The statement "It takes about 25 minutes to ride a round" describes the time it takes to complete one full cycle. This is referred to as the "period" of the motion. $25=\frac{2\pi}{b}$. b is $\frac{2\pi}{25}$.
What is the sinusoidal function that models the height of the rider?

Student: The rider goes on the ride at the lowest point.
I would like to use $y=-\cos x$ to model the height.
$y=-56\cos\left(\frac{2\pi}{25}x\right)+70$.

Teacher: The choice between sine and cosine will depend on the starting point. If the rider starts at the midline of the wheel, then a sine function might be more suitable. If the rider starts at the highest or lowest point, then a cosine function might be better.
You've correctly chosen $y=-\cos x$ to represent the height in this model.

老師： 搭摩天輪的時候，最低點和最高點會在哪裡呢？

學生： 最低點在 14 公尺，最高點在 126 公尺。

老師： 很好。

而振幅是波高的一半，能算出是多少嗎？

學生： 是 $\frac{126-14}{2}=56$。

老師： 很好！如果我們將中心放在最低點和最高點的中間，坐標是什麼？

學生： $\frac{126+14}{2}$ 計算得出，中點的 y 坐標為 70。

老師： 沒錯。現在，我們來找出週期。

「搭乘一圈約需 25 分鐘」這句話描述完成一個完整週期所需的時間。我們稱之為運動的「週期」。$25=\frac{2\pi}{b}$，因此 b 算出來是 $\frac{2\pi}{25}$.

摩天輪搭乘者高度的正弦函數是什麼？

學生： 搭乘者從最低點坐上摩天輪。

我想用 $y=-\cos x$ 來模擬高度。

$y=-56\cos\left(\frac{2\pi}{25}x\right)+70$。

老師： 正弦波或是餘弦波的選擇取決於起點。如果搭乘者從摩天輪的中線開始，那麼正弦函數會更適合；如果搭乘者從最高或最低點開始，則餘弦函數比較適合。用 $y=-\cos x$ 來表示摩天輪的模擬高度，是正確的選擇喔！

例題三

說明：這題利用正餘弦函數圖來模擬水深的變化。

（英文）The time it takes for cargo ships to enter the port and unload is influenced by the depth of the water. The water must be deep enough for ships to enter the port and unload. Someone measured the relationship between the time t (in hours) and water depth h (in meters) over a 24-hour period at a certain port.

The table below shows the data from midnight (0:00) to noon (12:00).

Time (t)	0:00	3:00	6:00	9:00	12:00
Depth (h)	12.5	15.2	12.5	9.8	12.5

For instance, the water depth measured at 0:00 is 12.5 meters. Based on the table, the water depth of this port on that day corresponds to the sine function $y = a\sin(bt) + 12.5$, where both a and b are positive numbers.

(a) Determine the values of a and b.

(b) If the cargo ship has the option to dock and unload in the afternoon, around what time would be optimal?

（中文）運輸船進港卸貨時間會受到水深的影響，水深夠深時，才能進港卸貨。某人測量某海港某日 24 小時之時間 t（單位：時）與水深 h（單位：公尺）的關係，下表為其當日凌晨 0:00 到中午 12:00 的部分數據。

時間 (t)	0:00	3:00	6:00	9:00	12:00
水深 (h)	12.5	15.2	12.5	9.8	12.5

例如：凌晨 0:00 測量水深為 12.5 公尺。根據上表，當日此海港的水深與測量時間符合正弦函數 $y = a\sin(bt) + 12.5$，其中 a 與 b 均為正數。

(1) 試求出 a 與 b 的值。

(2) 如果貨運船有在下午停靠和卸貨的選擇，大約什麼時候會是最佳的時間？

（改編自 111 年學測數學 B 第 12 題）

Teacher: What are the smallest and greatest water depths?

Student: According to the table, the smallest water depth is 9.8 meters and the greatest depth is 15.2

Teacher: Great. The amplitude is half the height of the wave.
What is the amplitude?

Student: It would be $\frac{15.2-9.8}{2}=2.7$

Teacher: Great! $a=2.7$.

If we plot the time against water depth on a coordinate plane, the duration from the greatest water depth to the smallest represents half the period.

What is the period?

Student: It takes 6 hours for the water depth to change from the greatest to the smallest.

The period is 12.

Teacher: Correct. $12=\frac{2\pi}{b}$. b is $\frac{\pi}{6}$.

Therefore, the equation to model the water depth is $y=2.7\sin\left(\frac{\pi}{6}t\right)+12.5$

According to the question description, the water must be deep enough for ships to enter the port and unload.

When does the water attain its greatest depth?

Student: The water attains its greatest depth at 3:00 in the morning, and its period is 12 hours.

As a result, it will attain its greatest depth at 15:00, which is 3:00 in the afternoon

Teacher: Correct.

If the cargo ship can enter the port to unload in the afternoon, the best approximate time would be around 15:00.

老師： 水深的最小值和最大值分別是多少？

學生： 依據表格資訊，最小水深是 9.8 公尺，最大水深是 15.2 公尺。

老師： 很好。振幅是波高的一半。是多少呢？

學生： 振幅是 $\frac{15.2-9.8}{2}=2.7$。

老師： 很好！$a=2.7$。

如果我們在坐標平面上繪製時間與水深的關係，從最大水深到最小水深的時間間隔代表半個週期。

週期是多少？

學生： 水深從最大到最小的變化需要 6 小時。週期是 12。

老師： 正確。$12 = \frac{2\pi}{b}$，b 為 $\frac{\pi}{6}$。

因此，模擬水深的方程式為 $y = 2.7\sin\left(\frac{\pi}{6}t\right) + 12.5$。

根據題目描述，水深必須夠深才能讓船隻進入港口卸貨。

水何時達到最大深度？

學生： 水在早上 3 點達到最大深度，週期為 12 小時。因此，將會在下午 3 點達到最大深度。

老師： 答對了！如果貨船可以在下午進入港口卸貨，最佳的時間應該是下午 3 點左右。

單元二　三角的合角和差角公式
The Angle Sum and Difference Formulas in Trigonometry

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■ 前言 Introduction

本節介紹三角的合角和差角公式，並進一步推導出二倍角與半角公式。合角和差角公式有助於簡化三角函數的表示式。在物理和工程中，特別是在處理波現象時，和差公式有助於將波形加在一起（如聲波或光波）。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
derive	推導	**formula**	公式

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ Using the formula for ________.

例句(1)：**Using the formula for** the angle difference formula of cosine, we can easily derive the formula for the angle sum formula of cosine by utilizing the odd-even identities.

利用餘弦的差角公式，我們可以輕鬆地利用函數的奇偶特性推導出餘弦的合角公式。

例句(2)：**Using the formula for** the cosine of a sum, we can easily derive the cosine formula for a double angle.

利用餘弦的合角公式，我們可以輕鬆地推導出餘弦的倍角公式。

❷ When a line forms an angle θ with _____, ______is defined as $\tan\theta$.

例句(1)：**When a line forms an angle** θ **with** the positive x-axis, its slope **is defined as** $\tan\theta$.

當一直線與 x 軸正方向形成角度 θ 時，該直線的斜率是 $\tan\theta$。

例句(2)：**When a road forms an angle** θ **with** the horizon, its grade **is defined as** $\tan\theta$ expressed as a percentage.

當一條道路與地平線形成 θ 角度時，其坡度被定義為 $\tan\theta$，並以百分比表示。

❸ _____ rotated counterclockwise by ____ about the origin (0, 0).

例句(1)：What is the slope of the line $x-y=0$ after it is **rotated counterclockwise by** 30° **about the origin (0, 0)?**

當直線 $x-y=0$ 繞著原點 (0, 0) 逆時針旋轉 30° 之後，其斜率為多少？

例句(2)：How does the cosine curve change after it is **rotated counterclockwise by** 180° **about the origin (0, 0)?**

當餘弦函數的圖形繞著原點 (0, 0) 逆時針旋轉 180° 之後，它會如何變化？

❹ When taking the square root, pay attention to ______.

例句(1)：**When taking the square root, pay attention to** which quadrant the angle is in to determine the sign of your result.
當求平方根時，請注意角度所在的象限以決定結果的正負。

例句(2)：**When taking the square root, pay attention to** the value under the radical to ensure it's non-negative.
當求平方根時，請注意根號下的值以確保它是非負的。

❺ To find the ____, first calculate _______ and then ______.

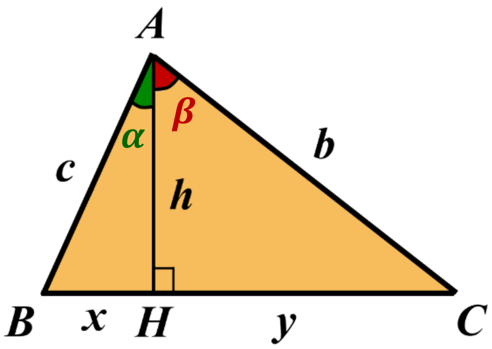
例句(1)：**To find the** tangent of the half-angle, **first calculate** the sine and cosine values **and then** use their ratio.
對於正弦波型函數，其週期是完成一個完整週期所需的定義域的長度。

例句(2)：**To find the** secant of an angle, **first calculate i**ts cosine value **and then** take its reciprocal.
要求一角的正割值，先計算其餘弦值，再取其倒數。

■ 問題講解 Explanation of Problems

☙ 說明 ❧

In this section, we'll use the angle difference and sum formulas of cosine to determine other trigonometric values. Understanding trigonometric function identities is crucial for this application.



Area of $\triangle ABC$ = Area of $\triangle ABH$ + Area of $\triangle ACH$

Therefore, $\frac{1}{2}bc\sin(\alpha+\beta)=\frac{1}{2}ch\sin\alpha+\frac{1}{2}bh\sin\beta$. Dividing both sides by $\frac{1}{2}bc$, we have

$\sin(\alpha+\beta)=\frac{h}{b}\sin\alpha+\frac{b}{c}\sin\beta$.

Because $\frac{h}{c}=\cos\alpha$ and $\frac{h}{b}=\cos\beta$,

the equation above can be represented as $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$.

$$\cos(\alpha+\beta)=\sin\left(\frac{\pi}{2}-(\alpha+\beta)\right)=\sin\left(\frac{\pi}{2}-\alpha\right)\cos(-\beta)+\cos\left(\frac{\pi}{2}-\alpha\right)\sin(-\beta)$$
$$=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

Next, the trigonometric ratios of $\alpha-\beta$ can be further obtained through conversion formulas.

$\sin(\alpha-\beta)=\sin(\alpha+(-\beta))=\sin\alpha\cos(-\beta)+\cos\alpha\sin(-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$;

$\cos(\alpha-\beta)=\cos(\alpha+(-\beta))=\cos\alpha\cos(-\beta)-\sin\alpha\sin(-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

Therefore, we have the sum and difference formulas for sine and cosine.

運算問題的講解

例題一

說明：本題是合角與差角公式的基本應用。

（英文）(a) Please calculate $\cos 75°$ by using the angle sum formula of cosine.
(b) Please calculate $\cos 75°$ by using the angle difference formula of cosine.

（中文）(a) 請使用餘弦的合角公式計算 $\cos 75°$。
(b) 請使用餘弦的差角公式計算 $\cos 75°$。

Teacher: How can $75°$ be expressed as the sum of two special angles?

Student: Thirty degrees plus forty-five degrees.

Teacher: Great!
Using the angle sum formula of cosine, how can we expand $\cos 75°$?

Student: $\cos 75° = \cos(30° + 45°) = \cos 30° \cos 45° - \sin 30° \sin 45°$

$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

Note: $\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2}$ reads as “square root of three over two times square root of two over two minus one-half times square root of two over two.”

$\frac{\sqrt{6}-\sqrt{2}}{4}$ read as “the quantity of the square root of 6 minus the square root of 2, divided by 4.”

Teacher: Well done.
How can 75° be expressed as the difference between two special angles?

Student: Let me try. It should be one hundred twenty degrees minus forty-five degrees equals seventy-five degrees.

Teacher: Correct.
Using the angle difference formula of cosine, how can we expand $\cos 75°$?

Student: $\cos 75° = \cos(120° - 45°) = \cos 120° \cos 45° + \sin 120° \sin 45°$

$$= \frac{-1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

Teacher: As you can see, both methods yield the same result.

老師： 75 度如何表示為兩個特殊角度之和？

學生： 30 度加上 45 度。

老師： 使用餘弦的和角公式，如何展開 $\cos 75°$?

學生： $\cos 75° = \cos(30° + 45°) = \cos 30° \cos 45° - \sin 30° \sin 45°$

$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$

老師： 做得很好。

接下來，如何將 75 度表示為兩個特殊角度之差？

學生： 讓我試試。應該是 120 度減 45 度。

老師： 使用餘弦的差角公式，如何展開 $\cos 75°$?

學生： $\cos 75° = \cos(120° - 45°) = \cos 120° \cos 45° + \sin 120° \sin 45°$

$= \frac{-1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$

老師： 如你所見，兩種方法產生的結果是相同的。

例題二

說明：本題是差角公式的進階應用。

（英文）Given $\frac{\pi}{2} < \alpha < \pi$ and $\frac{3\pi}{2} < \beta < 2\pi$, if $\sin\alpha = \cos\beta = \frac{4}{5}$, evaluate $\cos(\alpha - \beta)$, $\sin(\alpha - \beta)$ and $\tan(\alpha - \beta)$.

（中文）已知 $\frac{\pi}{2} < \alpha < \pi$ 和 $\frac{3\pi}{2} < \beta < 2\pi$。若 $\sin\alpha = \cos\beta = \frac{4}{5}$，試求 $\cos(\alpha - \beta)$, $\sin(\alpha - \beta)$ 和 $\tan(\alpha - \beta)$ 的值。

Teacher: Given $\frac{\pi}{2} < \alpha < \pi$ and $\sin\alpha = \frac{4}{5}$, what is the value of $\cos\alpha$?

Student: Refer to the triangle in the second quadrant, $\cos\alpha = \frac{-3}{5}$.

Teacher: Yes, that's correct. Given $\frac{3\pi}{2} < \beta < 2\pi$ and $\cos\beta = \frac{4}{5}$, what is the value of $\sin\beta$?

Student: Refer to the triangle in the fourth quadrant, $\sin\beta = \frac{-3}{5}$.

Teacher: You got it right. Now, let's use the angle difference formula of sine.

$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

What is the value of $\sin(\alpha-\beta)$?

Student: $\frac{4}{5}\cdot\frac{4}{5}-\frac{-3}{5}\cdot\frac{-3}{5}=\frac{16-9}{25}=\frac{7}{25}$.

Teacher: Well done. Now, let's use the angle difference formula of sine.

$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

What is the value of $\cos(\alpha-\beta)$?

Student: $\frac{-3}{5}\cdot\frac{4}{5}+\frac{4}{5}\cdot\frac{-3}{5}=\frac{-24}{25}$.

Teacher: To find the tangent of $\alpha-\beta$, find the ratio of sine to cosine. What is the answer?

Student: $\frac{7}{-24}$.

Teacher: Good job. By examining the values of sine, cosine, and tangent, we can determine that $\alpha-\beta$ falls within the second quadrant.

This aligns with the range $\frac{\pi}{2}-2\pi<\alpha-\beta<\pi-\frac{3\pi}{2}$.

老師： 已知 $\frac{\pi}{2}<\alpha<\pi$ 和 $\sin\alpha=\frac{4}{5}$，$\cos\alpha$ 的值是多少?

學生： 參考位於第二象限的三角形， $\cos\alpha=\frac{-3}{5}$

老師： 沒錯。已知 $\frac{3\pi}{2}<\beta<2\pi$ 和 $\cos\beta=\frac{4}{5}$， $\sin\beta$ 的值是多少?

學生： 參考位於第四象限的三角形，$\sin\beta=\frac{-3}{5}$。

老師： 答對了。

現在，讓我們使用正弦的差角公式：

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

$\sin(\alpha-\beta)$ 的值是多少?

學生： $\frac{4}{5}\cdot\frac{4}{5}-\frac{-3}{5}\cdot\frac{-3}{5}=\frac{16-9}{25}=\frac{7}{25}$

老師： 答對了。

現在，讓我們使用餘弦的差角公式：

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

$\cos(\alpha-\beta)$的值是多少？

學生： $\frac{-3}{5}\cdot\frac{4}{5}+\frac{4}{5}\cdot\frac{-3}{5}=\frac{-24}{25}$。

老師： 要找到 $\alpha-\beta$ 的正切值，找到正弦和餘弦的比率。
答案是什麼？

學生： $\frac{7}{-24}\frac{7}{-24}$。

老師： 很棒喔！
首先，根據正弦、餘弦和正切的值，我們可以判斷 $\alpha-\beta$ 落在第二象限內。
這與範圍 $\frac{\pi}{2}-2\pi<\alpha-\beta<\pi-\frac{3\pi}{2}$是一致的。

例題三

說明：本題是倍角公式的基本題型。

（英文）Given $0<\theta<\frac{\pi}{2}$ and $\cos\theta=\frac{4}{5}$, evaluate $\sin 2\theta$ and $\cos 2\theta$.

（中文）已知 $0<\theta<\frac{\pi}{2}$ 且 $\cos\theta=\frac{4}{5}$，試求 $\sin 2\theta$ 與 $\cos 2\theta$ 的值。

Teacher: Recall that the formula for $\sin 2\theta$ is $2\sin\theta\cos\theta$.
We need to find the value of $\sin\theta$ so that we can apply the formula.
What is the value of $\sin\theta$?

Student: Three, four, and five form a Pythagorean Triple.
$\sin\theta=\frac{3}{5}$.

Teacher: You got it right.
What is the value of $\sin 2\theta$?

Student: $2\sin\theta\cos\theta=2\times\frac{3}{5}\times\frac{4}{5}=\frac{24}{25}$

Teacher: How about the value of $\cos 2\theta$?

Student: $\cos 2\theta=2\cos^2\theta-1=2\cdot\frac{16}{25}-1=\frac{7}{25}$.

Teacher: Great. We can use the formula $\cos 2\theta=\cos^2\theta-\sin^2\theta=\frac{16}{25}-\frac{9}{25}=\frac{7}{25}$.
The result must be the same.

老師：　回想一下，$\sin 2\theta$ 的公式是 $2\sin\theta\cos\theta$.
　　　　我們需要找出 $\sin\theta$ 的值，才能應用這個公式。
　　　　$\sin\theta$ 的值是多少？

學生：　3、4、5 構成一組畢氏數組。
　　　　$\sin\theta = \frac{3}{5}$。

老師：　你答對了。所以 $\sin 2\theta$ 的值是多少？

學生：　$2\sin\theta\cos\theta = 2\times\frac{3}{5}\times\frac{4}{5} = \frac{24}{25}$。

老師：　$\cos 2\theta$ 呢？

學生：　$\cos 2\theta = 2\cos^2\theta - 1 = 2\cdot\frac{16}{25} - 1 = \frac{7}{25}$。

老師：　很棒。我們也可以使用另一個公式 $\cos 2\theta = \cos^2\theta - \sin^2\theta = \frac{16}{25} - \frac{9}{25} = \frac{7}{25}$，結果是一樣的。

例題四

說明：本題是半角公式的基本題型。

（英文）Given $\frac{3\pi}{2} < \theta < 2\pi$ and $\cos\theta = \frac{4}{5}$, evaluate $\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2}$.

（中文）已知 $\frac{3\pi}{2} < \theta < 2\pi$ 且 $\cos\theta = \frac{4}{5}$，試求 $\sin\frac{\theta}{2}$ 與 $\cos\frac{\theta}{2}$ 的值。

Teacher:　What is the range of $\frac{\theta}{2}$ given $\frac{3\pi}{2} < \theta < 2\pi$?

Student:　Divide the inequality by 2, and then we will have $\frac{3\pi}{4} < \frac{\theta}{2} < \pi$.

Teacher:　What are the signs of $\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2}$ given that $\frac{3\pi}{4} < \frac{\theta}{2} < \pi$?

Student:　The half-angle is in the second quadrant.
　　　　$\sin\frac{\theta}{2}$ is positive but $\cos\frac{\theta}{2}$ is negative.

Teacher: Now, we need to apply the half-angle formula.

$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$ and $\cos\frac{\theta}{2} = -\sqrt{\frac{1+\cos\theta}{2}}$

What are the values of $\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2}$?

Student: $\sin\frac{\theta}{2} = \sqrt{\frac{1-\frac{4}{5}}{2}} = \sqrt{\frac{1}{10}} = \frac{\sqrt{10}}{10}$.

$\cos\frac{\theta}{2} = \sqrt{\frac{1+\frac{4}{5}}{2}} = \sqrt{\frac{9}{10}} = \frac{3\sqrt{10}}{10}$.

Note: read $\sqrt{\frac{1-\frac{4}{5}}{2}}$ as "the square root of (1 minus $\frac{4}{5}$) divided by 2."

Teacher: When taking the square root, pay attention to which quadrant the angle is in to determine the sign of your result.

The half-angle is in the second quadrant, and the value of cosine is negative.

Therefore, $\cos\frac{\theta}{2} = \frac{-3\sqrt{10}}{10}$.

老師： 已知 $\frac{3\pi}{2} < \theta < 2\pi$，半角的範圍是多少？

學生： 把不等式的每一個項除以 2，$\frac{3\pi}{4} < \frac{\theta}{2} < \pi$。

老師： 如果 $\frac{3\pi}{4} < \frac{\theta}{2} < \pi$，半角的正弦和餘弦值是正的還是負的？

學生： 因為這個半角在第二項限，正弦值是正的，但是餘弦值是負的。

老師： 現在，讓我們來利用半角公式。

$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$ 以及 $\cos\frac{\theta}{2} = -\sqrt{\frac{1+\cos\theta}{2}}$

半角的正弦和餘弦值各是多少？

學生： $\sin\frac{\theta}{2} = \sqrt{\frac{1-\frac{4}{5}}{2}} = \sqrt{\frac{1}{10}} = \frac{\sqrt{10}}{10}$、$\cos\frac{\theta}{2} = \sqrt{\frac{1+\frac{4}{5}}{2}} = \sqrt{\frac{9}{10}} = \frac{3\sqrt{10}}{10}$。

老師： 當取平方根時，請注意角度所在的象限，以確定結果的正負號。

半角在第二象限，因此餘弦值為負。

$\cos\frac{\theta}{2} = \frac{-3\sqrt{10}}{10}$。

應用問題 / 學測指考題

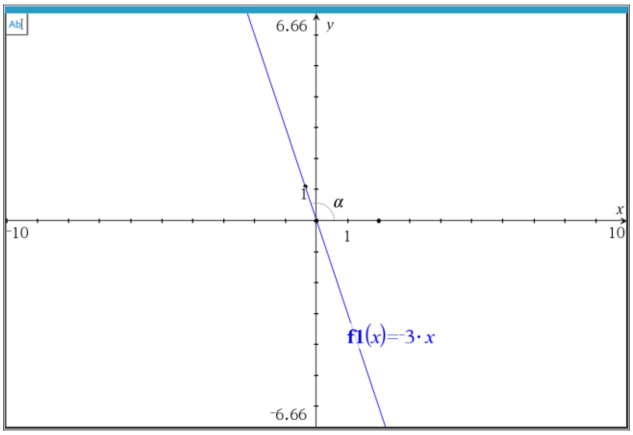
例題一

說明：這題利用直線斜率與正切值的關係求出兩條直線的夾角。

（英文）Find the approximate value of the acute angle θ formed by the intersection of the two lines defined by the equations $3x + y = 0$ and $x - 2y = 0$

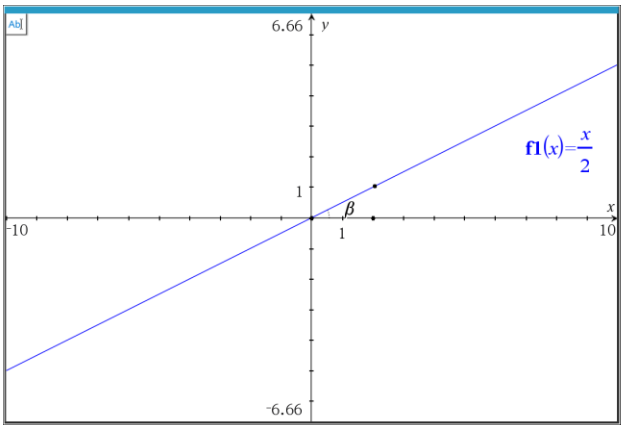
（中文）找出由方程式 $3x + y = 0$ 和 $x - 2y = 0$ 所定義的兩條直線交叉處形成的銳角 θ 的近似值。

Teacher: When a line forms an angle θ with the positive x-axis, its slope is defined by $\tan\theta$. Refer to the graph of $y = -3x$, what is the value of $\tan\alpha$?



Student: The value of the tangent is the same as the slope, $\tan\alpha = -3$

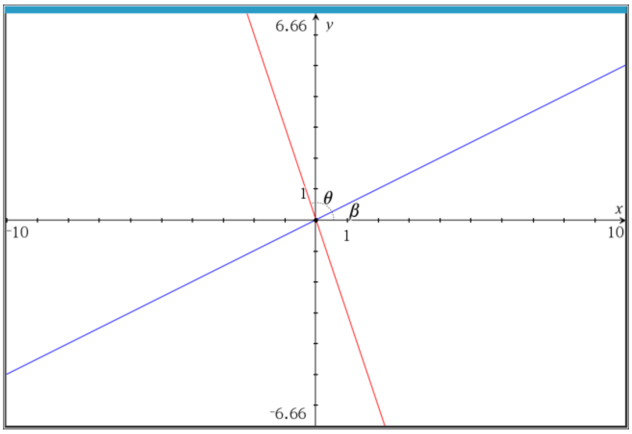
Teacher: Similarly, refer to the graph of $y = \frac{x}{2}$, what is the value of $\tan\beta$?



Student: $\tan\beta = \frac{1}{2}$.

Teacher: The angle θ, which is formed by the intersection of the two lines, is equal to $\alpha - \beta$.

How to find the value of $\alpha - \beta$ using the angle difference formula of tangent?



Student: $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \dfrac{-3 - 0.5}{1 - 3\cdot 0.5} = \dfrac{-3.5}{-0.5} = 7$

Note: $\frac{-3-0.5}{1-3\cdot 0.5}$ read as the quantity of subtracting negative three and negative 0.5, divided by the quantity of one minus three times zero point five.

Teacher: Using the calculator, $\tan^{-1} 7$ is about 81.87 degrees.

老師： 當一直線與 x 軸正方向形成角度 θ 時，其斜率由，該直線的斜率是 $\tan\theta$。
看看這條 $y = -3x$ 直線，他的正切值應該是多少?

學生： 正切值和斜率一樣，$\tan\alpha = -3$

老師： 同樣地，看看 $y = \frac{x}{2}$ 的直線，他的正切值應該是多少？

學生： $\tan\beta = \frac{1}{2}$。

老師： 由兩條直線交叉所形成的角度 θ，等於 $\alpha - \beta$
如何利用正切的差角公式計算$\alpha - \beta$ 的值？

學生： $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \dfrac{-3 - 0.5}{1 - 3\cdot 0.5} = \dfrac{-3.5}{-0.5} = 7$

老師： 使用計算機，$\tan^{-1} 7$ 大約是 81.87 度。

例題二

說明：這題利用正切值與坡度的關係，併用半角公式。

（英文）In the Extreme Sports Park, there's a skateboard ramp with varying degrees of inclination, depending on the skill level of the riders. Let's consider a particular ramp designed for beginner skateboarders. This ramp's angle of inclination θ is half the angle of a ramp designed for advanced skateboarders.

When a road forms an angle θ with the horizon, its grade is defined as $\tan\theta$ expressed as a percentage. If the ramp for advanced skateboarders has a grade of 120%, what is the grade for the beginner skateboard ramp?

（中文）在極限運動公園中，有一個滑板坡道，其傾斜角度會根據滑手的技能水平而有所不同。考慮一個專為初學者設計的滑板坡道。這個坡道的傾斜角度 θ 是為高級滑手設計的坡道角度的一半。

當一條路與地平線形成角度 θ 時，其坡度定義為以百分比表示的 $\tan\theta$。如果為高級滑手設計的坡道的坡度為 120%，那麼初學者滑板坡道的坡度是多少？

Teacher: When a road forms an angle θ with the horizon, its grade is defined as $tan\,\theta$ expressed as a percentage. The ramp for advanced skateboarders has a grade of 120%. It means that $\tan\theta = \frac{120}{100} = \frac{6}{5}$

Now, we need to apply the half-angle formula of tangent.

What is the formula?

Student: $\tan\frac{\theta}{2} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$.

Teacher: Great.

Originally, we needed to consider the quadrant in which the angle lies to determine the sign of our result.

However, since the ramp's inclination is always at an acute angle, we can take the result to be positive.

Let’s find the value of $\cos\theta$ first. What is the value of $\cos\theta$?

Student: If $\tan\theta = \frac{6}{5}$, $\cos\theta$ would be $\frac{5}{\sqrt{61}}$.

Teacher: Great!

Using the half angle formula of tangent, what is the value of $\tan\frac{\theta}{2}$?

Student: $\tan\frac{\theta}{2} = \sqrt{\frac{1-\frac{5}{\sqrt{61}}}{1+\frac{5}{\sqrt{61}}}} = \sqrt{\frac{\sqrt{61}-5}{\sqrt{61}+5}}$.

Teacher: Using a calculator, the grade of the ramp for the beginner is about 46.84%

老師： 當一條路與地平線形成角度 θ 時，其坡度定義為以百分比表示的 $\tan\theta$。

專為高級滑板手設計的坡道的坡度為 120%。

這意味著 $\tan\theta = \frac{120}{100} = \frac{6}{5}$

現在，我們需要應用半角的正切公式。

公式是什麼？

學生： $\tan\frac{\theta}{2} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$。

老師： 原本，我們需要考慮角度所在的象限，以確定結果的正負號。

但是，由於坡道的傾斜角度始終是鈍角，我們可以認為結果是正的。

首先，讓我們找出 $\cos\theta$ 的值。$\cos\theta$ 的值是多少？

學生： 如果 $\tan\theta = \frac{6}{5}$，$\cos\theta$ 就是 $\frac{5}{\sqrt{61}}$。

老師： 很棒。利用半角的正切公式，$\tan\frac{\theta}{2}$ 的值是多少？

學生： $\tan\frac{\theta}{2} = \sqrt{\frac{1-\frac{5}{\sqrt{61}}}{1+\frac{5}{\sqrt{61}}}} = \sqrt{\frac{\sqrt{61}-5}{\sqrt{61}+5}}$。

老師： 使用計算機，初學者的滑板斜坡坡度大約是 46.84%。

單元三　指數函數
Exponential Function

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■　前言 Introduction

本單元的內容為指數函數 $f(x) = a^x$的定義及其圖形，利用描點法作圖來觀察底數 $a > 1$ 及 $0 < a < 1$ 等指數函數的性質；接下來討論指數方程式、指數不等式的求解問題；最後是指數函數在生活中的應用。老師在介紹本單元指數函數的圖形後，先讓學生透過隨堂練習，觀察並了解指數函數圖形的特性。建議在示範例題時老師能反覆提問，讓學生能熟悉本章節的重要的英文名詞及用語後，最後以應用問題或 AMC (American Mathematics Competition) 測驗題來加深學生們對本單元的了解。

■　詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
exponential function	**指數函數**	**domain**	**定義域**
dot method	**描點法**	**range**	**值域**
smooth curve	平滑曲線	**simple interest**	**單利**
strictly increasing	**嚴格遞增的**	**compound interest**	**複利**
strictly decreasing	**嚴格遞減的**	principal	本金

Arbitrary	任意的	balance	本利和
demonstrate	顯示	**findings**	發現

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ Connect ___________ from left to right.

例句：A line graph is a graph that uses the line segments to **connect** points **from left to right** to demonstrate changes in value.

折線圖是將資料點用折線由左至右連接形成以顯示數值的變化。

❷ Plot _______ on _______.

例句：In this section, you will see how to **plot** points **on** the coordinate plane.

在本單元，你將看到如何將點描繪在坐標平面上。

❸ _______, where ________

例句：An exponential function is defined by $f(x) = a^x$**, where** the base a is a positive real number other than 1.

指數函數 $f(x) = a^x$ 中，底數 a 是不等於 1 的正實數。

❹ _______ are symmetric with respect to________.

例句：The graphs of $y = a^x$ and $y = a^{-x}$ **are symmetric with respect to** the y-axis.

函數 $y = a^x$ 及 $y = a^{-x}$ 的圖形對稱於 y 軸。

❺ convert ________ to ________

例句：We need to solve this equation by **convert**ing all the exponents **to** the same base.

我們需要將所有的指數轉換成相同的底數來解這個方程式。

❻ ________, and vice versa.

例句：A function $f(x)$ is said to be strictly decreasing if $f(b) < f(a)$ for all $b > a$, **and vice versa.**
函數 $f(x)$為嚴格遞減函數則「如果 $f(b) < f(a)$ 則 $b > a$；反之亦然」。

■ 問題講解 Explanation of Problems

∽ 說明 ∾

In this section, we will cover the exponential function $f(x) = a^x$ (or $y = a^x$), where the base a is a positive real number other than 1 and the exponent x is a real independent variable. The lesson starts with using the dot method to graph the exponential functions $y = a^x$ when $a > 1$ or $0 < a < 1$. Teachers can lead students to draw and observe the graphs of the exponential functions with different bases and introduce the concepts of strictly increasing and decreasing functions, domain and range, and asymptotes, etc.

The second part of the lesson covers the contents of solving the exponential equations and inequalities. Students need to change the exponents to the same base in solving the exponential functions. And students need to note the properties of strictly increasing (base $a > 1$) and decreasing functions ($0 < a < 1$) when solving inequalities. Finally, they will learn to solve real-life problems which include compound interest questions.

運算問題的講解

例題一

說明：本題是畫出指數函數 $y=a^x$ 的圖形。(a > 1)

（英文）Find the values of the exponential function $y=2^x$ in the table.
Then, use the dot method to graph $y=2^x$.

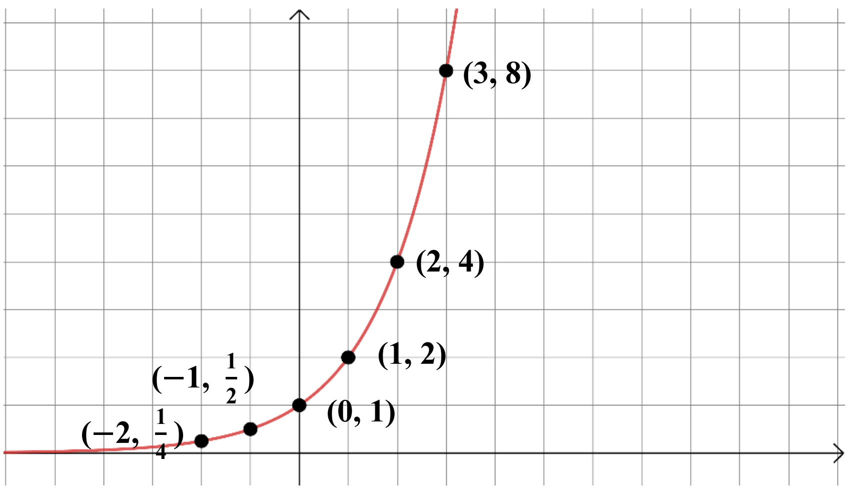
（中文）找出表格中的函數值，並用描點法畫出指數函數 $y=2^x$。

x	-3	-2	-1	0	1	2	3
$y=2^x$							

Teacher: Find all the values in the table now. What are they from left to right?

Student: $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}$, 1, 2, 4, 8.

Teacher: Yes, you are correct. I am glad that you remember what we covered last year. Next, draw a dot at each point from the table. Then we can use a smooth curve to connect the dots from left to right. This method is called the dot method.



Can you repeat the same steps to graph $y=3^x$?

Student: Yes.

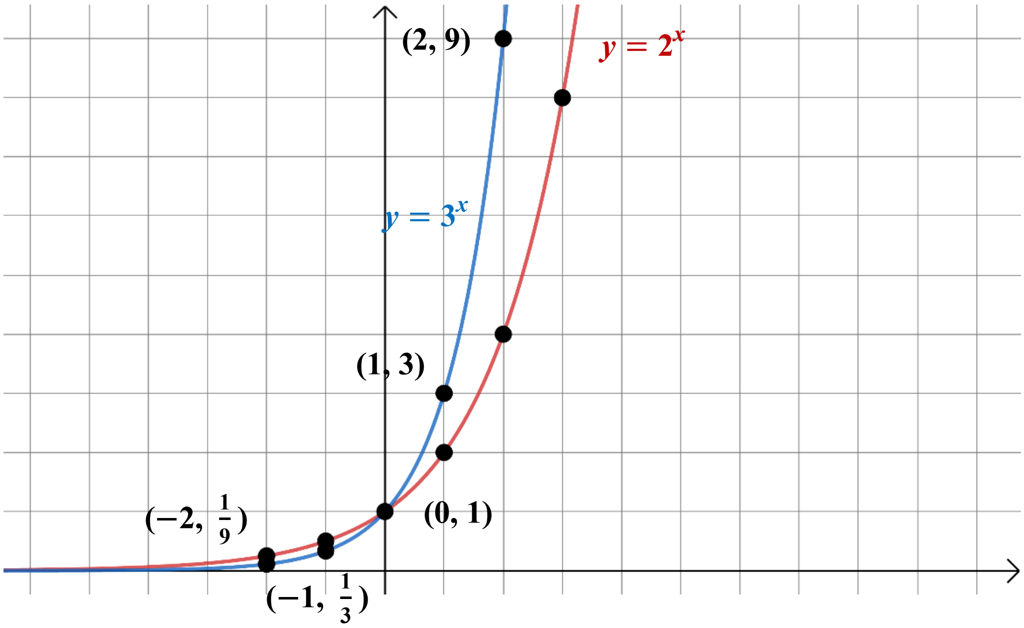
Teacher: Make the table of seven different values first. What are these values of y?

Student: $\frac{1}{27}, \frac{1}{9}, \frac{1}{3}$, 1, 3, 9, 27.

Teacher: Good. Then, plot these points and draw a smooth curve to connect them. Compare the two graphs and tell me your findings.

Student: Both of them are strictly increasing. The graph of $y = 3^x$ increases faster than $y = 2^x$.

Both of the graphs pass through (0, 1).



Teacher: Excellent. You are all correct.

The exponential functions $y = 2^x$ and $y = 3^x$ are **strictly increasing**.

A function is said to be strictly increasing if $f(x_2) > f(x_1)$ for all $x_2 > x_1$.

Second, the graph of $y = a^x$ passes through (0, 1) because $a^0 = 1$.

Besides, the graph of $y = a^x$ $(a > 1)$ is always above x-axis and gets closer to the x-axis when x becomes a smaller negative number. The x-axis is the asymptote of the exponential function $y = a^x$.

老師： 找出表格中的所有數值。從左至右分別是多少？

學生： $\frac{1}{8}$、$\frac{1}{4}$、$\frac{1}{2}$、1、2、4、8。

老師： 是的，你說得對。我很高興你還記得我們去年學過的內容。

接下來，將每個表格中的數對分別繪製成點。然後，我們可以使用平滑曲線將這些點從左到右連接起來。這種方法稱為描點法。

你能用相同的步驟來畫 $y = 3^x$ 嗎？

學生： 可以。

老師： 首先，製作七個不同值的表格。這些 y 的值是什麼？

學生： $\frac{1}{27}$、$\frac{1}{9}$、$\frac{1}{3}$、1、3、9、27

老師： 很好。然後，畫出這些點並用平滑曲線來連接它們。

比較這兩個圖形，告訴我你們發現了什麼。

學生：它們都是嚴格遞增的。函數 $y = 3^x$ 的圖形增長速度比 $y = 2^x$ 快。

這兩個圖形都通過 $(0, 1)$。

老師：很優秀！你們都答對了。

指數函數 $y = 3^x$ 和 $y = 2^x$ 都是嚴格遞增。如果對於所有 $x_2 > x_1$，$f(x_2)$ 都大於 $f(x_1)$ 那麼這個函數被稱為嚴格遞增函數。

再者，函數 $y = a^x$ 通過 $(0, 1)$，是因為 $a^0 = 1$。

此外，函數 $y = a^x (a > 1)$ 始終位於 x 軸之上，且當 x 變成更小的負數時，它會越來越接近 x 軸。x 軸是指數函數 $y = a^x$ 的漸近線。

例題二

說明：本題是畫出指數函數 $y = a^x$ 的圖形。$(0 < a < 1)$

（英文）Find the values of the exponential function $y = (\frac{1}{2})^x$ in the table.

Then graph $y = (\frac{1}{2})^x$.

（中文）找出表格中的函數值，並畫出 $y = (\frac{1}{2})^x$。

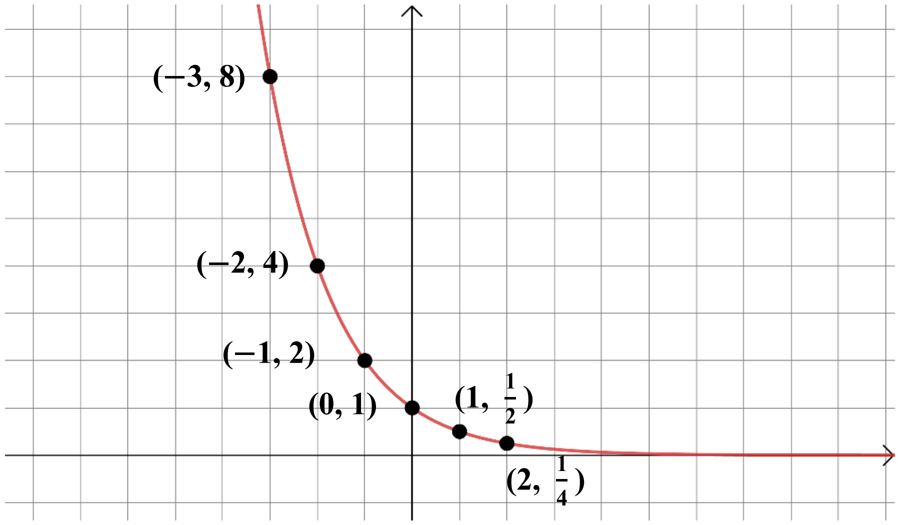
x	-3	-2	-1	0	1	2	3
$y = (\frac{1}{2})^x$							

Teacher: Find all the values in the table now. What are they from left to right?

Student: $8, 4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$.

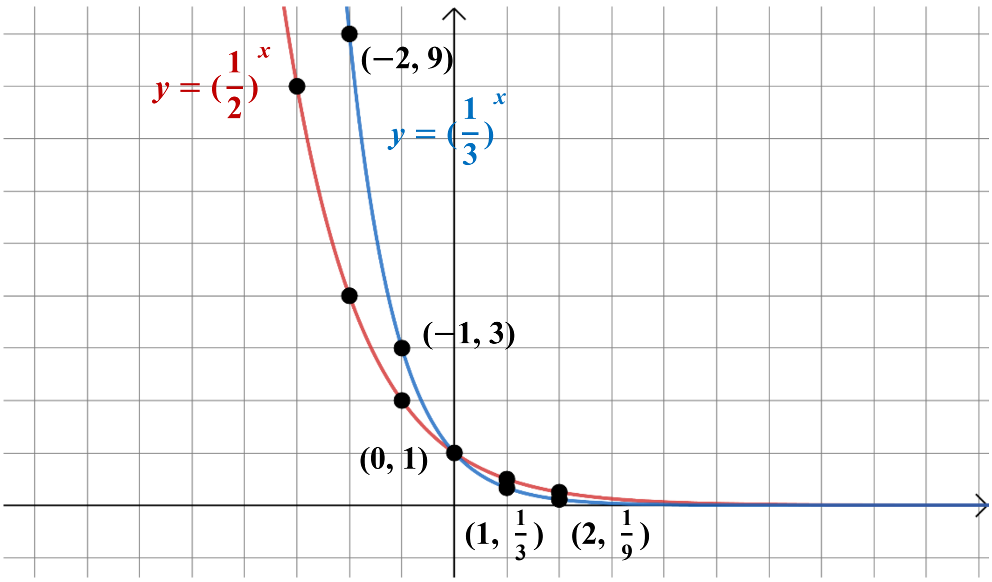
Teacher: Good.

Then, we can use the dot method to graph the function.



Repeat the same steps to graph $y=(\frac{1}{3})^x$ now.

Student: OK.



Teacher: Compare the two graphs and tell me your findings.

Student: Both of the graphs of the exponential functions $y=(\frac{1}{2})^x$ and $y=(\frac{1}{3})^x$ are strictly decreasing.

The graphs also pass through (0, 1).

The x-axis is the horizontal asymptote for both of the graphs.

Teacher: Well done.

Now, compare the graphs of $y=2^x$ and $y=(\frac{1}{2})^x$. We can find that the two graphs are symmetric to the *y*-axis.

The domain (the set of all x values) of the exponential function $y = a^x$ (a > 0) is all real numbers and the range (the set of all y values) is $y > 0$.

老師：找出表格中的所有數值。從左至右分別是多少？

學生：8、4、2、1、$\frac{1}{2}$、$\frac{1}{4}$、$\frac{1}{8}$。

老師：很好。

然後，我們可以使用描點法畫出這個函數圖形。

重複相同的步驟來繪製 $y = (\frac{1}{3})^x$。

學生：好的。

老師：比較這兩個圖形，告訴我你發現什麼。

學生：指數函數 $y = (\frac{1}{2})^x$ 和 $y = (\frac{1}{3})^x$ 的圖形都是嚴格遞減的。這兩個圖形也都通過 (0, 1)。x 軸是它們的水平漸近線。

老師：做得很好。

現在，比較函數 $y = 2^x$ 和$y = (\frac{1}{2})^x$ 的圖形。我們可以發現這兩個圖形對稱於 y 軸。指數函數 $y = a^x$ (a > 0) 的定義域（所有 x 值的集合）是所有實數，而值域（所有 y 值的集合）是 y > 0。

例題三

說明：本題是求解指數方程式的練習。

（英文）Solve the following exponential equations.

(1) $\sqrt{2}^{5x+1} = 32\sqrt{2}$　　(2) $4^x - 9 \cdot 2^x + 8 = 0$

（中文）試求下列指數方程式的解。

(1) $\sqrt{2}^{5x+1} = 32\sqrt{2}$　　(2) $4^x - 9 \cdot 2^x + 8 = 0$

Teacher: We know that $\sqrt{2} = 2^{\frac{1}{2}}$.

Then, $\sqrt{2}^{5x+1} = 32\sqrt{2}$ can be written as $2^{\frac{5x+1}{2}} = 2^{\frac{11}{2}}$.

In an exponential equation, both of the exponents are equal when the bases are the same. It means $\frac{5x+1}{2} = \frac{11}{2}$. What is the value of x?

Student: 2.

Teacher: Correct. Let us see the second equation: $4^x - 9 \cdot 2^x + 8 = 0$

There are two different bases of exponential functions in this equation.

Usually, we need to convert all of the exponents to the same base.

So, $4^x = (2^2)^x = 2^{2x} = 2^x \cdot 2^x$.

The equation can be factored as $(2^x - 1)(2^x - 8) = 0$.

Hence, $2^x = 1$ or $2^x = 8$

Can you find the solutions of x?

Student: Yes, $x = 0$ or 3.

Teacher: Very good.

老師： 我們知道$\sqrt{2} = 2^{\frac{1}{2}}$。所以，$\sqrt{2}^{5x+1} = 32\sqrt{2}$ 可以寫成 $2^{\frac{5x+1}{2}} = 2^{\frac{11}{2}}$。在一個指數方程式中，當底數相同時，兩個指數相等，也就是說$\frac{5x+1}{2} = \frac{11}{2}$。那麼 x 的是多少呢？

學生： 2。

老師： 正確。讓我們看看第二個方程式：$4^x - 9 \cdot 2^x + 8 = 0$。這個方程式中有兩個不同底數的指數。通常，我們需要將所有指數轉換為相同的底數。

所以，$4^x = (2^2)^x = 2^{2x} = 2^x \cdot 2^x$。方程式可以分解為 $(2^x - 1)(2^x - 8) = 0$。

因此，$2^x = 1$或 $2^x = 8$。請大家求出 x。

學生： $x = 0$ 或 3。

老師： 非常好。

例題四

說明：本題是求解指數不等式 $(a > 1)$ 的練習。

（英文）Solve the following exponential inequalities.

(1) $9^x \le 3^{x^2-x+2}$　(2) $4^x - 2^{x+1} - 8 > 0$

（中文）試求下列指數不等式的解。

(1) $9^x \le 3^{x^2-x+2}$　(2) $4^x - 2^{x+1} - 8 > 0$

Teacher: First, we change all exponential functions to the functions of the same base.
Then, we get $3^{2x} \le 3^{x^2-x+2}$ because $9^x = 3^{2x}$.
We know that $y = a^x$ $(a > 1)$ is strictly increasing.
So, $2x \le x^2 - x + 2$.
Solve the quadratic inequality now and tell me the solution.

Student: The solution is $x \ge 2$ or $x \le 1$.

Teacher: Very good. Let us see the next question: $4^x - 2^{x+1} - 8 > 0$
Similarly, we change $4^x = 2^{2x}$ and $2^{x+1} = 2 \cdot 2^x$.
So, the inequality can be rewritten as $(2^x)^2 - 2(2^x) - 8 > 0$.
The inequality can be factored as $(2^x + 2)(2^x - 4) > 0$.
Hence, $2^x < -2$ or $2^x > 4$
Can you find the solutions of the inequality $2^x < -2$?

Student: No, there is no solution because 2^x is always greater than 0.

Teacher: Yes, you are correct. How about the solution of the inequality $2^x > 4$.

Student: The solution is $x > 2$.

Teacher: Excellent.
The exponential functions $y = a^x$ (a > 0) is strictly increasing.
If $x_1 < x_2$, then $f(x_1) < f(x_2)$; and vice versa. (If $f(x_1) < f(x_2)$, then $x_1 < x_2$.)
Second, the exponential functions $y = a^x$ $(0 < a < 1)$ is strictly decreasing.
If $x_1 < x_2$, then $f(x_1) > f(x_2)$; and vice versa. (If $f(x_1) < f(x_2)$, then $x_1 > x_2$.)
Let us do another inequality question when the base a is between 0 and 1.
$(0 < a < 1)$

老師：首先，我們將所有的指數函數改成相同底數的函數。

老師：接著，因為 $9^x = 3^{2x}$，可以得到 $3^{2x} \le 3^{x^2-x+2}$。

我們知道 $y = a^x$ (a > 1) 是嚴格遞增，所以，$2x \le x^2 - x + 2$。現在解這個二次不等式，告訴我解是什麼。

學生： 解是 $x \ge 2$ 或 $x \le 1$

老師： 非常好。讓我們看下一個問題：$4^x - 2^{x+1} - 8 > 0$。

同樣地，我們將4^x 改成 2^{2x}； 2^{x+1}改成 $2 \cdot 2^x$。因此，這個不等式可以重寫為

$(2^x)^2 - 2(2^x) - 8 > 0$。

老師： 這個不等式可以分解為 $(2^x + 2)(2^x - 4) > 0$。因此，$2^x < -2$ 或 $2^x > 4$。

你能找到不等式 $2^x < -2$ 的解嗎？-

學生： 不行，因為 2^x 會恆大於 0。

老師： 沒錯，你說得對。那麼不等式 $2^x > 4$ 的解是什麼？

學生： 解是 $x > 2$。

老師： 太棒了。

指數函數 $y = a^x$ (a > 0) 是嚴格遞增的。如果 $x_1 < x_2$，那麼 $f(x_1) < f(x_2)$；反之亦然（如果$f(x_1) < f(x_2)$，那麼 $x_1 < x_2$）。

接著，指數函數 $y = a^x$ $(0 < a < 1)$是嚴格遞減的。

如果$x_1 < x_2$，那麼$f(x_1) > f(x_2)$；反之亦然（如果 $f(x_1) > f(x_2)$，那麼 $x_1 < x_2$）。

老師： 讓我們做另一個當底數 a 介於 0 和 1 之間的不等式問題。$(0 < a < 1)$

例題五

說明：本題是求解指數不等式 $(0 < a < 1)$ 的練習。

（英文）Solve the following exponential inequality.

(1) $0.25^x > 0.5^{x^2-3}$　(2) $(\frac{1}{9})^x + 2(\frac{1}{3})^{x+1} - 3 \le 0$

（中文）試求下列指數不等式的解。

(1) $0.25^x > 0.5^{x^2-3}$　(2) $(\frac{1}{9})^x + 2(\frac{1}{3})^{x+1} - 3 \le 0$

Teacher: First, we rewrite the inequality to $0.5^{2x} > 0.5^{x^2-3}$ because $0.25^x = 0.5^{2x}$.
We know that $y = a^x$ $(0 < a < 1)$ is strictly decreasing.
So, $2x < x^2 - 3$.
Solve the quadratic inequality now and tell me the solution.

Student: The solution is $x > 3$ or $x < -1$.

Teacher: Correct. Next, find the solution to the second question.

Student: How to solve the inequality $-3 \le (\frac{1}{3})^x \le 1$?

Teacher: Good question.
Because a^x is always greater than zero, the inequality $-3 \le (\frac{1}{3})^x$ is always true.
The solution of this left inequality is all real numbers.
Can you find the solutions of the other part $(\frac{1}{3})^x \le 1$?

Student: The solution is $x \ge 0$.

Teacher: Excellent. $(\frac{1}{3})^x \le 1$ is the same as $(\frac{1}{3})^x \le (\frac{1}{3})^0$.
Since the exponential functions $y = a^x$ $(0 < a < 1)$ is strictly decreasing, the solution is $x \ge 0$.
The intersection of both sides of the inequality is still $x \ge 0$.

老師：首先，因為$0.25^x = 0.5^{2x}$，我們將不等式改寫為 $0.5^{2x} > 0.5^{x^2-3}$。
我們知道$y = a^x$ $(0 < a < 1)$ 是嚴格遞減的。
所以，$2x < x^2 - 3$。
現在解這個二次不等式，告訴我解是什麼。

學生：解是 $x>3$ 或 $x<-1$。

老師：正確。接下來，找出第 2 小題的解。

學生：如何解不等式 $-3\le\left(\frac{1}{3}\right)^x\le 1$？

老師：好問題。

老師：因為 a^x 總是大於零，不等式 $-3\le\left(\frac{1}{3}\right)^x$ 始終成立。這個左邊不等式的解是所有實數。

你能找到另一部分 $\left(\frac{1}{3}\right)^x\le 1$ 的解嗎？

學生：解是 $x\ge 0$。

老師：很棒。$\left(\frac{1}{3}\right)^x\le 1$等同於 $\left(\frac{1}{3}\right)^x\le\left(\frac{1}{3}\right)^0$。由於指數函數 $y=a^x\ (0<a<1)$ 是嚴格遞減的，解是 $x\ge 0$。

不等式兩邊的交集仍然是 $x\ge 0$。

🙞 應用問題 / 學測指考題🙜

例題一

說明：這題利用指數函數計算銀行複利問題。

（英文）Ming deposited 2 million NT in a bank that pays 4% annual interest. What is the balance after two years when the interest is compounded semiannually?

（中文）阿明將 200 萬元新台幣存入銀行，年利率 4%，每半年複利計息一次。請問兩年後其本利和為多少元？

Teacher: The annual interest rate is 4% and the interest is compounded semiannually.
So, the semiannual interest rate is 2%, and the amount will be compounded 4 times after 2 years.
After the end of the first half year, the balance A
$= P$ (initial amount) + $P\cdot 0.02$ (interest)
$= 1.02P$
After the end of the first year, the new balance $A = 1.02P + 1.02P\cdot 0.02 = P\cdot 1.02^2$
Finally, after 2 years, the balance is $2000000\,(1+2\%)^4 = 2000000\cdot 1.02^4$.
Now, please use a calculator to find the balance after two years.

Student: The balance is 2,164,864.

Teacher: Very good.
If an initial amount P is deposited in an account that pays interest at an annual rate r, compounded n times per year. The balance A in the account after t years is given by

$$A = P\left(1+\frac{r}{n}\right)^{nt}.$$

老師： 題目說，年利率是 4%，並且利息每半年複利一次。所以，半年複利的利率是 2%，在 2 年後將複利 4 次。
在第一半年結束後，餘額 A= P（初始金額）+ $P\cdot 0.02$（利息）=1.02P
在第一年結束後，新的餘額 A=$1.02P + 1.02P\cdot 0.02 = P\cdot 1.02^2$
因此在 2 年後，餘額是 $2000000(1+2\%)^4 = 2000000\cdot 1.02^4$。
現在，請使用計算器算出 2 年後的餘額。

學生： 餘額為 2,164,864。

老師： 非常好。

如果將初始金額 P 存入一個每年支付年利率 r，每年複利 n 次的帳戶。t 年後帳戶中的餘額 A 由以下公式算出：

$$A = P\ (1+\frac{r}{n})^{nt}$$

例題二

說明：本題是指數方程式求解。

（英文）For what value of x does $10^x \cdot 100^{2x} = 1000^5$.

(A) 1　(B) 2　(C) 3　(D) 4　(E) 5

（中文）試求下列指數方程式中 x 的值：$10^x \cdot 100^{2x} = 1000^5$

(A) 1　(B) 2　(C) 3　(D) 4　(E) 5

(2023 AMC 10A problem 2)

Teacher: We can change all of the terms in the base of 10 to get a clear expression.

The left-hand side is $10^x \cdot 100^{2x} = 10^x \cdot (10^2)^{2x} = 10^x \cdot 10^{4x} = 10^{5x}$.

Can you convert the right side to the same base 10?

Student: Yes, the right-hand side 1000^5 can be simplified to 10^{15}.

Teacher: Excellent. So, $5x = 15$ and $x = 3$. The answer is (C).

老師： 我們可以把每一項的底數換成 10，得到一個清楚的表達式。

左側是 $10^x \cdot 100^{2x} = 10^x \cdot (10^2)^{2x} = 10^x \cdot 10^{4x} = 10^{5x}$。

你們可以將右側簡化為底數為 10 的函數嗎?

它的指數是多少？

學生： 可以。右側1000^5可以簡化成10^{15}。

老師： 太棒了。接著，計算 $5x = 15, x = 3$。答案是 (C).

例題三

說明：這題利用指數函數計算通貨膨脹問題。

（英文）In the past 8 years, the price of Wow Burger could be approximated by the model $y= 85(1.06)^t$, where y is the price (NT) and t is the number of years since 2015 (t = 0).

(1) What is the price of Wow Burger in 2023?

(2) If the price model still works for the following 10 years, estimate the year when the price of Wow Burger will be over 170 NT.

(Round your answer to the nearest whole number.)

（中文）過去 8 年，Wow 漢堡的價格變化可以用模型 $y = 85(1.06)^t$ 來估算，其中 *y* 是漢堡的價格（新台幣）而 *t* 表示自 2015 年以來的年數(2015 年時 t = 0)。

(1) 試問 Wow 漢堡 2023 年的價格是多少？

(2) 若此函數模型仍適用於未來 10 年，預計哪一年 Wow 漢堡價格將超過新台幣 170 元？

（請將答案四捨五入至整數）

Teacher: Use a calculator to do the first question. In 2023, t=8. What is the price of Wow Burger in 2023?

Student: 135 NT.

Teacher: Very good. Let us move on to the next question. We can use the calculator of TI-89 to run “solve ($85(1.06)^t > 150$, t)” to find the value of t. Or you can use another calculator as well. Please find the value of t now.

Student: t = 12.

Teacher: Excellent. The price will exceed 170 NT in the year of 2027 (t = 12).

老師： 大家用計算機來算第一個問題。在 2023 年時，t=8。2023 年 Wow 漢堡的價格是多少？

學生： 新台幣 135 元。

老師： 很好。讓我們繼續看下一小題。

我們可以使用 TI-89 計算機運行 “solve (85(1.06)^t> 150, t)” 來找到 t 的值。
或者你也可以使用其他的計算機計算。
現在請求出 t 的值。

學生： t = 12。

老師： 太棒了。價格將會在 2027 年（t = 12）超過新台幣 170 元。

單元四　對數與對數律
Logarithms and Laws of Logarithms

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■ 前言 Introduction

本單元的內容為對數的定義及其基本性質；接下來討論對數律與換底公式在對數的運算練習；最後是練習與對數有關的學測題。老師在介紹對數的定義與指數函數的關係後，先讓學生透過隨堂練習，熟悉對數運算的特性。建議在示範例題時老師能讓學生瞭解對數與指數的關係，運用已學的指數性質來熟悉對數的運算與性質，最後以應用問題或學測題來加深學生們對本單元的了解。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
logarithm	**對數**	illustrate	闡明
antilogarithm	**真數**	denote	表示
common logarithm	**常用對數**	condense	壓縮
logarithmic	**對數的**	expand	展開
laws of logarithms	**對數律**		

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ ___________ is read as ___________.

例句：The expression of $\log_2 128$is read as “log of 128 with base 2.”

$\log_2 128$讀作「以 2 為底數時，128 的對數」。

❷ _______ denoted by _______.

例句：The exponential equation $10^x = 7$ has a unique real solution which can be **denoted by** $\log 7$.

指數方程式$10^x = 7$有唯一實數解，可以用$\log 7$表示。

❸ derive ________

例句：How can you use properties of exponents to **derive** properties of logarithms?

如何用指數的性質來導出對數的性質?

❹ condense________

例句：Please **condense** the following logarithmic expressions and get the value.

請將下列的對數式壓縮並求出其值。

■ 問題講解 Explanation of Problems

☙ 說明 ❧

In this section, we will define and evaluate logarithms, then apply the laws of logarithms to expand, condense, or simplify the logarithms, and finally evaluate logarithms with different bases by applying the change-of-base formula. First, students will use the definition of logarithms to rewrite logarithms in exponential form and evaluate the logarithms with different bases. Remind students to follow the definition (If $y= b^x$, then $x = \log_b y$) when they have difficulties in evaluating the logarithms. The second part of the lesson is to apply the laws of logarithms:

1. Product Law: $\log_a x + \log_a y = \log_a xy$
2. Quotient Law: $\log_a x - \log_a y = \log_a \frac{x}{y}$
3. Power Law: $k \log_a x = \log_a x^k$

The properties are used to perform calculations, along with expanding and condensing logarithmic expressions. Teachers can use the properties of exponents to derive the laws of logarithms. Finally, the change-of-base formula is introduced to evaluate the logarithms with different bases. Logarithms with any base other than 10 or *e* can be written in terms of common or natural logarithms using the change-of-base formula.

運算問題的講解

例題一

說明：本題為運用對數的定義求對數值。

（英文）Evaluate each logarithm.

(1) $\log_2 128$　(2) $\log_3 9\sqrt{3}$　(3) $\log_{\frac{1}{5}} 25$

（中文）試求下列對數值。

(1) $\log_2 128$　(2) $\log_3 9\sqrt{3}$　(3) $\log_{\frac{1}{5}} 25$

Teacher: By the definition of the logarithm: $\log_a y = x$ if and only if $a^x = y$.
For each logarithm $\log_a y = x$, you can rewrite its exponential form $a^x = y$ to get the value of x.
So, to evaluate$\log_2 128 = x$, you can find the value of x by using $2^x = 128$.
What is the value of x?

Student: 7.

Teacher: Yes, you are correct. So, $\log_2 128 = 7$.
Let us see the next question $\log_3 9\sqrt{3}$.
Find the value of x when we convert it to the exponential form $3^x = 9\sqrt{3}$.

Student: $x = \frac{5}{2}$.

Teacher: Very good.
In the last question, how to rewrite the logarithm in exponential form?

Student: $\left(\frac{1}{5}\right)^x = 25$.

Teacher: Excellent. Then, we rewrite each side as a power of the same base 5.
Then, you can get $\left(\frac{1}{5}\right)^{\mathrm{x}} = 5^{-\mathrm{x}} = 5^2$.
What is the value of x?

Student: -2

Teacher: Correct. Let us move on to the next question.

老師： 根據對數的定義：若且唯若$a^x = y$時，$\log_a y = x$。每個對數$\log_a y = x$，你都可以重新寫成它的指數形式$a^x = y$，以求 x 的值。

老師： 因此，要求$\log_2 128 = x$，你可以透過$2^x = 128$來找到 x 的值。

x 的值是多少？

學生： 7。

老師： 沒錯。所以$\log_2 128 = 7$。

讓我們來看第 2 小題$\log_3 9\sqrt{3}$。將它轉換為指數形式 $3^x = 9\sqrt{3}$後，求出 x 的值。

學生： $x=\frac{5}{2}$。

老師： 非常好。最後一小題，如何將對數重新寫成指數形式？

學生： $(\frac{1}{5})^x = 25$。

老師： 優秀。然後，我們將兩邊都寫成相同底數 5 的次方。然後得到$(\frac{1}{5})^x = 5^{-x} = 5^2$。

x 的值是多少？

學生： -2

老師： 正確。我們繼續看下一個例題。

例題二

說明：本題是運用對數的基本性質求對數值。。

（英文）Evaluate each logarithm.

(1) $\log_5 1$ (2) $\log_7 7$ (3) $10^{\log 5}$ (4) $\log_7 49^x$

（中文）試求下列對數值。

(1) $\log_5 1$ (2) $\log_7 7$ (3) $10^{\log 5}$ (4) $\log_7 49^x$

Teacher: Instead of using the exponential form to evaluate the logarithms, you can find the value of $\log_5 1$directly by asking yourself what power of 5 gives you 1

Student: 0.

Teacher: Good. So, we get $\log_5 1 = 0$.

And what power of 7 is 7? Then, it is easy to get $\log_7 7 = 1$.

The first two parts illustrate two special logarithmic values:

Logarithm of 1: $\log_b 1 = 0$.

Logarithm of b with base b: $\log_b b = 1$.

So, what is the value of $\log_{2023} 2023$?

Student: 1.

Teacher: Yes, you are correct. Let us see the next one $10^{\log 5}$.

$\log 5$ is a common logarithm whose base is 10.

Assume $\log 5 = x$, then we get $10^x = 5$.

That means $10^{\log 5} = 5$.

This is another property of logarithm: $a^{\log_a b} = b$.

To evaluate $\log_7 49^x$, we can express 49 as a power with base 7.

So, $\log_7 49^x = \log_7 (7^2)^x = \log_7 7^{2x}$.

What is the answer to the last question $\log_7 49^x$

Student: $2x$.

Teacher: Excellent.

This is also a property of logarithm: $\log_b b^x = x$.

What power of b gives you b^x? It is natural that the answer is x.

老師：不必使用指數形式來求對數，你可以直接問自己「5 的多少次方等於 1？」來找到$\log_5 1$的值。

學生：0。

老師：很好。所以，我們得到$\log_5 1 = 0$。

那麼，7 的多少次方等於 7 呢？這很容易得到$\log_7 7 = 1$

老師：前兩部分說明了兩個特別的對數值：

1 的對數：$\log_b 1 = 0$。

以 b 為底數，b 的對數：$\log_b b = 1$.

那麼老師考你們，$\log_{2023} 2023$的值是多少？

學生：1。

老師：是的，沒錯。讓我們看下一小題$10^{\log 5}$。

$\log 5$是底數為 10 的常用對數。假設$\log 5 = x$，那麼我們得到$10^x = 5$，也就是說$10^{\log 5} = 5$。

這是對數的另一個性質：$\boldsymbol{a^{\log_a b} = b}$。

老師：　要算$\log_7 49^x$，我們可以將 49 表示成底數為 7 的次方。

因此，$\log_7 49^x = \log_7 (7^2)^x = \log_7 7^{2x}$，所以最後一小題$\log_7 49^x$的答案是多少？

學生：　$2x$。

老師：　非常好。這也是對數的性質之一：$\mathbf{\log_b b^x = x}$。
b 的多少次方等於b^x？答案自然會是 x。

例題三

說明：本題是運用對數律化簡對數。

（英文）Evaluate each logarithm.

(1) $\log_6 18 + \log_6 12$　　(2) $\log_3 9\sqrt{5} - \log_3 \sqrt{45}$

(3) $3\log 6 - 2\log 3 - \log \frac{3}{125}$

（中文）試求下列各式的值。

(1) $\log_6 18 + \log_6 12$　　(2) $\log_3 9\sqrt{5} - \log_3 \sqrt{45}$

(3) $3\log 6 - 2\log 3 - \log \frac{3}{125}$

Teacher:　We can apply $\mathbf{\log_a x + \log_a y = \log_a xy}$ (Product Law) to evaluate the expression.
(1) $\log_6 18 + \log_6 12 = \log_6 18 \cdot 12 = \log_6 216 = 3$
And we can use a similar way to solve the second question by applying

$\mathbf{\log_a x - \log_a y = \log_a \frac{x}{y}}$ (Quotient Law)

What is the answer to the second part?

Student:　1.

Teacher:　Excellent. By applying the law of logarithm, you can get:

$\log_3 9\sqrt{5} - \log_3 \sqrt{45} = \log_3 \frac{9\sqrt{5}}{\sqrt{45}} = \log_3 3 = 1.$

Please do the last question with your partner.
(3 minutes later) Tell me what you get.

Student:　3

Teacher: Great.Your answer is correct.

This question also needs to apply the Power Law $k\log_a x = \log_a x^k$.

So, $3\log 6 - 2\log 3 - \log\frac{3}{125}$ is condensed to $\log\frac{6^3}{3^2}\cdot\frac{125}{3}$ which is equal to 3.

老師： 我們可以應用$\log_a x + \log_a y = \log_a xy$（乘法法則）來算第 1 個式子。

學生： (1) $\log_6 18 + \log_6 12 = \log_6 18\cdot 12 = \log_6 216 = 3$

老師： 我們可以用類似的方法來解第 2 小題，應用$\log_a x - \log_a y = \log_a\frac{x}{y}$（商法則）。

答案是多少？

學生： 1。

老師： 很好。通過應用對數律，你可以得到：

$\log_3 9\sqrt{5} - \log_3\sqrt{45} = \log_3\frac{9\sqrt{5}}{\sqrt{45}} = \log_3 3 = 1$。

請和你的搭檔一起完成最後 1 小題。（3 分鐘後）告訴我答案是多少。

學生： 3

老師： 很好，答對了。這個問題還需要應用到對數律，$k\log_a x = \log_a x^k$。

所以，$3\log 6 - 2\log 3 - \log\frac{3}{125}$可以簡化為$\log\frac{6^3}{3^2}\cdot\frac{125}{3}$，答案是 3。

例題四

說明：本題是運用換底公式求解。

（英文）Evaluate each logarithm.

(1) $\log_3 8 \times \log_2 9$　(2)$\log_a b \times \log_b c \times \log_c a$

（中文）試求下列各式的值。

(1) $\log_3 8 \times \log_2 9$　(2) $\log_a b \times \log_b c \times \log_c a$

Teacher: When the bases of the logarithms are different, we need to change them to the logarithms with the same base.

Here is the rule for changing the base: $\log_a b = \frac{\log_c b}{\log_c a}$ or $\log_a b = \frac{\log b}{\log a}$

So, $\log_3 8 \times \log_2 9 = \frac{\log 8}{\log 3} \times \frac{\log 9}{\log 2} = \frac{3\log 2}{\log 3} \times \frac{2\log 3}{\log 2} = 6$

Do you have any questions?

Student: No.

Teacher: OK. Then, please do the second question now.

What is the answer?

Student: 1

Teacher: Yes, you are excellent.

It is convenient to use the common logarithm (base= 10).

$$\log_a b \times \log_b c \times \log_c a = \frac{\log b}{\log a} \cdot \frac{\log c}{\log b} \cdot \frac{\log a}{\log c} = 1$$

老師：當對數的底數不同時，我們需要將它們換為相同底數的對數。

這就是換底公式：$\log_a b = \frac{\log_c b}{\log_c a}$ 或者是 $\log_a b = \frac{\log b}{\log a}$

因此，$\log_3 8 \times \log_2 9 = \frac{\log 8}{\log 3} \times \frac{\log 9}{\log 2} = \frac{3\log 2}{\log 3} \times \frac{2\log 3}{\log 2} = 6$。

有沒有任何問題？

學生：沒有。

老師：好的，那麼現在請回答第 2 小題。答案是多少？

學生：1

老師：沒錯，做得很好。換成常用對數（底數 ＝ 10）會讓計算更便利。

$$\log_a b \times \log_b c \times \log_c a = \frac{\log b}{\log a} \cdot \frac{\log c}{\log b} \cdot \frac{\log a}{\log c} = 1$$

應用問題／學測指考題

例題一

說明：本題是對數的應用練習。

（英文）The five terms of a real number sequence a_1, a_2, a_3, a_4, and a_5 are all greater than 1, and one number is twice the other number in each adjacent two terms. How many possible values are there for a_5 if $a_1 = \log_{10} 36$?

(1) 3 (2) 4 (3) 5 (4) 7 (5) 8

（中文）五項實數數列a_1，a_2，a_3，a_4，a_5的每一項都大於 1，且每相鄰的兩項中，都有一數是另一數的兩倍。若$a_1 = \log_{10} 36$，則 a_5 有多少種可能的值？

(1) 3 (2) 4 (3) 5 (4) 7 (5) 8

（110 年學測單選題第 2 題）

Teacher: By the given information, you can get $a_2 = 2a_1$ or $a_2 = \frac{1}{2}a_1$ since one number is twice the other number in each adjacent two terms.

When $a_1 = \log_{10} 36$, $a_2 = \frac{1}{2}a_1 = \frac{1}{2}\log_{10} 36 = \log_{10} 6$ is less than 1 which contradicts the given information.

So, we have $a_2 = 2a_1$.

We can use a tree diagram to show the possible values:

$$a_1 \left\langle \begin{array}{l} a_2 = \frac{1}{2}a_1 = \log_{10} 6\ (\times) \\ a_2 = 2a_1\ (\checkmark) \end{array} \right.$$

What are the possible values for a_3?

Student: $a_3 = 2a_2 = 4a_1$ or $a_3 = \frac{1}{2}a_2 = a_1$

Teacher: Correct. The tree diagram can show the answer clearly.

Tree diagram: $a_2 \left\langle \begin{array}{l} a_3 = \frac{1}{2}a_2 = a_1(\checkmark) \\ a_3 = 2a_2 = 4a_1\ (\checkmark) \end{array} \right.$

Next, find the possible values for a_4.

What will you derive from $a_3 = 4a_1$?

Student: $a_3(=4a_1)$ $\left\langle \begin{array}{l} a_4 = \frac{1}{2}a_3 = 2a_1(\checkmark) \\ a_4 = 2a_3 = 8a_1\ (\checkmark) \end{array} \right.$

Teacher: Good. And we can also get $a_4 = 2a_1$ by the other condition $a_3 = a_1$.

When $a_3(=a_1)$ $\left\langle \begin{array}{l} a_4 = \frac{1}{2}a_3 = \frac{1}{2}a_1(\times) \\ a_4 = 2a_3 = 2a_1\ (\checkmark) \end{array} \right.$

So, there are two possible values of a_4: $a_4 = 8a_1$ or $a_4 = 2a_1$. Please find the possible values of a_5 now.

Student: When $a_4(=8a_1)$ $\left\langle \begin{array}{l} a_5 = \frac{1}{2}a_4 = 4a_1(\checkmark) \\ a_5 = 2a_4 = 16a_1\ (\checkmark) \end{array} \right.$

When $a_4(=2a_1)$ $\left\langle \begin{array}{l} a_5 = \frac{1}{2}a_4 = a_1(\checkmark) \\ a_5 = 2a_4 = 4a_1\ (\checkmark) \end{array} \right.$.

Teacher: Excellent.

So, there are 3 possible values for a_5 $(a_5 = a_1, 4a_1, or\ 16a_1)$. The answer is (1) 3.

老師： 題目說每相鄰的兩項中，都有一數是另一數的兩倍，所以可以得出$a_2 = 2a_1$或$a_2 = \frac{1}{2}a_1$。

老師： 當$a_1 = \log_{10} 36$時，假設$a_2 = \frac{1}{2}a_1 = \frac{1}{2}\log_{10} 36 = \log_{10} 6$，則會小於 1，與題目給的條件互相矛盾。因此，應該會是$a_2 = 2a_1$。

我們可以用樹狀圖來呈現可能的結果。

a_1 $\left\langle \begin{array}{l} a_2 = \frac{1}{2}a_1 = \log_{10} 6\ (\times) \\ a_2 = 2a_1\ (\checkmark) \end{array} \right.$

a_3可能的值會是多少？

學生： $a_3 = 2a_2 = 4a_1$或$a_3 = \frac{1}{2}a_2 = a_1$。

老師： 正確。用樹狀圖來呈現$a_3 = 4a_1$或$a_3 = a_1$會更清楚。

a_2 $\left\langle \begin{array}{l} a_3 = \frac{1}{2}a_2 = a_1(\checkmark) \\ a_3 = 2a_2 = 4a_1\ (\checkmark) \end{array} \right.$

接下來，找出a_4可能的值。

你能從$a_3 = 4a_1$推出什麼？

學生： $a_3(=4a_1)$ $\left\langle \begin{array}{l} a_4 = \frac{1}{2}a_3 = 2a_1(\checkmark) \\ a_4 = 2a_3 = 8a_1\ (\checkmark) \end{array} \right.$

老師： 很好。我們還可以根據另一個條件$a_3 = a_1$得到$a_4 = 2a_1$。

$a_3(=a_1)\begin{cases} a_4=\frac{1}{2}a_3=\frac{1}{2}a_1(\times) \\ a_4=2a_3=2a_1\ (\checkmark) \end{cases}$

因此，a_4有兩個可能的值：$8a_1$或$2a_1$。

請現在找出a_5的可能值。

學生：當$a_4=8a_1$時，$\begin{cases} a_5=\frac{1}{2}a_4=4a_1(\checkmark) \\ a_5=2a_4=16a_1\ (\checkmark) \end{cases}$。

當$a_4=2a_1$時，$\begin{cases} a_5=\frac{1}{2}a_4=a_1(\checkmark) \\ a_5=2a_4=4a_1\ (\checkmark) \end{cases}$。

老師：很好。所以，a_5有 3 個可能的值: $a_5=16a_1$， $4a_1$或 a_1。答案是 (1) 3。

例題二

說明：這題是利用對數律求解。

（英文）If $ab^2=10^5$ and $a^2b=10^3$ $(a>0, b>0)$, then $\log b=$?
(Write the fraction in the lowest terms.)

（中文）有兩個正實數 a、b，已知$ab^2=10^5$，$a^2b=10^3$，則$\log b=$_______。（化為最簡分數）

（112 年學測數學 A 選填題第 13 題）

Teacher: We can take the common log of each side of the original equations $\begin{cases} ab^2=10^5 \\ a^2b=10^3 \end{cases}$.

Then, we get $\begin{cases} \log ab^2=\log a+2\log b=5 \\ \log a^2 b=2\log a+\log b=3 \end{cases}$.

Can you find the value of $\log b$?

Student: Yes, $\log b=\frac{7}{3}$.

Teacher: Very good.

老師：　我們可以對原方程式 $ab^2=10^5$ 和 $a^2b=10^3$ 共同取常用對數。

接著得到 $\begin{cases}\log ab^2=\log a+2\log b=5\\ \log a^2 b=2\log a+\log b=3\end{cases}$。

能算出 $\log b$是多少嗎？

學生：　$\log b=\frac{7}{3}$。

老師：　非常好。

例題三

說明：本題是運用對數的換底公式。

（英文）A student used a calculator of a certain brand to find $\log_a b$ ($a>1$ and $b>1$) by keying in $\boxed{\log(a,b)}$. But the student did it in the wrong order by keying in $\boxed{\log(b,a)}$ and he got the value which was $\frac{9}{4}$ times the correct value. Find the relationship between a and b.

(1) $a^2=b^3$　(2) $a^3=b^2$　(3) $a^4=b^9$　(4) $2a=3b$　(5) $3a=2b$

（中文）某品牌計算機在計算對數$\log_a b$時需按$\boxed{\log(a,b)}$。**某生在計算**$\log_a b$時（其中 $a>1$ 且 $b>1$）順序弄錯，誤按$\boxed{\log(b,a)}$，所得為正確值的$\frac{9}{4}$倍。試選出 a,b 間的關係式。

(1) $a^2=b^3$　(2) $a^3=b^2$　(3) $a^4=b^9$　(4) $2a=3b$　(5) $3a=2b$

（111 年學測數學 A 單選題第 2 題）

Teacher:　First write an equation using the given information. $\log_b a=\frac{9}{4}\log_a b$

Change both of them into a common logarithm. $\frac{\log a}{\log b}=\frac{9}{4}\cdot\frac{\log b}{\log a}$

Then, we get $4(\log a)^2=9(\log b)^2$. So, $2\log a=3\log b$.

Student:　Why $2\log a\neq -3\log b$?

Teacher:　Good question.

Both values of $\log a$ and $\log b$ are positive if $a>1$ and $b>1$.

So, $2\log a=-3\log b$ is not the answer.

Can you find the relationship between a and b now?

Student: Yes, the answer is (1) $a^2 = b^3$.

Teacher: Excellent. $2\log a = 3\log b$ can be converted to $\log a^2 = \log b^3$.

So, $a^2 = b^3$. The answer is (1).

老師： 首先，根據題目所給的資訊，列出方程式：$\log_b a = \frac{9}{4}\log_a b$。接著將它們都轉換成常用對數：$\frac{\log a}{\log b} = \frac{9}{4} \cdot \frac{\log b}{\log a}$。然後，我們得到$4(\log a)^2 = 9(\log b)^2$。

所以，$2\log a = 3\log b$。

學生： 為什麼$2\log a \neq -3\log b$？

老師： 很好的問題。如果 a > 1 且 b > 1，那麼$\log a$和$\log b$的值都是正的。

因此，$2\log a = -3\log b$不是答案。

老師： 你現在能求出a和b之間的關係嗎？

學生： 可以，答案是 (1) $a^2 = b^3$。

老師： 太棒了。$2\log a = 3\log b$可以轉換為$\log a^2 = \log b^3$。

學生： 所以，$a^2 = b^3$。答案是 (1)。

例題四

說明：這題是解指數與對數的關係求解。

（英文）If $x^{-\frac{1}{3}}y^2 = 1$ and $2\log y = 1$ $(x > 0, y > 0)$, then find the value of $\frac{x-y^2}{10}$.

（中文）若 x, y 為兩正實數，且滿足$x^{-\frac{1}{3}}y^2 = 1$及$2\log y = 1$，則$\frac{x-y^2}{10} =$ __________

（111 年學測數學 B 選填題第 13 題）

Teacher: Directly multiplying $x^{\frac{1}{3}}$on each side of$x^{-\frac{1}{3}}y^2 = 1$, you can get$y^2 = x^{\frac{1}{3}}$and then$x = y^6$.

It is your turn to find the value of $y$from the other equation $2\log y = 1$.

Student: OK. $y^2 = 10$.

Teacher: Good. How about the value of x?

Student: $x =$ 1000.

Teacher: Then, what is the value of $\frac{x-y^2}{10}$?

Student: $\frac{x-y^2}{10} = 99$.

Teacher: Great.

老師： 直接將 $x^{\frac{1}{3}}$ 乘在方程式 $x^{-\frac{1}{3}}y^2 = 1$ 的兩側，你可以得到 $y^2 = x^{\frac{1}{3}}$，然後 $x = y^6$。

換你們試著從 $2\log y = 1$中找到 y 的值。

學生： 好的。$y^2 = 10$。

老師： 很好。那麼 x 是多少？

學生： $x = 1000$。

老師： 非常好。那麼 $\frac{x-y^2}{10}$ 的值是多少？

學生： $\frac{x-y^2}{10} = 99$。

老師： 太棒了。

單元五　對數函數
Logarithmic Functions

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■ 前言 Introduction

本單元的內容為對數函數 $f(x) = \log_a x$ 的定義及其圖形，利用描點法作圖來觀察底數 $a > 0$，$a \neq 1$且$x > 0$ 等對數函數的定義及圖形；接下來討論對數方程式及對數不等式的求解問題；最後是對數函數在生活中的應用。老師在介紹本單元對數函數的圖形後，先讓學生透過隨堂練習，觀察並了解對數函數與指數函數的反函數關係與圖形對稱的特性。建議在示範例題時老師能反覆提問，讓學生能熟悉本章節的重要的英文名詞及用語後，最後以應用問題或學測題來加深學生們對本單元的了解。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
logarithmic function	**對數函數**	collinear	共線的
contradict	與..矛盾	format	格式
sketch	畫出	blank	空格
huge	巨大的	exponentiate	指數化

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ ___________ is read as ___________.

例句：$\log_2 128$ **is read as** "log base 2 of 128."

$\log_2 128$ 讀作「以 2 為底數時，128 的對數」。

❷ fill in _______.

例句：Please **fill in** your answer in the corresponding space after you find out the solution.

找出答案後，請將答案填在相對應的空格上。

❸ exponentiate ________

例句：**Exponentiate** each side with base 10, then you can get the equation in the exponential form.

將等號兩側轉為以 10 為底數的指數函數，就可以得到指數方程式。

❹ contradict ___________.

例句：Your solution **contradicts** the definition of the logarithmic function.

你的答案與對數函數的定義相矛盾。

■ 問題講解 Explanation of Problems

☙ 說明 ❧

In this section, we will cover the logarithmic function $f(x) = \log_a x$, where $a > 0, a \neq 1$ and $x > 0$. The lesson starts with the graphs of logarithmic functions for a > 1 and 0 < a < 1. The graph of the function $\log_a x$ is strictly increasing when a > 1, and strictly decreasing when 0 < a < 1. The second part of the lesson involves solving logarithmic equations and inequalities.

When we are solving logarithmic equations, the logarithms can be set to the logarithms with the same base. We should check the solutions because the domain of a logarithmic function is positive numbers only. When solving logarithmic inequalities, $\log_a x > \log_a y$ if and only if $x > y$ when the base a > 1. Finally, we will do some real-life problems.

꧁運算問題的講解꧂

例題一

說明：本題是畫出對數函數 $y = \log_a x$ 的圖形 $(a > 1)$。

（英文）Find the values of the arithmetic function $y = \log_2 x$ in the table.
Then, use the dot method to graph $y = \log_2 x$.

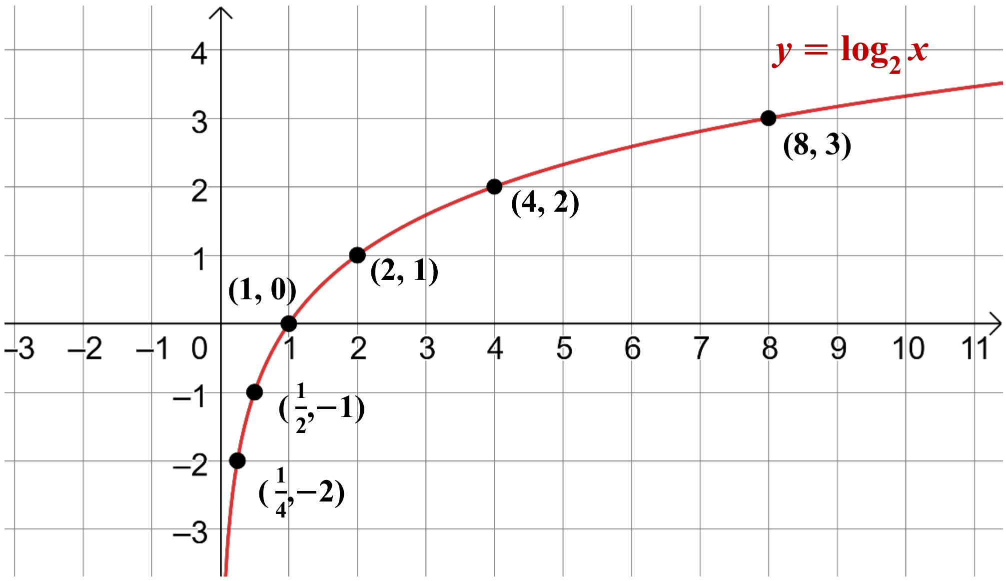
（中文）找出表格中的函數值，並用描點法畫出指數函數 $y = \log_2 x$。

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$y = \log_2 x$							

Teacher: Complete the table for the given logarithmic function.
Tell me the values from left to right.

Student: $-3, -2, -1, 0, 1, 2, 3.$

Teacher: Next, plot all the points and use a smooth curve to connect the dots from left to right.



Now please repeat the same steps to sketch the graph of $y = \log_3 x$ in the same coordinate plane.

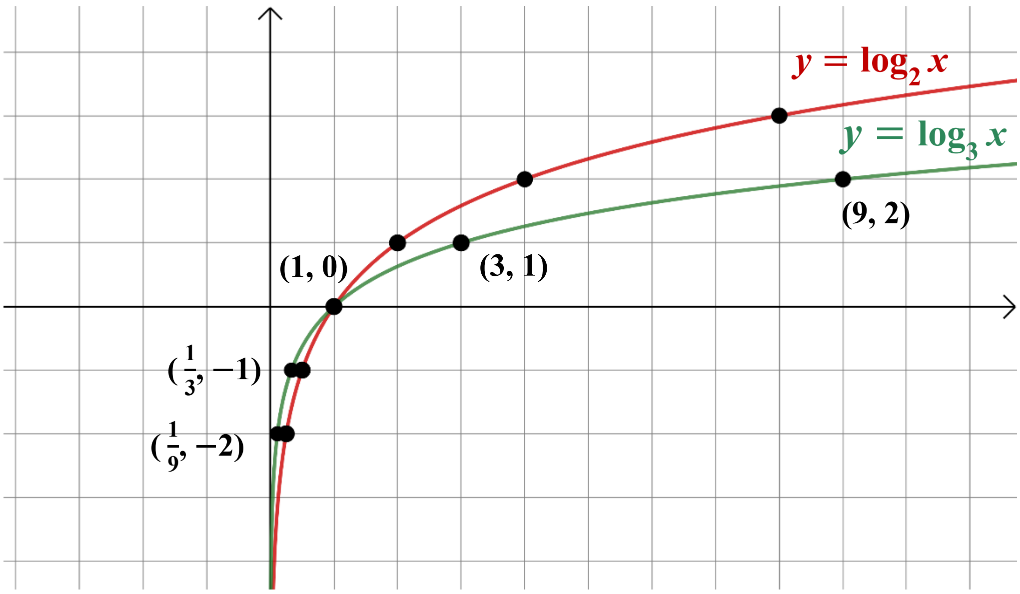
(A few minutes later.)

Compare the two graphs and tell me your findings.

Student: 1. Both graphs of $y = \log_2 x$ and $y = \log_3 x$ are strictly increasing.

2. The graph of $y = \log_3 x$ increases slower than $y = \log_2 x$.

3. Both of the graphs pass through (1, 0)



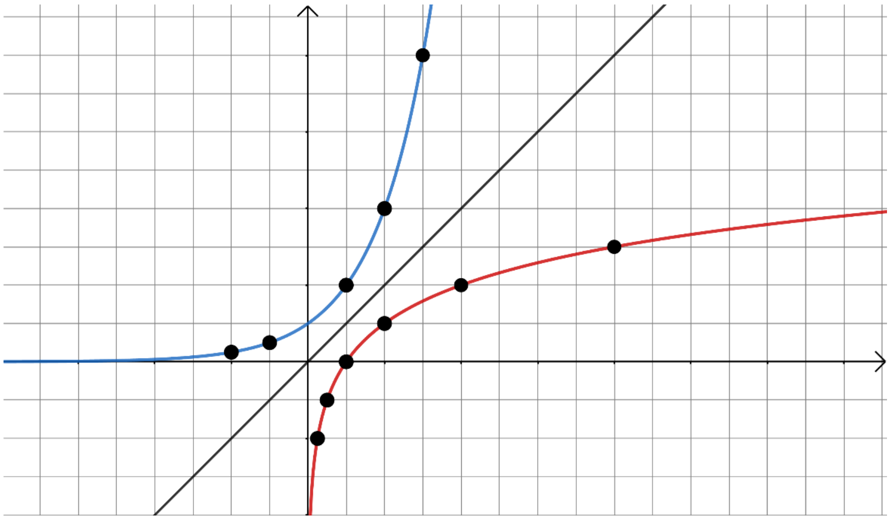
Teacher: Excellent. You are all correct.

The logarithmic functions $y = \log_2 x$ and $y = \log_3 x$ are strictly increasing.

Besides, you can find that the y-axis is the vertical asymptote of the logarithmic function $y = \log_a x$.

Then, compare the graphs of $y = \log_2 x$ and $y = 2^x$.

We know that the exponential function and the logarithmic function are inverse functions of each other. The graphs are symmetric with respect to the line $y = x$.



Finally, the domain (the set of all x values) of the logarithmic function $y = \log_a x$ $(a > 0, a \neq 1)$ is all real positive numbers ($x > 0$) and the range (the set of all y values) is all real numbers.

老師： 透過給定的對數函數完成表格。告訴我從左到右分別是多少？

學生： −3、 −2、 −1、0、1、2、3。

老師： 接下來，畫出所有點，用平滑曲線將點從左到右連接起來。

老師： 現在請重複相同的步驟，在同一坐標平面上畫出 $y = \log_3 x$ 的圖形。
（幾分鐘後）比較這兩個圖形，告訴我你發現了什麼。

學生： $y = \log_2 x$ 和 $y = \log_3 x$ 的圖形都是嚴格遞增的。
$y = \log_3 x$ 的圖形增長速度比 $y = \log_2 x$ 慢。
這兩個圖形都通過 $(1, 0)$ 點。

老師： 非常好。你們都答對了。

老師： 對數函數 $y = \log_2 x$ 和 $y = \log_3 x$ 都是嚴格遞增的。此外，你可以發現 y 軸是對數函數 $y = \log_a x$ 的垂直漸近線。

老師： 然後來比較 $y = \log_2 x$ 和 $y = 2^x$ 的圖形。我們知道指數函數和對數函數是彼此的反函數。它們的圖形對稱於直線 $y = x$。

老師： 最後，對數函數 $y = \log_a x$ $(a > 0, a \neq 1)$ 的定義域（所有 x 值的集合）是所有正實數（$x > 0$），而值域（所有 y 值的集合）是所有實數。

例題二

說明：本題是畫出對數函數 $y = \log_a x\,(0 < a < 1)$ 的圖形。。

（英文）Find the values of the exponential function $y = \log_{\frac{1}{2}} x$ in the table.

Then, use the dot method to sketch $y = \log_{\frac{1}{2}} x$.

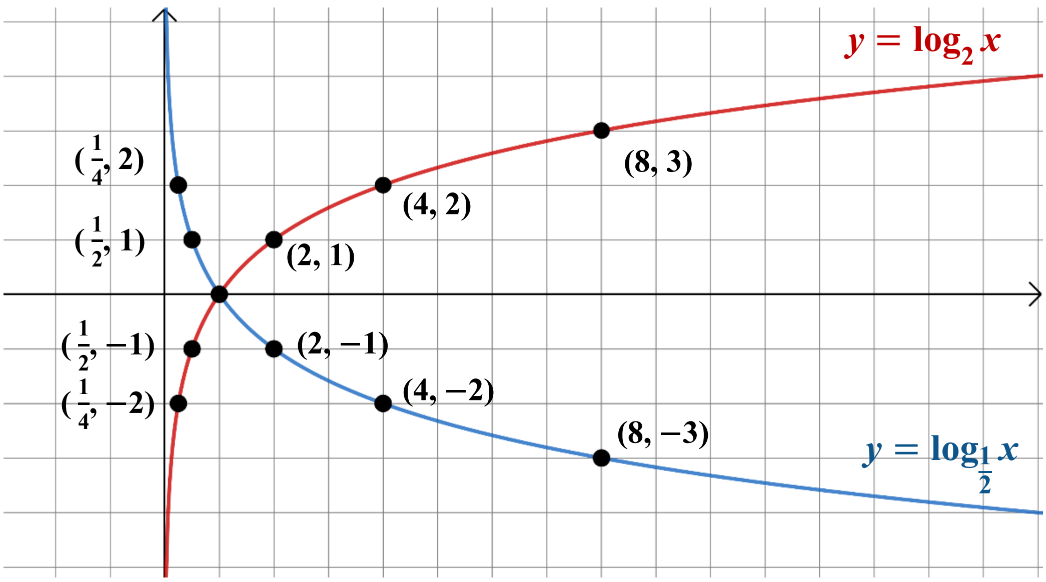
（中文）找出表格中的函數值，並用描點法畫出指數函數 $y = \log_{\frac{1}{2}} x$。

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$y = \log_{\frac{1}{2}} x$							

Teacher: Find all the values in the table now. What are the values from left to right?

Student: 3, 2, 1, 0, − 1, − 2, − 3

Teacher: Draw the graph of $y = \log_{\frac{1}{2}} x$ in the same coordinate plane with the graph of $y = \log_2 x$.



Compare the two graphs and tell me your findings.

Student: $y = \log_2 x$ and $y = \log_{\frac{1}{2}} x$ are symmetric with the x-axis.

Teacher: Yes, you are correct.

Besides, the graph of $y = \log_2 x$ is strictly increasing and the graph of $y = \log_{\frac{1}{2}} x$ is strictly decreasing.

Both of the graphs pass through the point (1, 0) and they both have *y*-axis as the vertical asymptote.

老師：現在找出表格中的所有數值。從左到右分別是多少？

學生：3、2、1、0、－1、－2、－3。

老師：在同一個坐標平面上畫出$y = \log_2 x$ 和 $y = \log_{\frac{1}{2}} x$ 的圖形。

比較這兩個圖形，告訴我你發現了什麼。

學生：$y = \log_2 x$ 跟 $y = \log_{\frac{1}{2}} x$ 對稱於 x 軸。

老師：是的，沒錯。

此外，$y = \log_2 x$ 是嚴格遞增函數，而 $y = \log_{\frac{1}{2}} x$ 是嚴格遞減函數。兩個圖形都通過點 (1, 0)，且都以 y 軸為垂直漸近線。

例題三

說明：本題是對數方程式求解。

（英文）Solve each logarithmic equation.

(1) $\log_2 2x = \log_2(x^2 - x - 10)$ (2) $\log_3 x + \log_3(2x - 3) = 3$

（中文）試求下列各對數方程式的解。

(1) $\log_2 2x = \log_2(x^2 - x - 10)$ (2) $\log_3 x + \log_3(2x - 3) = 3$

Teacher: In the first question, we get $2x = x^2 - x - 10$ because they have the same base. What is the answer to the quadratic equation?

Student: $x = 5$ or -2.

Teacher: Did you remember to check the solutions?

Remember that the antilogarithm b is a positive real number in the logarithm $\log_a b$.

Which of the values contradicts the definition?

Student: -2.

Teacher: Yes, you are correct. So, the solution is $x = 5$.

Next, we can apply the product law of logarithms to get

$\log_3 x(2x - 3) = 3$.

Rewrite it with the exponential form $x(2x - 3) = 27$.

Then, find the solution.

(3 minutes later) What is the solution?

Student: $\frac{9}{2}$.

Teacher: Great. You can exclude the value $x = -3$ because the value of the antilogarithm is always positive.

老師： 第一小題，因為兩邊底數相同，我們知道 $2x = x^2 - x - 10$

這個二次方程式的答案是多少？

學生： $x = 5$ 或-2。

老師： 有記得再檢查解嗎？

請記住，對數 $\log_a b$ 中的底數 b 是正實數。哪個不符合這個定義？

學生： -2。

老師： 很好沒錯。所以，解是 $x = 5$。

老師：　下一小題，我們可以應用對數的乘法法則，得到 $\log_3 x(2x-3)=3$。

將它重寫為指數形式 $x(2x-3)=27$。然後求出解。

（三分鐘後）答案是多少？

學生：　$\frac{9}{2}$。

老師：　太棒了。解要排除 $x=-3$，因為對數的真數始終是正的。

例題四

說明：本題是對數不等式求解。

（英文）Evaluate each logarithm.

(1) $\log_5(12x-4)<\log_5(25x^2-8x)$　　(2) $\log_4(-x)+\log_4(x+10)\geq 2$

（中文）試求下列對數不等式的解。

(1) $\log_5(12x-4)<\log_5(25x^2-8x)$　　(2) $\log_4(-x)+\log_4(x+10)\geq 2$

Teacher:　First, we get $12x-4<25x^2-8x$ because the logarithms are of the same base.

So, $25x^2-20x+4>0$. Or $(5x-2)^2>0$

What is the solution if $(5x-2)^2>0$?

Student:　$x\neq\frac{2}{5}$.

Teacher:　Good. Let's see the second question $\log_4(-x)+\log_4(x+6)\geq 2$.

First, you can get $-6<x<0$ because the antilogarithms are positive.

Then, combine the logarithms. $\log_4(-x)(x+6)\geq 2=\log_4 16$.

$-x^2-6x\geq 16$ or $x^2+6x-16\leq 0$.

Factor and find out the solution to the inequality now.

Student:　$-8\leq x\leq 2$.

Teacher:　The answer is not correct because you forget to check the answer.

$(-x)$and $(x+6)$are both greater than 0.

So, $-6<x<0$.

Find the intersection of the intervals $-8\leq x\leq 2$and $-6<x<0$.

What is the solution?

Student:　$-6<x<0$.

Teacher: Yes, you are correct.

老師：首先，因為兩對數有相同的底數，所以我們可以看成 $12x-4<25x^2-8x$。因此，$25x^2-20x+4>0$；或$(5x-2)^2>0$。

如果是$(5x-2)^2>0$的話，解是多少？

學生：$x\neq\frac{2}{5}$。

老師：很好。接下來看第 2 小題， $\log_4(-x)+\log_4(x+6)\geq 2$。

老師：首先，你可以得到 $-6<x<0$，因為真數必須是正的。然後，合併對數。$\log_4(-x)(x+6)\geq 2=\log_4 16$。得到$-x^2-6x\geq 16$ 或 $x^2+6x-16\leq 0$。

現在因數分解，然後找出不等式的解。

學生：$-8\leq x\leq 2$

老師：答案不對，你忘記再檢查一次答案了。

$(-x)$ 和 $(x+6)$都大於 0。

所以，$-6<x<0$。接著找出區間 $-8\leq x\leq 2$ 和 $-6<x<0$ 的交集。答案是多少？

學生：$-6<x<0$

老師：是的，答對了。

例題五

說明：本題是運用對數來比較巨大數字的大小。

（英文）Compare the three numbers 3^{1500}, 5^{1200} and 8^{700}. List them from least to greatest.
$(\log 2\approx 0.3010,\ \log 3\approx 0.4771)$

（中文）比較下列三數 3^{1500}、5^{1200}、8^{700}。由小至大排列此三數。
$(\log 2\approx 0.3010,\ \log 3\approx 0.4771)$

Teacher: We can take the common logarithms when comparing the huge numbers 3^{1500}, 5^{1200} and 8^{700}.

So, $\log 3^{1500}=1500\cdot\log 3=715.65$ by the given information $\log 3=0.4771$.

Please find the common log of the other numbers.

Student: How to get $\log 5^{1200}$ because $\log 5$ is not listed in the given information?

Teacher: Good question.

Here we can get $\log 5$ by using $\log 2=0.3010$.

Because $\log 5 = \log\frac{10}{2} = \log 10 - \log 2 = 1 - 0.3010 = 0.6990$.

Now, find the value of $\log 5^{1200}$ and $\log 8^{700}$.

Student: $\log 5^{1200} = 838.8$ and $\log 8^{700} = 632.1$.

Teacher: Excellent. $\log 8^{700} = 700 \cdot \log 8 = 700 \cdot (3\log 2) = 632.1$.

So, we get $\log 8^{700} < \log 3^{1500} < \log 5^{1200}$.

Please list the numbers in order now.

Student: They are $8^{700}, 3^{1500}, 5^{1200}$.

Teacher: Great. You are correct.

老師： 當我們比較這些大數字 3^{1500}, 5^{1200} 和 8^{700} 時，我們可以取對數。

根據題目給的 $\log 3 = 0.4771$，可以求出 $\log 3^{1500} = 1500 \cdot \log 3 = 715.65$，。

請大家算出其他數字的對數值。

學生： 怎樣才能得到 $\log 5^{1200}$呢？ 題目沒有給 $\log 5$ 是多少。

老師： 好問題。這裡，我們可以用 $\log 2 = 0.3010$ 來求 $\log 5$。

因為 $\log 5 = \log\frac{10}{2} = \log 10 - \log 2 = 1 - 0.3010 = 0.6990$。

現在，找出 $\log 5^{1200}$ 和 $\log 8^{700}$ 的值。

學生： $\log 5^{1200} = 838.8$ ， $\log 8^{700} = 632.1$

老師： 太棒了。$\log 8^{700} = 700 \cdot \log 8 = 700 \cdot (3\log 2) = 632.1$。

所以，我們得到 $\log 8^{700} < \log 3^{1500} < \log 5^{1200}$。現在請由小到大排列此三數。

學生： 8^{700}、3^{1500}、5^{1200}。

老師： 很棒，沒錯。

☙ 應用問題 / 學測指考題 ❧

例題一

說明：這題是利用對數律求解。

（英文）In a coordinate plane, the three points $(3,\ \log 3)$, $(6,\ \log 6)$and $(12, y)$ are on the same line, then $y = \log$ ___ ?

（中文）已知坐標平面上三點 $(3,\ \log 3)$、$(6,\ \log 6)$ 與 $(12, y)$在同一直線上，則 $y = \log$ ___ 。

（107 年學測選填題 A）

Teacher: If the three points $(3,\ \log 3)$, $(6,\ \log 6)$ and $(12,\ y)$ are collinear, then the slopes formed by any two points are the same.

$$m = \frac{\log 6 - \log 3}{6-3} = \frac{y - \log 6}{12-6}$$

Then, we get $2(\log 6 - \log 3) = y - \log 6$.

What is the value of y?

Student: $y = 3\log 6 - 2\log 3$.

Teacher: Well, your answer is not simplified because $\log 6 = \log 2 + \log 3$.

Besides, you need to condense it in one logarithm to fit the required format.

What is the solution?

Student: $y = 3\log 2 + \log 3 = \log 24$

Teacher: Excellent. Please fill in the blank with 24.

老師： 如果三個點 $(3,\ \log 3)$、$(6,\ \log 6)$ 和 $(12, y)$ 共線，那麼由任意兩點形成的斜率是相同的。

$$m = \frac{\log 6 - \log 3}{6-3} = \frac{y - \log 6}{12-6}$$

因此，我們得到 $2(\log 6 - \log 3) = y - \log 6$。求 y 的值是多少？

學生： $y = 3\log 6 - 2\log 3$

老師： 嗯，不過你的答案還沒化到最簡，因為 $\log 6 = \log 2 + \log 3$。而且，你需要將它簡化成題目要求的對數。算出答案是多少？

學生： $y = 3\log 2 + \log 3 = \log 24$

老師： 很好。請將空格填上 24。

例題二

說明：本題是對數不等式的應用問題。

（英文）When a six-sided die is rolled twice, the outcomes of the points are shown as *a* and *b* in order. What is the probability of $\log(a^2)+\log b>1$?

(1) $\frac{1}{3}$　(2) $\frac{1}{2}$　(3) $\frac{2}{3}$　(4) $\frac{3}{4}$　(5) $\frac{5}{6}$

（中文）連續投擲一公正骰子兩次，設出現的點數依序為 a, b。試問發生 $\log(a^2)+\log b>1$的機率為多少？

(1) $\frac{1}{3}$　(2) $\frac{1}{2}$　(3) $\frac{2}{3}$　(4) $\frac{3}{4}$　(5) $\frac{5}{6}$

（109 年學測單選題第 6 題）

Teacher: By applying the product law of logarithms, $\log(a^2)+\log b>1$ is equivalent to $\log(a^2b)>1$.
Exponentiate each side using base 10 and get $a^2b>10$.
How many different outcomes of (a, b) that satisfies $a^2b>10$?
List them out.

Student: $a=2\rightarrow b=3, 4, 5, 6$
$a=3\rightarrow b=2, 3, 4, 5, 6$
$a=4\rightarrow b=1, 2, 3, 4, 5, 6$
$a=5\rightarrow b=1, 2, 3, 4, 5, 6$
$a=6\rightarrow b=1, 2, 3, 4, 5, 6$
There are 27 outcomes.

Teacher: Good.
When you roll a six-sided die, there are 36 possible outcomes.
What is the probability?

Student: $\frac{27}{36}=\frac{3}{4}$. The answer is (4) $\frac{3}{4}$.

Teacher: Excellent.

老師： 運用對數的乘法法則，$\log(a^2)+\log b>1$ 等同於 $\log(a^2b)>1$。

接著兩邊用以 10 為底數的指數做運算，得到 $a^2b>10$。

有多少個不同的 (a, b) 組合滿足 $a^2b>10$ 呢？請列出來。

學生： $a=2\rightarrow b=3,4,5,6$

$a=3\rightarrow b=2,3,4,5,6$

$a=4\rightarrow b=1,2,3,4,5,6$

$a=5\rightarrow b=1,2,3,4,5,6$

$a=6\rightarrow b=1,2,3,4,5,6$

共有 27 種結果。

老師： 很好。

當你擲一個六面骰子時，有 36 種可能的結果。機率是多少？

學生： $\frac{27}{36}=\frac{3}{4}$，答案是(4) $\frac{3}{4}$。

老師： 非常好！

例題三

說明：這題是解指數與對數的聯立方程式。

（英文）If $(\log 100)(\log b)+\log 100+\log b=7$ with $b>0$, then which of the following statements is true?

(1) $1\le b\le\sqrt{10}$　(2) $\sqrt{10}\le b\le 10$　(3) $10\le b\le 10\sqrt{10}$

(4) $10\sqrt{10}\le b\le 100$　(5) $100\le b\le 100\sqrt{10}$

（中文）設正實數 b 滿足 $(\log 100)(\log b)+\log 100+\log b=7$。試選出正確的選項。

(1) $1\le b\le\sqrt{10}$　(2) $\sqrt{10}\le b\le 10$　(3) $10\le b\le 10\sqrt{10}$

(4) $10\sqrt{10}\le b\le 100$　(5) $100\le b\le 100\sqrt{10}$

（108 年學測單選題第 5 題）

Teacher: Simplify the given equation,we get $2\log b+2+\log b=7$.

$3\log b=5$ or $\log b=\frac{5}{3}$.

Find the value of b.

Student: $b=10^{\frac{5}{3}}$.

Teacher: Good. Then, choose the correct answer and explain the reason.

Student: The answer is (4) $10\sqrt{10} \le b \le 100$.

Because $10\sqrt{10} = 10^{\frac{3}{2}}$ and $100 = 10^2$, and $b = 10^{\frac{5}{3}}$ is between $10^{\frac{3}{2}}$ and 10^2.

Teacher: Great.

老師： 化簡題目給的方程式，我們得到 $2\log b + 2 + \log b = 7$。

$3\log b = 5$ 或是 $\log b = \frac{5}{3}$。求出 b 的值。

學生： $b = 10^{\frac{5}{3}}$.

老師： 很好。接下來，選出正確答案並解釋一下原因。

學生： 答案是 (4) $10\sqrt{10} \le b \le 100$。

因為 $10\sqrt{10} = 10^{\frac{3}{2}}$ 以及 $100 = 10^2$， $b = 10^{\frac{5}{3}}$ 就介於 $10^{\frac{3}{2}}$ 和 10^2 之間。

老師： 很棒。

例題四

說明：這題是解指數與對數的聯立方程式。

（英文）In an arithmetic sequence <a_1>, the first term a_1 and the common difference d are both positive real numbers. The three terms $\log a_1$, $\log a_3$, $\log a_6$ are also arithmetic. Find the common difference of the sequence $\log a_1$, $\log a_3$, $\log a_6$.

（中文）設等差數列<a_1>之首項 a_1 與公差 d 皆為正數，且 $\log a_1$, $\log a_3$, $\log a_6$ 依序也成等差數列。試選出數列 $\log a_1$, $\log a_3$, $\log a_6$ 的公差。

(1) $\log d$ (2) $\log\frac{2}{3}$ (3) $\log\frac{3}{2}$ (4) $\log 2d$ (5) $\log 3d$

（111 年學測數學 A 單選題第 4 題）

Teacher: Assume $a_3 = a_1 + 2d$ and $a_6 = a_1 + 5d$.

Also, $\log a_6 - \log a_3 = \log a_3 - \log a_1$ because $\log a_1$, $\log a_3$, $\log a_6$ are also arithmetic.

$$\log\frac{a_6}{a_3} = \log\frac{a_3}{a_1} \to \frac{a_6}{a_3} = \frac{a_3}{a_1} \to a_1(a_1 + 5d) = (a_1 + 2d)^2$$

What is the relationship between a_1 and d?

Student: $a_1 = 4d$.

Teacher: Very good. Then the sequence $\log a_1$, $\log a_3$, $\log a_6$ can be rewritten as $\log 4d$, $\log 6d$, $\log 9d$. Then, what is the common difference of the sequence $\log a_1$, $\log a_3$, $\log a_6$?

Student: The common difference is $\log 3 - \log 2$.

Teacher: Great. So, the answer is (3) because $\log 6\mathrm{d} - \log 4\mathrm{d} = \log\frac{3}{2}$.

老師： 假設 $a_3 = a_1 + 2d$ 且 $a_6 = a_1 + 5d$。因為 $\log a_1$、$\log a_3$、$\log a_6$ 也成等差，所以 $\log a_6 - \log a_3 = \log a_3 - \log a_1$。

$$\log\frac{a_6}{a_3} = \log\frac{a_3}{a_1} \rightarrow \frac{a_6}{a_3} = \frac{a_3}{a_1} \rightarrow a_1(a_1 + 5d) = (a_1 + 2d)^2$$

a_1 和 d 之間的關係是什麼？

學生： $a_1 = 4d$。

老師： 非常好。然後數列 $\log a_1$、$\log a_3$、$\log a_6$ 可以重新寫為 $\log 4d$、$\log 6d$、$\log 9d$。那麼，數列 $\log a_1$、$\log a_3$、$\log a_6$ 的公差是多少？

學生： 公差是 $\log 3 - \log 2$。

老師： 很好。$\log 6\mathrm{d} - \log 4\mathrm{d} = \log\frac{3}{2}$ 所以答案是 (3)。

單元六　平面向量的運算
Vector Operations in the Plane

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■　前言 Introduction

本單元以英文介紹向量定義及表示法，向量的係數積、加法、減法等運算及其在坐標平面上的幾何意義，最後介紹向量的線性組合，以例子引導出分點公式。

■　詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
directed line segment	有向線段	initial point	起點
terminal point	終點	magnitude	量
vector	向量	zero vector	零向量
x component	x 分量	y component	y 分量
section formula	分點公式	resultant vector	結果向量
parallelogram law	平行四邊形法		

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ **_____ be denoted by _______**

例句：The vector can **be denoted by** $\vec{u}$.

向量可以表示成 $\vec{u}$。

❷ **_____ is increased by a factor of _____.**

例句：The vector **is increased by a factor of** 2.

這個向量變成 2 倍長。

❸ **_____ is twice/ triple ______ of _____.**

例句：The new vector $3\vec{u}$ **is triple** the magnitude **of** the original vector $\vec{u}$.

新向量 $3\vec{u}$ 的長是原本向量 $\vec{u}$ 長的三倍。

❹ **______ divides ____ in the ratio of ______.**

例句：The point P **divides** $\overline{AB}$ **in the ratio of** 2:3

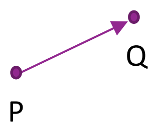
點 P 把 $\overline{AB}$ 分為兩線段，線段長的比為 2:3.

■ 問題講解 Explanation of Problems

☙ 說明 ❧

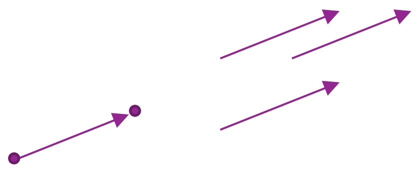
[Definitions of vectors]

In physics, a force can cause the motion of an object and change its velocity. For example, the forces that enable you to push a ball or pull a door, the gravitational force that pulls objects toward each other, and the frictional force that resists motion. A force has magnitude and direction, just as the velocity of an object also has its magnitude and direction. In this section, we are going to introduce such quantities with magnitude and direction. In mathematics, we use directed line segments to represent such quantities. In the following graph:

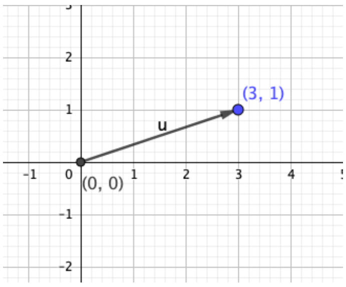


$\overrightarrow{PQ}$ is a directed line segment, while P is the initial point and Q is the terminal point. The arrowhead shows the direction, from P to Q, from the tail to the head. The magnitude is the size, referring to the length of the directed line segment, and can be denoted by $|\overrightarrow{PQ}|$. This absolute value sign represents the magnitude of $|\overrightarrow{PQ}|$.

I draw several directed line segments here, with the same size as $\overrightarrow{PQ}$. Their directions are the same. Since these directed line segments have the same magnitude and the same direction as $\overrightarrow{PQ}$, they are all equivalent. The set of all these equivalent directed line segments can be denoted by a vector $\vec{u}$. We use the directed line segment to represent "vector", so $\vec{u} = \overrightarrow{PQ}$. The direction and the magnitude of the directed line segment $\overrightarrow{PQ}$ are the direction and the magnitude of the vector $\vec{u}$. A vector can be denoted by a single lowercase letter, such as $\vec{u}$, $\vec{v}$, etc.

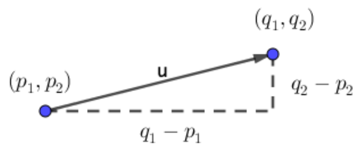


There is a directed line segment in this g figure. The initial point is the origin $(0, 0)$, and the terminal point is $(3, 1)$. From the tail $(0, 0)$ to the head $(3, 1)$, the horizontal change is 3, and the vertical change is 1. We say that $\vec{u} = (3, 1)$. 3 is the change in x, and 1 is the change in y. The vector $\vec{u}$ can be directly represented by the coordinates of the terminal point. $\vec{u} = (3, 1)$ looks like the coordinates on the coordinate plane, but actually they are the changes in x and y. If the vector $\vec{u}$ has the initial point at the origin $(0, 0)$, and the terminal point (u_1, u_2), then $\vec{u} = (u_1, u_2)$. It is the component form of a vector $\vec{u}$. The first number is the x component of $\vec{u}$, which is the change in x. The second number is the y component of $\vec{u}$, which is the change in y.



The directed line segment whose initial point is the origin is convenient to represent a set of the equivalent directed line segments. This particular directed line segment is in the standard position.

What if the initial point doesn't start at the origin?



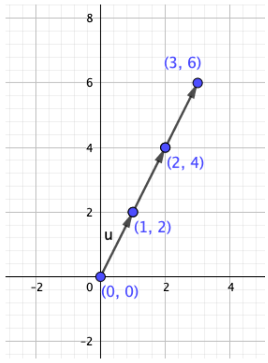
In the left figure, the initial point is $P(p_1, p_2)$ and the terminal point is $Q(q_1, q_2)$. We can construct a right triangle to find out the horizontal change and the vertical change. The horizontal change is $q_1 - p_1$, the x value of the terminal point minus the x value of the initial point. The vertical change is $q_2 - p_2$, the y value of the terminal point minus the y value of the initial point.

The component form of the vector is given by $\overrightarrow{PQ} = (q_1 - p_1, q_2 - p_2) = (u_1, u_2) = \vec{u}$. If the initial point is the same as the terminal point, such as $\overrightarrow{PP}$, then there's no horizontal or vertical change. $\overrightarrow{PP}$ is a zero vector, denoted by $\vec{0} = (0, 0)$.

Please note that the component form represents "a family of vectors". The vector $(3, 1)$ can be considered a vector with the initial point $(0, 0)$ and the terminal point $(3, 1)$ as well as a vector with the initial point $(1, 1)$ and the terminal point $(4,2)$, and other equivalent vectors. The vector is not defined byan exact position, you can shift the vector around and it is still the same vector.

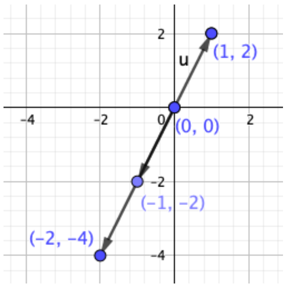
The magnitude of the vector, which is the length of the vector, can be figured out by the distance formula. The distance formula comes from the Pythagorean theorem. Two legs are $q_1 - p_1$ and $q_2 - p_2$. The length of hypotenuse is $|\overrightarrow{PQ}| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2} = \sqrt{{u_1}^2 + {u_2}^2} = |\vec{u}|$. If $|\vec{u}| = 1$, we call $\vec{u}$ the unit vector. If $|\vec{u}| = 0$, we call $\vec{u}$ the zero vector.

[Scalar multiplication: Scale a vector]



Scalar multiplication is to scale a vector: change its magnitude, or flip its direction by $180°$.

Let's draw a vector $\vec{u}=(1,2)$ in the standard position. Put the initial point at the origin, and the terminal point at $(1,2)$. Multiply the vector by 3, and 3 is the scalar. We multiply each component by 3, and we will have $3\vec{u}=(3,6)$. Draw $(3,6)$ in the standard position. The initial point at the origin, and the terminal point at $(3,6)$. We can tell that the direction doesn't change, but the magnitude changes. It is increased by a factor of 3.



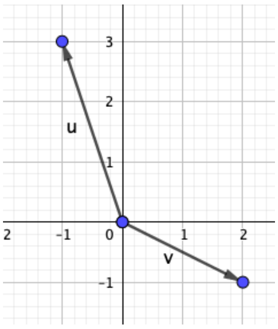
Let's try another scalar multiplication. Multiply the vector by -2. We multiply each component by -2, and we will have $-2\vec{u}=(-2\,,-4)$. Draw the vector in the standard position and we can observe the relative position between $\vec{u}$ and $-2\vec{u}$. We can tell the direction has been flipped by $180°$, and it is in the opposite direction. They still sit on the same line. The magnitude of the new vector is increased by a factor of 2. This is twice the magnitude of the original vector. Therefore, please note that the negative scalar will flip the direction by $180°$

[Vector Operations: addition and subtraction]

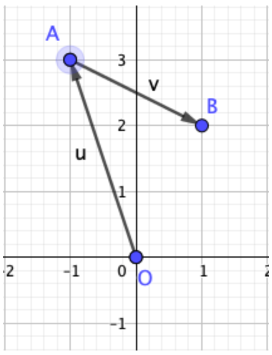
Assume that $\vec{u}=(-1,3)$ and $\vec{v}=(2,-1)$, what would $\vec{u}+\vec{v}$ be? To take the sum of two vectors, we add up their x components to get the new x components: $-1+2=1$, as well as their y components to get the new y components: $3+(-1)=2$, and get the resulting vector $(1,2)$. It would not be a problem for you to do the simple calculation. However, the geometric meaning of adding vectors is more interesting. Let's draw $\vec{u}=(-1,3)$ and $\vec{v}=(2,-1)$ in the standard position, see figure (1). As we add $\vec{v}$ and $\vec{u}$, we have to shift vector $\vec{v}$ over, so that its initial point starts at vector $\vec{u}$'s terminal point, see figure (2). The resulting vector is going from the initial point of $\vec{u}$ to the terminal point of $\vec{v}$. The new vector, $\overrightarrow{OB}$ in figure (3), is called the resultant of vector addition. It has horizontal component 1 and vertical component 2, and the vector is $(1,2)$.

Follow the direction of arrowheads, the addition can be expressed as $\overrightarrow{OA}+\overrightarrow{AB}=\overrightarrow{OB}$. Point A is the connecting point.

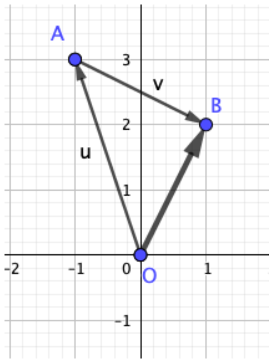
(1)



(2)



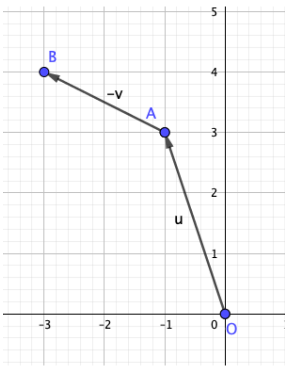
(3)



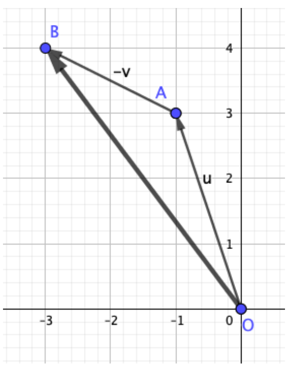
Assume that $\vec{u} = (-1, 3)$ and $\vec{v} = (2, -1)$, what would $\vec{u} - \vec{v}$ be? The expression $\vec{u} - \vec{v}$ can written as $\vec{u} + (-\vec{v})$. The vector $-\vec{v}$ has the same magnitude as $\vec{v}$, but in the opposite direction. $-\vec{v} = (-2, 1)$. Instead of 2 to the right and 1 down, we go 2 to the left and 1 up. To add $-\vec{v}$ to $\vec{u}$, we shift vector $-\vec{v}$ over so that its initial point starts at vector $\vec{u}$'s terminal point, see figure (4). We add the x components and y components correspondingly, and we will have

$\vec{u} + (-\vec{v}) = (-1, 3) + (-2, 1) = (-3, 4)$. In figure (5), the resultant vector $\overrightarrow{OB} = (-3, 4)$. The change in x is 3 units to the left, and the change in y is 4 units up.

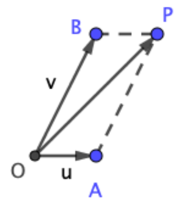
(4)



(5)



[Linear combination of vectors]

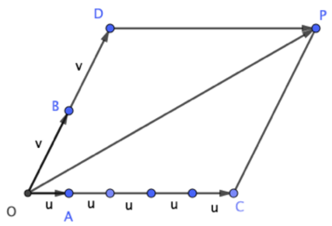


Given $\overrightarrow{OA}$ and $\overrightarrow{OB}$, how to construct $\overrightarrow{OA}+\overrightarrow{OB}$? We can shift $\overrightarrow{OB}$ around to connect to the terminal point of $\overrightarrow{OA}$, like we did above. However, here's another method, which is called "Parallelogram Law". We construct a parallelogram with $\overrightarrow{OA}$ and $\overrightarrow{OB}$ as the two sides.

In the parallelogram, the opposite sides are parallel and congruent, therefore the vectors are equivalent: $\overrightarrow{OB}=\overrightarrow{AP}$, $\overrightarrow{OA}=\overrightarrow{BP}$. Each pair of vectors has the same magnitude and the same direction. Following the direction of arrowheads, $\overrightarrow{OP}=\overrightarrow{OA}+\overrightarrow{AP}$.

Replacing $\overrightarrow{AP}$ with $\overrightarrow{OB}$, and we have $\overrightarrow{OP}=\overrightarrow{OA}+\overrightarrow{AP}=\overrightarrow{OA}+\overrightarrow{OB}$. The diagonal vector from point O is $\overrightarrow{OP}$, which is equivalent to $\overrightarrow{OA}+\overrightarrow{OB}$.

Let's see another example. In the parallelogram below, $\overline{OC}=5\overline{OA}$. These segments have the same direction (their arrowheads all point at the same direction), so $\overrightarrow{OC}=5\overrightarrow{OA}$. Similarly, $\overrightarrow{OD}=2\overrightarrow{OB}$. With some substitution, we would get $\overrightarrow{OP}=\overrightarrow{OC}+\overrightarrow{CP}=\overrightarrow{OC}+\overrightarrow{OD}=5\overrightarrow{OA}+2\overrightarrow{OB}$.



$\overrightarrow{OP}=\overrightarrow{OA}+\overrightarrow{OB}$ and $\overrightarrow{OP}=5\overrightarrow{OA}+2\overrightarrow{OB}$. show that $\overrightarrow{OP}$ is expressed in a linear function of $\overrightarrow{OA}$ and $\overrightarrow{OB}$, and it is called the linear combination. A linear combination of vectors is a result combining the vector using scalar multiplications. It can combine more than two vectors, but in the textbook, we only deal with the combination of two vectors.

It is denoted by $\overrightarrow{OP}=a\overrightarrow{OA}+\mathrm{b}\overrightarrow{OB}$, while $\overrightarrow{OA}$, $\overrightarrow{OB}$ are non-zero vectors, unparalleled, coplanar vectors, and a, b are scalars.

運算問題的講解

例題一

說明：以坐標表示法表示向量，並求出向量的值。

（英文）Find the component form and magnitude of the vector $\vec{v}$ that has the initial point $(4,-7)$ and the terminal point $(0,5)$.

（中文）以坐標表示法表示起點為 $(4,-7)$ 及終點為 $(0,5)$ 的向量，並求其值。

Teacher: You can plot the points on the coordinate plane to see their relative positions. The change in x is from 4 to 0. What is the horizontal change?

Student: $0-4=-4$

Teacher: Ok. -4 is the horizontal component, or we can say x component. What about the vertical change? From which number to which number?

Student: From -7 to 5. The vertical change is $5-(-7)=12$

Teacher: You have found out the x component and the y component. You can combine them into a vector.

Student: $(-4,12)$.

Teacher: Correct. The magnitude of the vector is the length. You can apply the Pythagorean theorem to get it.

Student: $\sqrt{(-4)^2+12^2}=\sqrt{160}=4\sqrt{10}$. It is the simplest form.

Teacher: Good.

老師： 在坐標平面上畫出這些點，觀察它們的相對位置。x 的變化從 4 變到 0，水平變化是多少？

學生： $0-4=-4$

老師： 好的。-4 是水平分量，或者我們可以說是 x 分量。垂直變化呢？從哪個數到哪個數？

學生： 從 -7 到 5。垂直變化是 $5-(-7)=12$。

老師： 你已經找到了 x 分量和 y 分量，接著可以將它們結合成一個向量。

學生： $(-4,12)$。

老師： 沒錯。向量的大小就是長度，再應用畢氏定理求出答案。

學生： $\sqrt{(-4)^2+12^2}=\sqrt{160}=4\sqrt{10}$，是最簡的形式了。

老師： 很好。

例題二

說明：以坐標表示法表示向量加法，並在坐標平面上畫出向量和。

（英文）Assume that $\vec{u}=(4,2)$ and $\vec{v}=(0,-1)$, find $\vec{u}+3\vec{v}$ and sketch the resulting vector.

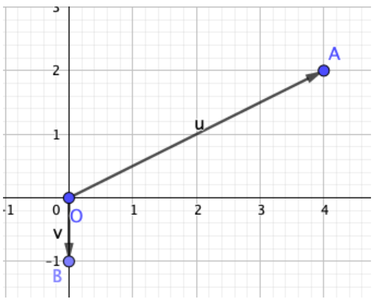
（中文）設 $\vec{u}=(4,2)$，$\vec{v}=(0,-1)$，表示下列向量 $\vec{u}+3\vec{v}$ 並畫圖。

Teacher: First please draw these two vectors in the standard position. Put the initial points at the origin.

Student: How do I draw $\vec{v}=(0,-1)$? The x component is 0.

Teacher: The x component is 0, which means that there's no horizontal change. The y component is -1, which means that 1 unit down. Start from the origin, how do you plot 1 unit down?

Student:



Like this?

Teacher: Correct. The vector $\vec{v}$ is vertically downward. You can use simple math to find out the answer. What is $3\vec{v}$?

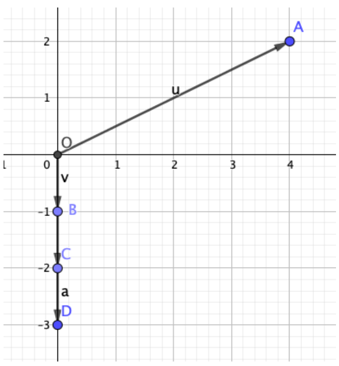
Student: $3\vec{v}=\big(3\times 0, 3\times(-1)\big)=(0,-3)$.

Teacher: What is the $\vec{u}+3\vec{v}$?

Student: $(4,2)+(0,-3)=(4,-1)$

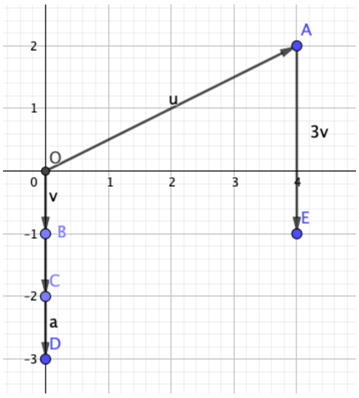
Teacher: Correct. Let's sketch the vectors. $3\vec{v}$ is triple the magnitude of $\vec{v}$. You can extend the vector $\vec{v}$ and triple the length.

Student:



like this?

Teacher: Yes. Now add $3\vec{v}$ to $\vec{u}$. You shift the vector $3\vec{v}$, so that its initial point meets the terminal point of $\vec{u}$. Please show me your graph.

Student: 

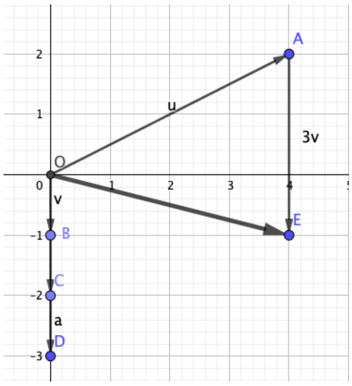
Like this?

Teacher: Yes. How do you move over the vector?

Student: It is like a parallelogram. I draw a line parallel to $\overline{OD}$, and measure the length of $3\vec{v}$. $\overline{AE} = \overline{OD}$

Teacher: What is the result?

Student: The result is the $\overrightarrow{OE} = (4, -1)$. Here is my graph:



Teacher: How do you know you are correct?

Student: In vector $(4, -1)$, the change in x is 4 to the right, and the change in y is 1 down. It matches the direction of $\overrightarrow{OE}$ in the graph.

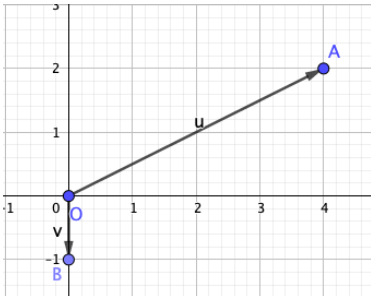
Teacher: It's always good to double check the answer with a graph.

老師： 首先，請在標準位置繪製這兩個向量。將起始點放在原點。

學生： 我要如何畫 $\vec{v} = (0, -1)$？x 分量是 0。

老師： x 分量為 0，表示水平方向沒有變化。y 分量為 -1，表示向下 1 單位。從原點開始，要怎麼向下畫 1 單位？

學生： 像這樣嗎？



老師：　正確。向量 $\vec{v}$ 是垂直向下的，你可以使用簡單的數學計算出答案。$3\vec{v}$ 是多少呢？

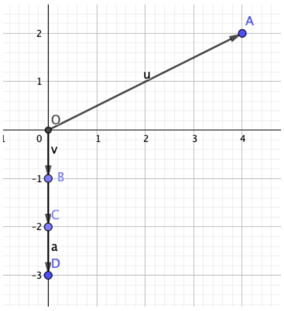
學生：　$3\vec{v} = \big(3 \times 0, 3 \times (-1)\big) = (0, -3)$。

老師：　$\vec{u} + 3\vec{v}$ 等於？

學生：　$(4, 2) + (0, -3) = (4, -1)$

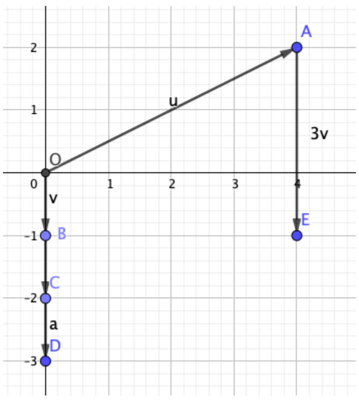
老師：　正確。我們一起來畫出這些向量。$3\vec{v}$ 的大小是 $\vec{v}$ 的三倍。你可以延伸向量 $\vec{v}$ 並使其長度變為三倍。

學生：　像這樣嗎？



老師：　是的，現在將 $3\vec{v}$ 加到 $\vec{v}$ 上。你移動向量 $3\vec{v}$，使其起始點與 $\vec{v}$ 的終點重合。讓我看看你們畫的圖。

學生：　像這樣嗎？



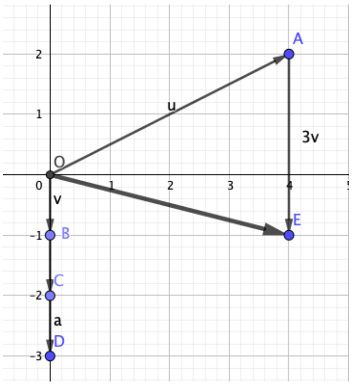
老師：　沒錯。你如何移動這個向量？

學生：　這就像一個平行四邊形。我畫了一條平行於 $\overline{OD}$ 的線，然後測量了$3\vec{v}$ 的長度。

$\overline{AE} = \overline{OD}$。

老師：　結果是多少？

學生：　結果是 $\overrightarrow{OE} = (4, -1)$。這是我的圖：



老師：　你怎麼知道你是正確的？

學生：　在向量 (4, −1) 中，x 的變化是向右 4 單位，y 的變化是向下 1 單位。這與圖中的 $\overrightarrow{OE}$ 方向一致。

老師：　利用圖形檢查答案通常是件好事。

例題三

說明：平面向量的線性組合。

（英文）Show that $\vec{c}=(5,4)$ is a linear combination of $\vec{u}=(4,2)$ and $\vec{v}=(0,-1)$.

（中文）將 $\vec{c}=(5,4)$表示成向量 $\vec{u}=(4,2)$和 $\vec{v}=(0,-1)$ 的線性組合

Teacher: $\vec{c}$ can be expressed as a linear combination of $\vec{u}$ and $\vec{v}$, since $\vec{u}$ and $\vec{v}$ are not parallel. We can assume the scalars as x and y. How to write $\vec{c}$ as a linear function of $\vec{u}$ and $\vec{v}$?

Student: $\vec{c}=x\vec{u}+y\vec{v}$.

Teacher: Correct. So we have $\vec{c}=x(4,2)+y(0,-1)$.
Please apply the scalar multiplication and addition.

Student: $\vec{c}=x(4,2)+y(0,-1)=(4x+0,2x-y)$. Is it right?

Teacher: Yes, with the scalar multiplication, we have
$\vec{c}=x(4,2)+y(0,-1)=(4x,2x)+(0,-y)$. Then apply the addition, add the x components: $4x+0$, and add y components: $2x-y$.
Next, $\vec{c}=(4x,\ 2x-y)=(5,4)$. By comparing the components, we get two equations $4x=5$ and $2x-y=4$. It is a system of linear equations written as
$\begin{cases}4x=5\\2x-y=4\end{cases}$. Please solve for x and y.

Student: $x=1.25$ and $y=-1.5$

Teacher: $\vec{c}=1.25\vec{u}-1.5\vec{v}$

老師： 因為 $\vec{u}$ 和 $\vec{v}$ 不平行，$\vec{c}$ 可以表示為 $\vec{u}$ 和 $\vec{v}$ 的線性組合。我們可以假設純量為 x 和 y。如何將 $\vec{c}$ 寫成 $\vec{u}$ 和 $\vec{v}$ 的線性函數呢？

學生： $\vec{c}=x\vec{u}+y\vec{v}$。

老師： 正確。我們得到 $\vec{c}=x(4,2)+y(0,-1)$。請應用純量乘法和加法。

學生： $\vec{c}=x(4,2)+y(0,-1)=(4x+0,2x-y)$ 是這樣嗎？

老師：　沒錯，通過純量乘法，得到 $\vec{c} = x(4,2) + y(0,-1) = (4x, 2x) + (0, -y)$。
然後應用加法，將 x 分量相加：$4x + 0$，y 分量相加：$2x - y$。
接下來，$\vec{c} = (4x, 2x - y) = (5,4)$。通過比較分量，我們得到兩個方程式
$4x = 5$ 和 $2x - y = 4$。寫成線性方程組 $\begin{cases} 4x = 5 \\ 2x - y = 4 \end{cases}$
請解出 x 和 y。

學生：　$x = 1.25$ 和 $y = -1.5$。

老師：　答對了，$\vec{c} = 1.25\vec{u} - 1.5\vec{v}$。

應用問題 / 學測指考題

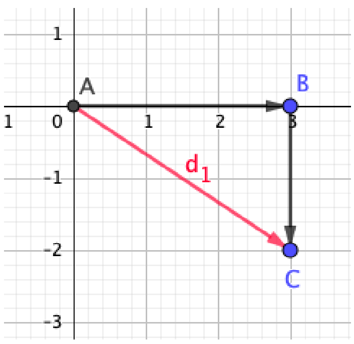
例題一

說明：利用向量加法解決日常問題。

（英文）A hiker left the camp with a compass. He walked 3 km E and 2 km S on the first day, 2 km E and 5 km N on the second day, 4 km W and 4 km S on the third day. At the end of the third day, the hiker got lost. With the method of vectors, please calculate how far the hiker is from the camp.

（中文）一個健行者帶著指南針離開營地。第一天他向東走了 3 公里，向南走了 2 公里，第二天他向東走了 2 公里，向北走了 5 公里，第三天向西走了 4 公里，向南走了 4 公里。第三天結束後他迷路了。使用向量的方法，計算他離營地還有多遠的距離。

Teacher: On the first day, the hiker walked 3 km E and 2 km S. E is the East, and S is the south. We can plot his route on the plane. Let's plot the camp at the origin. The horizontal change is $+3$, and the vertical change is -2. We usually use "+" to represent the East and the North, and use "−" to represent the West and the South. The vector of day 1 can be denoted as $\overrightarrow{d_1} = (3, -2)$.



Can you find out the vector of day 2 and day 3?

Student: On day 2, the horizontal change is $+2$, and the vertical change is $+5$.

The vector of day 2 can be denoted as $\overrightarrow{d_2}=(2,5)$.

Student: On day 3, the vector is $\overrightarrow{d_3}=(-4,-4)$.

Teacher: What is the resultant vector of $\overrightarrow{d_1}+\overrightarrow{d_2}+\overrightarrow{d_3}$? What does it mean?

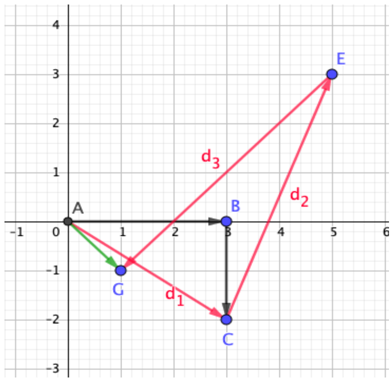
Student: The x component is 3+2−4=1, and the y component is $-2+5-4=-1$.

$\overrightarrow{d_1}+\overrightarrow{d_2}+\overrightarrow{d_3}=(1,-1)$. It means the total distance within the 3 days.

Teacher: No, it should be the displacement. The change in position is the displacement.

At the end of three days, his displacement is 1 km E and 1 km S since $\overrightarrow{d_1}+\overrightarrow{d_2}+\overrightarrow{d_3}=\overrightarrow{AG}=(1,-1)$.

"How far is the hiker from the camp?" is asking the magnitude of the vector.



Student: $|\overrightarrow{AG}|=\sqrt{1+1}=\sqrt{2}$.

老師： 第一天，這位健行者向東行走了 3 公里，然後向南行走了 2 公里。E 代表東，S 代表南。我們可以在平面上繪製他的路線。

首先將營地設在原點。水平變化是 $+3$，垂直變化是 -2。通常我們使用「+」來表示東和北，使用「−」來表示西和南。第一天的向量，表示為 $\overrightarrow{d_1}=(3,-2)$。

你能找出第 2 天和第 3 天的向量嗎？

學生： 第 2 天，水平變化是 $+2$，垂直變化是 $+5$。

第 2 天的向量，表示為 $\overrightarrow{d_2}=(2,5)$。

學生： 第 3 天，向量是 $\overrightarrow{d_3}=(-4,-4)$。

老師： $\overrightarrow{d_1}+\overrightarrow{d_2}+\overrightarrow{d_3}$ 的合成向量是什麼？這代表什麼意思呢？

學生： x 分量是 3+2−4=1，y 分量是$-2+5-4=-1$。

$\overrightarrow{d_1}+\overrightarrow{d_2}+\overrightarrow{d_3}=(1,-1)$，是三天內的總距離。

老師： 不，這是位移。位置的變化就是位移。三天結束時，他的位移是向東 1 公里，向南 1 公里。$\overrightarrow{d_1}+\overrightarrow{d_2}+\overrightarrow{d_3}=\overrightarrow{AG}=(1,-1)$。

學生： 遠足者距離營地有多遠？這是在問向量的大小。

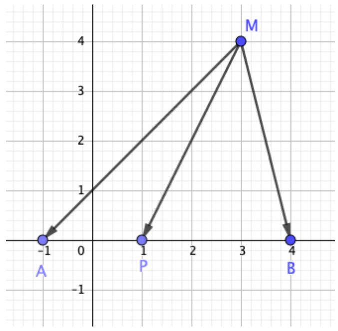
老師： $|\overrightarrow{AG}|=\sqrt{1+1}=\sqrt{2}$。

例題二

說明：利用向量的線性組合討論分點公式。

（英文）$M(3,4)$, $A(-1,0)$, $B(4,0)$ and $P(1,0)$ are four points on the coordinate plane. It is known that $\overrightarrow{MP} = x\overrightarrow{MA} + y\overrightarrow{MB}$, find the value of x and y.

（中文）$M(3,4)$、$A(-1,0)$、$B(4,0)$、$P(1,0)$為坐標平面上的四點，已知 $\overrightarrow{MP} = x\overrightarrow{MA} + y\overrightarrow{MB}$，求 x, y 之值。



Teacher: $\overrightarrow{MP}$ is a linear combination of $\overrightarrow{MA}$ and $\overrightarrow{MB}$. Since we know their coordinates, we can find out their vectors, substituted into the equations, and we will get a system of linear equations.
Please work on your own, and later we will discuss the answer.
(few minutes later) Would any of you like to explain your process?

Student: First, I find out the three vectors. $\overrightarrow{MP} = (-2,-4)$, $\overrightarrow{MA} = (-4,-4)$, $\overrightarrow{MB} = (1,-4)$. Then I substitute them into the equation $\overrightarrow{MP} = x\overrightarrow{MA} + y\overrightarrow{MB}$, and I have $(-2,-4) = x(-4,-4) + y(1,-4)$.

Teacher: Good. Please stop here. Does any of you want to follow up the process?

Student: I do multiplication and add up the vectors.
I have $-2 = -4x + y$, and $-4 = -4 - 4y$. Then I get $x = 0.6$ and $y = 0.4$.

Teacher: Great. Now I would like to show you another method. The initial point of $\overrightarrow{AP}$ starts at $\overrightarrow{MA}$'s terminal point, so I can write $\overrightarrow{MP} = \overrightarrow{MA} + \overrightarrow{AP}$. Please observe the vectors $\overrightarrow{AP}$ and $\overrightarrow{AB}$. How are they related?

Student: Their directions are the same, pointing east.

Student: The magnitude of $\overrightarrow{AP}$ is two-fifths of the magnitude of $\overrightarrow{AB}$.

Teacher: How do you know $\frac{2}{5}$?

Student: I counted it.

Teacher: That works. According to your inputs, I can tell that $\overrightarrow{AP}=\frac{2}{5}\overrightarrow{AB}$. I replace $\overrightarrow{AP}$ in the previous equation, and I have $\overrightarrow{MP}=\overrightarrow{MA}+\overrightarrow{AP}=\overrightarrow{MA}+\frac{2}{5}\overrightarrow{AB}$.

How are $\overrightarrow{AB}$, $\overrightarrow{MA}$ and $\overrightarrow{MB}$ related? Can you tell from the graph?

Student: They form a triangle.

Teacher: Ok, but I mean vectors. How are these vectors related?

Student: $\overrightarrow{MA}+\overrightarrow{AB}=\overrightarrow{MB}$.

Teacher: Great. I can say that $\overrightarrow{AB}=\overrightarrow{MB}-\overrightarrow{MA}$, by subtracting $\overrightarrow{MA}$ on both sides of equal sign.

Then, I replace $\overrightarrow{AB}$ in the previous equation, and I have

$$\overrightarrow{MP}=\overrightarrow{MA}+\overrightarrow{AP}=\overrightarrow{MA}+\frac{2}{5}\overrightarrow{AB}=\overrightarrow{MA}+\frac{2}{5}\left(\overrightarrow{MB}-\overrightarrow{MA}\right)=\overrightarrow{MA}+\frac{2}{5}\overrightarrow{MB}-\frac{2}{5}\overrightarrow{MA}$$

$$=\frac{3}{5}\overrightarrow{MA}+\frac{2}{5}\overrightarrow{MB}.$$

$\overrightarrow{MP}=\frac{3}{5}\overrightarrow{MA}+\frac{2}{5}\overrightarrow{MB}$ is the final answer! We had $x=0.6$ and $y=0.4$. Do they match?

Student: Yes.

Teacher: In the graph, the point P divides $\overrightarrow{AB}$ internally in the ratio of $2:3$. Point M can be generalized to any point on the plane. The position vector of P can be expressed by $\overrightarrow{MP}=\frac{3}{5}\overrightarrow{MA}+\frac{2}{5}\overrightarrow{MB}$. I would like to generalize this equation.

If the point P divides internally the segment joining A and B in the ratio of $x:y$, then the position vector of P is $\overrightarrow{MP}=\frac{y}{x+y}\overrightarrow{MA}+\frac{x}{x+y}\overrightarrow{MB}$. This is the Section Formula for internal division of vectors.

老師： $\overrightarrow{MP}$ 是 $\overrightarrow{MA}$ 和 $\overrightarrow{MB}$ 的線性組合。因為我們知道它們的坐標，所以我們可以找出它們的向量，代入方程式，然後我們會得到一個線性方程組。

請大家先自己算，稍後我們會討論答案。

（幾分鐘後）有人想解釋一下你的過程嗎？

學生： 首先，我找出了三個向量。$\overrightarrow{MP}=(-2,-4)$, $\overrightarrow{MA}=(-4,-4)$, $\overrightarrow{MB}=(1,-4)$。

然後代入方程 $\overrightarrow{MP}=x\overrightarrow{MA}+y\,\overrightarrow{MB}$，寫成 $(-2,-4)=x(-4,-4)+y(1,-4)$。

老師： 好的，請停在這裡。有其他人想繼續接下去嗎？

學生：接著使用乘法並相加這些向量，得到 $-2=-4x+y$，和 $-4=-4-4y$。解出 $x=0.6$ 和 $y=0.4$。

老師：太好了。現在我提供你們另一種方法。$\overrightarrow{AP}$ 的初始點在 $\overrightarrow{MA}$ 的終點，所以我可以寫成 $\overrightarrow{MP}=\overrightarrow{MA}+\overrightarrow{AP}$。請觀察向量 $\overrightarrow{AP}$ 和 $\overrightarrow{AB}$。它們有什麼關係？

學生：它們的方向相同，都指向東。

學生：$\overrightarrow{AP}$ 的大小是 $\overrightarrow{AB}$ 大小的 $\frac{2}{5}$。

老師：你怎麼知道是 $\frac{2}{5}$？

學生：我數過了。

老師：很好。根據你的算出的結果，我可以列出 $\overrightarrow{AP}=\frac{2}{5}\overrightarrow{AB}$。將 $\overrightarrow{AP}=\frac{2}{5}\overrightarrow{AB}$ 代入前一個方程式，得到 $\overrightarrow{MP}=\overrightarrow{MA}+\overrightarrow{AP}=\overrightarrow{MA}+\frac{2}{5}\overrightarrow{AB}$。

$\overrightarrow{AB}$、$\overrightarrow{MA}$ 和 $\overrightarrow{MB}$ 三者有什麼關係？你可以從圖中看出來嗎？

學生：它們形成一個三角形。

老師：是的，不過我的意思是指向量。這些向量之間有什麼關係？

學生：$\overrightarrow{MA}+\overrightarrow{AB}=\overrightarrow{MB}$。

老師：太好了。接著兩邊減去 $\overrightarrow{MA}$，得到 $\overrightarrow{AB}=\overrightarrow{MB}-\overrightarrow{MA}$，然後將 $\overrightarrow{AB}$ 代入前一個的方程式 $\overrightarrow{MP}=\overrightarrow{MA}+\overrightarrow{AP}=\overrightarrow{MA}+\frac{2}{5}\overrightarrow{AB}$

$=\overrightarrow{MA}+\frac{2}{5}(\overrightarrow{MB}-\overrightarrow{MA})=\overrightarrow{MA}+\frac{2}{5}\overrightarrow{MB}-\frac{2}{5}\overrightarrow{MA}=\frac{3}{5}\overrightarrow{MA}+\frac{2}{5}\overrightarrow{MB}$。

$\overrightarrow{MP}=\frac{3}{5}\overrightarrow{MA}+\frac{2}{5}\overrightarrow{MB}$ 就是最終答案！我們得到 $x=0.6$ 和 $y=0.4$。

它們相符嗎？

學生：是的。

老師：在圖中，P 點在 $\overrightarrow{AB}$ 內部以 $2:3$ 的比例分割。M 點可以推廣到平面上的任意點。P 的位置向量可以表示為 $\overrightarrow{MP}=\frac{3}{5}\overrightarrow{MA}+\frac{2}{5}\overrightarrow{MB}$。

我把這個方程式寫成通式。

老師：如果 P 點在連接 A 和 B 的線段上以比例 $x:y$ 內部分割，那麼 P 的位置向量就是 $\overrightarrow{MP}=\frac{y}{x+y}\overrightarrow{MA}+\frac{x}{x+y}\overrightarrow{MB}$。這是向量的內部分割的分段公式。

單元七　平面向量的內積
The Inner Product of Two Vectors

國立新竹科學園區實驗高級學校　吳珮蓁老師

■ 前言 Introduction

本單元介紹以英文講解向量夾角，從物理的『功』介紹向量分量，正射影及向量內積定義，說明正射影公式及向量內積的公式。介紹以坐標表示法求得向量內積的值，並以此判斷向量是否正交。並介紹平行向量的表示法，及柯西不等式等號成立時的條件。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
counterclockwise	逆時針	**vector component**	**向量分量**
inner (dot) product	**內積**	**orthogonal**	**正交**
projection	**正射影**	law of cosines	餘弦定理
Cauchy’s inequality	柯西不等式		

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ measure _____ counterclockwise/clockwise ______.

例句：An angle **measured counterclockwise** is positive.

逆時針測量的角度是正的。

❷ decompose _____ into ______.

例句：Please **decompose** the given vector $\vec{u}$ **into** the sum of two vector components.

請將這個向量 $\vec{u}$ 分解成兩個向量分量的和。

❸ ____ is the projection of ____ onto _____.

例句：$\overrightarrow{w_1}$ **is the projection of** $\vec{v}$ **onto** $\vec{u}$.

$\overrightarrow{w_1}$ 是 $\vec{v}$ 在 $\vec{u}$ 的正射影。

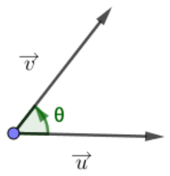
■ 問題講解 Explanation of Problems

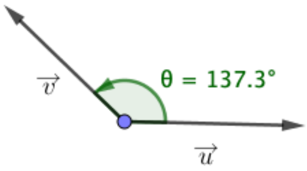
☙ 說明 ❧

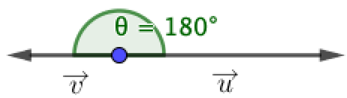
[The angle between two vectors]

Let's start with the angle between two vectors. There are two nonzero vectors $\vec{u}$ and $\vec{v}$, sharing the same initial points (as shown in the following figures). The angle between $\vec{u}$ and $\vec{v}$ is θ, $0 \le \theta \le \pi$. The angle is measured counterclockwise from $\vec{u}$ to $\vec{v}$, and the value of θ is positive. Not only in this section, we also measure the angle counterclockwise in trigonometry. The angle can be an acute angle, an obtuse angle, or even up to $180°$. If the angle between two vectors is $180°$, then the two vectors point in the opposite direction. Can you imagine what the vectors look like if the angle is $0°$? Overlapping? Parallel? Please think about it.

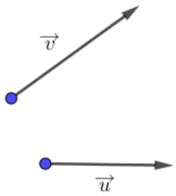
How do we measure the angle if they are parallel? They don't have joint initial points.

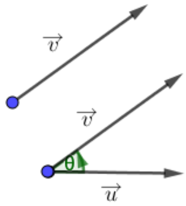






If the two vectors are not connected with each other (no joint initial points), you can shift one of them and make their initial points overlap. Now, the angle can be measured counterclockwise.

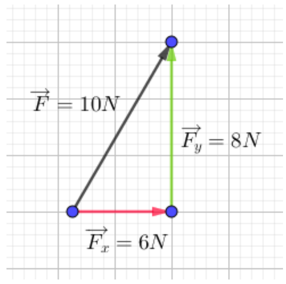




[Vector components and definition of projection]

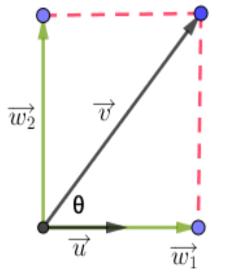
In physics, the constant force F has a magnitude and a direction. It is just like the "vector" we are learning now. Each force can be decomposed into two components- the component parallel and perpendicular to the displacement.

In the figure, a constant force of 10 Newton (denoted as 10N) has been applied on an object. The force can be decomposed into a horizontal component 6 N (parallel to the displacement) and a vertical component 8 N (perpendicular to the displacement). Do you see the right triangle? It is closely related to the "decomposed" vector.



The horizontal component describes the rightward influence of the force and the vertical component describes the upward influence of the force. The "work" done by the force is the product of "the component of the force parallel to the displacement" multiplied by "the displacement". If this force of 10N has moved the object 3 meters to the right, what is the "work" of the force? The object moves to the right, and this direction is parallel to the horizontal component $\overrightarrow{F_x}$. The "work" of the force is $6 \times 3 = 18$ joules (J). You will learn more applications in physics class.

We can decompose a vector into the sum of two vector components. Let $\vec{u}$ and $\vec{v}$ be nonzero



vectors. We can extend $\vec{u}$, and construct a rectangle with $\vec{v}$ as the diagonal. $\overrightarrow{w_1}$ and $\overrightarrow{w_2}$ are perpendicular, but we call them orthogonal especially referring to vectors. $\overrightarrow{w_1}$ overlaps $\vec{u}$, and is longer than $\vec{u}$. We can say that $\overrightarrow{w_1}$ is parallel to $\vec{u}$.

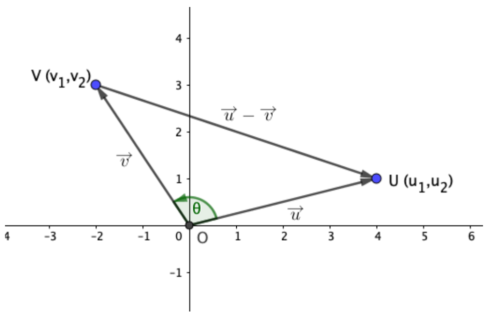
$\vec{v} = \overrightarrow{w_1} + \overrightarrow{w_2}$, where $\overrightarrow{w_1}$ and $\overrightarrow{w_2}$ are orthogonal. The vector $\overrightarrow{w_1}$ and $\overrightarrow{w_2}$ are called vector components of $\vec{v}$.

$\overrightarrow{w_1}$ is the projection of $\vec{v}$ onto $\vec{u}$. Pretend that $\vec{u}$ is on the ground, and the sun is directly above $\vec{v}$. The shadow of $\vec{v}$ will be on the ground, and is the projection of $\vec{v}$ onto $\vec{u}$. The angle between $\vec{u}$ and $\vec{v}$ is θ. The length of the projection of $\vec{v}$ onto $\vec{u}$ (the length of the shadow) is one leg of the right triangle, and its value is $|\vec{v}|cos\theta$. The product $|\vec{v}|cos\theta \cdot |\vec{u}| = |\vec{v}||\vec{u}|cos\theta$ is defined as the inner product of two vectors $\vec{u}$ and $\vec{v}$. We sometimes use "dot product".
We write "·" for "inner (dot) product," and it is a new operation: $\vec{u} \cdot \vec{v} = |\vec{v}||\vec{u}|cos\theta$.

Is it familiar with the work of the force? The "work" of the force is $6 \times 3 = 18$ joules (J). Please compare this equation to the formula $|\vec{v}|cos\theta \cdot |\vec{u}|$. You could see that $|\vec{v}|cos\theta$ is the horizontal component of the force, and $|\vec{u}|$ is equivalent to the "displacement" of the object. The "work" is the result of the inner (dot) product. Both values of $|\vec{v}|cos\theta$ and $|\vec{u}|$ are real numbers, not vectors. The product of two real numbers is also a real number. Therefore, the inner (dot) product of two vectors is denoted by $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|cos\theta$, which is a real number.

[Inner (Dot) product in terms of coordinates]

Plot $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$ at the origin, and the angle between these two vectors is θ.
We have point $U\ (u_1, u_2)$ and $V\ (v_1, v_2)$.
Since $\overrightarrow{OV} + \overrightarrow{VU} = \overrightarrow{OU}$, $\overrightarrow{VU} = \overrightarrow{OU} - \overrightarrow{OV} = \vec{u} - \vec{v}$. According to the law of cosines,



$$\overline{VU}^2 = \overline{OU}^2 + \overline{OV}^2 - 2\,\overline{OU} \cdot \overline{OV} \cdot cos\theta$$

The segment squared is equivalent to the square of the vector's magnitude.

Therefore, we can have equation

$|\vec{u}-\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}|\cdot|\vec{v}|\cdot cos\theta$.

$\vec{u}-\vec{v} = (u_1-v_1, u_2-v_2)$, its magnitude is $|\vec{u}-\vec{v}| = \sqrt{(u_1-v_1)^2+(u_2-v_2)^2}$.

$|\vec{u}| = \sqrt{(u_1)^2+(u_2)^2}$, $|\vec{v}| = \sqrt{(v_1)^2+(v_2)^2}$.

Meanwhile, the definition of the inner (dot) product tells that $\vec{u}\cdot\vec{v} = |\vec{u}|\cdot|\vec{v}|\cdot cos\theta$.

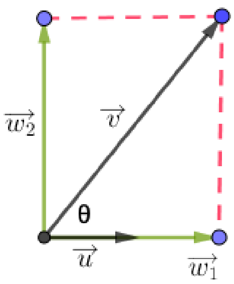
Let's substitute everything into the above equation. We will have

$$(u_1-v_1)^2+(u_2-v_2)^2 = (u_1)^2+(u_2)^2+(v_1)^2+(v_2)^2-2(\vec{u}\cdot\vec{v})$$

Simplify the equation and we will get $2u_1v_1+2u_2v_2 = 2(\vec{u}\cdot\vec{v})$. Dividing each side by 2, and we will have $\vec{u}\cdot\vec{v} = u_1v_1+u_2v_2$.

Therefore, the inner (dot) product of $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$ is given by $\vec{u}\cdot\vec{v} = u_1v_1+u_2v_2$. That's how we write the inner (dot) product in terms of "coordinates".

[Projection]



We mention the projection of $\vec{v}$ onto $\vec{u}$, and its length $|\vec{v}|cos\theta$. Now we would like to know its vector, $\vec{w_1}$.

Because $\vec{w_1}$ and $\vec{u}$ are parallel, $\vec{w_1}$ is a scalar multiple of $\vec{u}$. We can assume $\vec{w_1} = c\vec{u}$, c is the scalar.

$\vec{v} = \vec{w_1}+\vec{w_2}$, and we replace $\vec{w_1}$. We have $\vec{v} = c\vec{u}+\vec{w_2}$

Then, take the inner (dot) product of each side with $\vec{u}$.

We have $\vec{v}\cdot\vec{u} = (c\vec{u}+\vec{w_2})\cdot\vec{u} = c\vec{u}\cdot\vec{u}+\vec{w_2}\cdot\vec{u}$

Because the angle between $\vec{w_2}$ and $\vec{u}$ is $90°$, $\vec{w_2}\cdot\vec{u} = |\vec{w_2}||\vec{u}|cos90° = 0$

Therefore, $\vec{v}\cdot\vec{u} = c|\vec{u}|^2$. We isolate c, and have $c = \frac{\vec{v}\cdot\vec{u}}{|\vec{u}|^2}$.

The projection of $\vec{v}$ onto $\vec{u}$ is $\vec{w_1} = c\vec{u} = (\frac{\vec{v}\cdot\vec{u}}{|\vec{u}|^2})\vec{u}$.

[Parallel vectors and orthogonal vectors]

If vectors are in the same direction, but vary in magnitude (different sizes), then these vectors are parallel. If $\vec{u}$ and $\vec{v}$ are parallel, then we write $\vec{u} \parallel \vec{v}$. Since one vector is a multiple of the other, we can say that $\vec{u} = t\vec{v}$, while t is a non-zero real number.

Let's assume $\vec{u} = (x_1, y_1)$, $\vec{v} = (x_2, y_2)$. $(x_1, y_1) = t(x_2, y_2)$, then we have

$$\begin{cases} x_1 = tx_2 \\ y_1 = ty_2 \end{cases}$$

The ratio of $\frac{x_1}{y_1}$ can be expressed as $\frac{tx_2}{ty_2} = \frac{x_2}{y_2}$. We have $\frac{x_1}{y_1} = \frac{x_2}{y_2}$, or we rewrite the equation to $\frac{x_1}{x_2} = \frac{y_1}{y_2}$. We can conclude that if two vectors are parallel, the ratio of their x components is equivalent to the ratio of their y components. We can use this conclusion $\frac{x_1}{x_2} = \frac{y_1}{y_2}$ to examine whether the vectors are parallel or not.

For orthogonal vectors, we can use the inner (dot) product to examine whether two vectors are orthogonal or not. If the angle between $\vec{u} = (x_1, y_1)$ and $\vec{v} = (x_2, y_2)$ is $90°$, the inner (dot) product $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|cos90° = 0$. We can conclude that if the inner(dot) product of two vectors is 0, then these two vectors are orthogonal.

[Cauchy's Inequality]

Let $\vec{a}$, $\vec{b}$ be two non-zero vectors. Since $-1 \le cos\theta \le 1$, we multiply each term by $|\vec{a}||\vec{b}|$.

We have $-|\vec{a}||\vec{b}| \le |\vec{a}||\vec{b}|cos\theta \le |\vec{a}||\vec{b}|$

We replace $|\vec{a}||\vec{b}|cos\theta$ by $\vec{a} \cdot \vec{b}$, and we have $-|\vec{a}||\vec{b}| \le \vec{a} \cdot \vec{b} \le |\vec{a}||\vec{b}|$.

This inequality $|\vec{a} \cdot \vec{b}| \le |\vec{a}||\vec{b}|$ is called Cauchy's inequality. When does the equal sign happen?

Case (1) if $\vec{a} = 0$ or $\vec{b} = 0$: the Cauchy's inequality becomes $0 = 0$.

Case (2) if $\vec{a} \parallel \vec{b}$: There exists a constant t, such that $\vec{a} = t\vec{b}$, $t \ne 0$

The left-hand side is $|\vec{a} \cdot \vec{b}| = |t\vec{b} \cdot \vec{b}|$, and the right-hand side is $|\vec{a}||\vec{b}| = |t\vec{b}||\vec{b}|$. They are equivalent.

Therefore, Cauchy's inequality with equality happened if and only if

(1) $\vec{a}=0$ or $\vec{b}=0$ or (2) $\vec{a}\parallel\vec{b}$

Let's assume $\vec{a}=(x_1,y_1)$, $\vec{b}=(x_2,y_2)$,

$\vec{a}\cdot\vec{b}=x_1x_2+y_1y_2$, $|\vec{a}|=\sqrt{{x_1}^2+{y_1}^2}$, $|\vec{b}|=\sqrt{{x_2}^2+{y_2}^2}$

Then applying this to the Cauchy's inequality.

$$|x_1x_2+y_1y_2|\le\sqrt{{x_1}^2+{y_1}^2}\times\sqrt{{x_2}^2+{y_2}^2}$$

Square both sides, and we have $(x_1x_2+y_1y_2)^2\le({x_1}^2+{y_1}^2)({x_2}^2+{y_2}^2)$.

When does the equal sign work?

Case (1) if $\vec{a}=(x_1,y_1)=(0,0)$ or $\vec{b}=(x_2,y_2)=(0,0)$. Try to substitute in the inequality, and see whether the equal sign happened.

Case (2) if $\vec{a}\parallel\vec{b}$: there exists a constant t, such that $\vec{a}=t\vec{b}$, $t\neq0$.

$(x_1,y_1)=t(x_2,y_2)$. Compare the components and we have $\begin{cases}x_1=tx_2\\y_1=ty_2\end{cases}$

The ratio $\frac{x_1}{y_1}=\frac{tx_2}{ty_2}=\frac{x_2}{y_2}$, $\frac{x_1}{y_1}=\frac{x_2}{y_2}$, therefore $x_1y_2=x_2y_1$.

This is when Cauchy's inequality with equality happened.

運算問題的講解

例題一

說明：練習找出向量的夾角，並利用公式求出內積

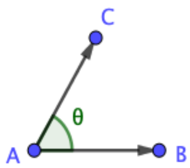
（英文）$\triangle ABC$ is an equilateral triangle with side 10cm. Find the dot product of (1) $\overrightarrow{AB}\cdot\overrightarrow{AC}$ and (2) $\overrightarrow{AB}\cdot\overrightarrow{BC}$.

（中文）已知$\triangle ABC$ 為邊長 10 公分的正三角形，求內積之值 (1) $\overrightarrow{AB}\cdot\overrightarrow{AC}$ (2) $\overrightarrow{AB}\cdot\overrightarrow{BC}$。

Teacher: $\triangle ABC$ is an equilateral triangle, what is the measure of each angle?

Student: 60°.

Teacher:



Let me plot $\overrightarrow{AB}$, $\overrightarrow{AC}$ and its angle.
The inner (dot) product $\overrightarrow{AB}\cdot\overrightarrow{AC}$ is $\overrightarrow{AB}\cdot\overrightarrow{AC}=|\overrightarrow{AB}||\overrightarrow{AC}|cos\theta$.
What is $|\overrightarrow{AC}|$? What is $|\overrightarrow{AB}|$?

Student: Are they 10?

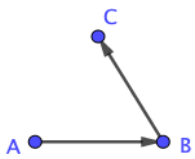
Teacher: Yes. $|\overrightarrow{AC}|$ is the magnitude, and it is the length of the side. Please find the dot product.

Student: $|\overrightarrow{AB}||\overrightarrow{AC}|cos\theta=10\cdot10\cdot cos60°=10\cdot10\cdot\frac{1}{2}=50.$

Teacher: Next, what is the angle between $\overrightarrow{AB}$ and $\overrightarrow{BC}$?

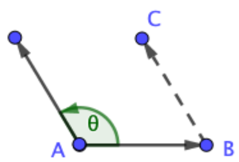
Student: 60°.

Teacher: Are the initial points of these two vectors overlapped?



Student: No. The initial point is connected to the terminal point.

Teacher: We have to shift $\overrightarrow{BC}$, and make their initial points overlapped. Then measure the angle counterclockwise. What is the angle?



Student: 120°

Teacher: Find the inner product $\overrightarrow{AB}\cdot\overrightarrow{BC}$. What is the formula?

Student: $|\overrightarrow{AB}||\overrightarrow{BC}|cos120° = 10 \cdot 10 \cdot cos120 = 10 \cdot 10 \cdot (-0.5) = -50.$

Teacher: Correct.

老師： △ABC 是一個等邊三角形，每個角的度數是多少？

學生： 60°。

老師： 讓我畫出 $\overrightarrow{AB}$、$\overrightarrow{AC}$ 和它們之間的角度。

內積 $\overrightarrow{AB} \cdot \overrightarrow{AC}$ 是 $\overrightarrow{AB} \cdot \overrightarrow{AC} = |\overrightarrow{AB}||\overrightarrow{AC}|cos\theta$。$|\overrightarrow{AC}|$、$|\overrightarrow{AB}|$ 分別是多少？

學生： 它們都是 10 嗎？

老師： 是的。$|\overrightarrow{AC}|$ 是大小，代表邊的長度。請計算內積。

學生： $|\overrightarrow{AB}||\overrightarrow{AC}|cos\theta = 10 \cdot 10 \cdot cos60° = 10 \cdot 10 \cdot \frac{1}{2} = 50$。

老師： 接下來，$\overrightarrow{AB}$ 和 $\overrightarrow{BC}$ 之間的角度是多少？

學生： 60°。

老師： 這兩個向量的起始點是否重合？

學生： 不是。起始點是跟終點相連。

老師： 因此我們必須先移動 $\overrightarrow{BC}$，使它們的起始點重疊。然後逆時針測量角度，角度是多少？

學生： 120°。

老師： 找出 $\overrightarrow{AB} \cdot \overrightarrow{BC}$ 的積。公式是什麼？

學生： $|\overrightarrow{AB}||\overrightarrow{BC}|cos120° = 10 \cdot 10 \cdot cos120 = 10 \cdot 10 \cdot (-0.5) = -50$

老師： 正確。

例題二

說明：利用內積判斷兩向量是否正交

（英文）Determine whether the vectors are orthogonal.

(1) $\vec{u}=(10,-5)$ and $\vec{v}=(4,8)$ (2) $\vec{u}=(2,-3)$ and $\vec{v}=(1,-2)$

（中文）判斷以下的向量是否正交？

(1) $\vec{u}=(10,-5)$ 和 $\vec{v}=(4,8)$ (2) $\vec{u}=(2,-3)$ 和 $\vec{v}=(1,-2)$

Teacher: How do we know whether the vectors are orthogonal?

Student: Can we plot the vectors on the coordinate plane? We can see whether they are perpendicular or not.

Teacher: I want you to practice the inner (dot) product. If the vectors are orthogonal, the angle between them is $90°$. What is the value of $cos\ 90°$?

Student: 0.

Teacher: Yes. The inner (dot) product $\vec{u}\cdot\vec{v}=|\vec{u}||\vec{v}|\cdot 0=0$. You can find the inner (dot) product by the formula $\vec{u}\cdot\vec{v}=u_1v_1+u_2v_2$.

Student: $(10,-5)\cdot(4,8)=10\cdot 4+(-5)\cdot 8=0$. They are orthogonal!

Teacher: Good. What about the next one?

Student: $(2,-3)\cdot(1,-2)=2\cdot 1+(-3)\cdot(-2)=8$. They are not orthogonal.

Teacher: Good.

老師： 要怎麼知道這些向量是否正交？

學生： 我們能在坐標平面上畫出這些向量嗎？可以看出它們是否垂直。

老師： 我想讓你們練習內積。如果向量正交，它們之間的角度是 90°。那麼 cos 90° 是多少？

學生： 0。

老師： 是的。內積 $\vec{u}\cdot\vec{v}=|\vec{u}||\vec{v}|\cdot 0=0$。
你可以使用公式 $\vec{u}\cdot\vec{v}=u_1v_1+u_2v_2$ 來計算內積。

學生： $(10,-5)\cdot(4,8)=10\cdot 4+(-5)\cdot 8=0$。它們是正交！

老師： 很好。下一個呢？

學生： $(2,-3)\cdot(1,-2)=2\cdot 1+(-3)\cdot(-2)=8$。它們不是正交。

老師： 答對了。

例題三

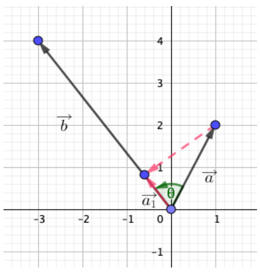
說明：畫出向量的分量，並練習正射影公式。

（英文）Let $\vec{a} = (1,2)$ and $\vec{b} = (-3,4)$. Find the projection of $\vec{a}$ onto $\vec{b}$, and the vector components of $\vec{a}$ orthogonal to $\vec{b}$.

（中文）$\vec{a} = (1,2)$ 和 $\vec{b} = (-3,4)$. 找 $\vec{a}$ 在 $\vec{b}$ 上的正射影，及正交於 $\vec{b}$ 的 $\vec{a}$ 向量分量。

Teacher: You can plot the vectors in standard position. Then plot the projection of $\vec{a}$ onto $\vec{b}$.

Student: This is my graph. I draw a perpendicular line from point $(1,2)$. $\vec{a_1}$ is the projection of $\vec{a}$ onto $\vec{b}$.



Teacher: The projection of $\vec{a}$ onto $\vec{b}$ is $(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2})\vec{b}$. Please find the projection by replacing the exact vectors in the forms. What is your answer?

Student: $\vec{a}\cdot\vec{b} = -3+8 = 5$, $|\vec{b}| = \sqrt{9+16} = 5$, so the projection is

$$\vec{a_1} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b} = \frac{5}{25}\cdot(-3,4) = \left(\frac{-3}{5},\frac{4}{5}\right).$$

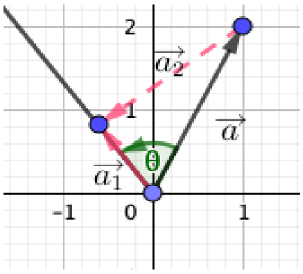
Teacher: The project of $\vec{a}$ onto $\vec{b}$ is also called as the vector components of $\vec{a}$ parallel to $\vec{b}$. Next question asks for the vector components of $\vec{a}$ orthogonal to $\vec{b}$.
Can you point out this vector component?

Student: The red dashed one?

Teacher: Yes.
Let me name the vector $\vec{a_2}$. Since $\vec{a_1}$ is orthogonal to $\vec{a_2}$, how do you get $\vec{a_2}$?

Student: I only know $\vec{a_1}\cdot\vec{a_2} = 0$. I don't know any component of $\vec{a_2}$.

Teacher:



Focus on the small triangle. Have you noticed the directions of the arrowhead?

The graph shows that $\vec{a}+\overrightarrow{a_2}=\overrightarrow{a_1}$, so $\overrightarrow{a_2}=\overrightarrow{a_1}-\vec{a}$. Can you get the answer now?

Student: Since $\overrightarrow{a_1}=\left(\frac{-3}{5},\frac{4}{5}\right)$ and $\vec{a}=(1,2)$, $=\overrightarrow{a_1}-\vec{a}=\left(\frac{-3}{5},\frac{4}{5}\right)-(1,2)=\left(\frac{-8}{5},\frac{-6}{5}\right)$

Teacher: Correct.

老師： 你可以將向量放在標準位置上。然後繪製 $\vec{a}$ 在 $\vec{b}$ 上的正射影。

學生： 這是我的圖。我從(1, 2) 畫了一條垂直線。$\overrightarrow{a_1}$ 是 $\vec{a}$ 在 $\vec{b}$ 上的正射影。

老師： $\vec{a}$ 在 $\vec{b}$ 上的正射影是$(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2})\vec{b}$。請代入向量找出正射影。答案是多少？

學生： $\vec{a}\cdot\vec{b}=-3+8=55$，$|\vec{b}|=\sqrt{9+16}=5$。

所以正射影是 $\overrightarrow{a_1}=\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}=\frac{5}{25}\cdot(-3,4)=\left(\frac{-3}{5},\frac{4}{5}\right)$。

老師： $\vec{a}$ 在 $\vec{b}$ 上的投影也稱為 $\vec{a}$ 平行於 $\vec{b}$ 的分量。

下一個問題要求向量 $\vec{a}$ 垂直於 $\vec{b}$ 的分量。你能指出這個向量分量嗎？

學生： 是那條紅色虛線嗎？

老師： 是的。我們把它命名為 $\overrightarrow{a_2}$。$\overrightarrow{a_1}$ 垂直於 $\overrightarrow{a_2}$。你怎麼求 $\overrightarrow{a_2}$？

學生： 我只知道 $\overrightarrow{a_1}\cdot\overrightarrow{a_2}=0$，不知道 $\overrightarrow{a_2}$ 的分量。

老師： 仔細看小三角形。你有注意到箭頭的方向嗎？

圖中顯示 $\vec{a}+\overrightarrow{a_2}=\overrightarrow{a_1}$，所以 $\overrightarrow{a_2}=\overrightarrow{a_1}-\vec{a}$。現在你能求出答案了嗎？

學生： 因為 $\overrightarrow{a_1}=\left(\frac{-3}{5},\frac{4}{5}\right)$，$\vec{a}=(1,2)$。$\overrightarrow{a_1}-\vec{a}=\left(\frac{-3}{5},\frac{4}{5}\right)-(1,2)=\left(\frac{-8}{5},\frac{-6}{5}\right)$。

老師： 正確。

應用問題 / 學測指考題

例題一

說明：利用內積解決日常生活的問題。

（英文）The vector $\vec{m} = (325, 200)$ gives the numbers of cheeseburgers and peanut-butter burgers sold in one week at a food stand. The vector $\vec{n} = (175, 195)$ gives the corresponding prices in NTD of a cheeseburger and a peanut-butter burger. Find the inner (dot) product $\vec{m} \cdot \vec{n}$ and explain its meaning in the context.

（中文）$\vec{m}$ =(325, 200)表示一個小吃攤一週賣出的起司漢堡和花生醬漢堡的數量，$\vec{n} = (175, 195)$表示起司漢堡和花生醬漢堡的單價。找出 $\vec{m} \cdot \vec{n}$ 並解釋其在題目中代表的意思。

Student: I don’t know that the vector can represent the number of hamburgers. I thought that vectors only show the horizontal movement and vertical movement of a point.

Teacher: Indeed. The vectors not only show movements of a point but also represent other things. We will learn more in the following section. Here, $\vec{m} = (325, 200)$ gives the number of hamburgers, and $\vec{n} = (175, 195)$ gives the corresponding prices. What is the inner (dot) product?

Student: $\vec{m} \cdot \vec{n} = (325, 200) \cdot (175, 195) = 325 \cdot 175 + 200 \cdot 195 = \cdots$.

Teacher: What does $325 \cdot 175$ mean?

Student: The number of cheeseburgers sold in one week times the price of a cheeseburger, and the result is the money we get by selling cheeseburgers.

Teacher: What about $200 \cdot 195$?

Student: The money we get by selling peanut-butter burgers.

Teacher: Ok. Can you explain the meaning of the inner (dot) product in the context?

Student: The total value of cheeseburgers and peanut-butter burgers sold in one week.

Teacher: Good. Please calculate the answer.

Student: 95875.

學生： 我不知道原來向量可以代表漢堡的數量，我以為向量只表示點的水平移動和垂直移動。

老師： 確實。向量不僅能表示點的移動，還能代表其他事物。我們接下來會學到更多。這裡，$\vec{m}=(325,200)$ 代表漢堡的數量，而 $\vec{n}=(175,195)$ 代表相對應的漢堡價格。
內積是多少？

學生： $\vec{m}\cdot\vec{n}=(325,200)\cdot(175,195)=325\cdot175+200\cdot195=\cdots$

老師： 那麼 $325\cdot175$ 代表什麼意思？

學生： 一週內賣出的起司漢堡的數量乘以價格，就是我們賣出起司漢堡所賺到的錢。

老師： 那 $200\cdot195$ 呢？

學生： 賣花生醬漢堡賺到的錢。

老師： 好的。你能解釋內積在這個情境中的意義嗎？

學生： 一週內賣出的起司漢堡和花生醬漢堡的總金額。

老師： 很好。請計算答案。

學生： 答案是 95875。

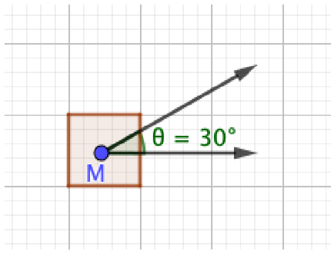
例題二

說明：利用向量分量解決日常生活問題。

（英文）A strong man dragged a cement block with a rope tied to it with a constant force of 230 pounds at an angle of 30° with the horizontal. The block has moved the distance *d*. Find the work done in terms of *d*. If *d* is 20 ft, find the work done.

（中文）一個強壯的人以 30° 的角度，用 230 磅的力拉一條綁著水泥磚的繩子，磚塊移動 *d*。試以 *d* 表示所作的功。如果移動距離為 20ft，請問所作的功為何？

Teacher: First, let’s plot the question and visualize it. Do you remember the formula of “work”?



Student: The force parallel to the displacement times the displacement.

Teacher: Yes. The “work” done by the force is the product of “the component of the force parallel to the displacement” multiplied by “the displacement”. The block has been moved horizontally *d* units, so we have to find out the horizontal component of the force.

Student: I can draw a right triangle, and the horizontal component is the leg adjacent to the angle. $230 \cdot cos30° = 115\sqrt{3}$.

Teacher: What about the work in terms of *d*?

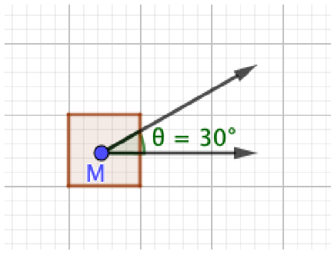
Student: $115\sqrt{3}d$.

Teacher: If *d* is 20 ft, then what is the answer?

Student: $115\sqrt{3}d = 2300\sqrt{3}$. It is about 3983.6.

Teacher: The unit can be written as “lb-ft”, meaning “pound-feet”. If you use “newtons” and “meters” as the unit of the force and the distance, then the unit of the work will be represented as “NT-m”, meaning “NT-meter”. In the physics book, you will learn that one NT-m is equal to one joule.

老師： 首先，畫圖將題目形象化。你們還記得功的公式嗎？



學生： 平行於位移的力乘上位移。

老師： 沒錯。力所做的「功」是「平行於位移的力分量」乘以「位移」的乘積。這個水泥磚水平移動了 d 個單位，所以我們必須找出力的水平分量。

學生： 我可以畫一個直角三角形，水平分量是鄰邊。$230 \cdot cos30° = 115\sqrt{3}$。

老師： 那以 d 表示，功是多少？

學生： $115\sqrt{3}d$。

老師： d 是 20，那答案是？

學生： $115\sqrt{3}d = 2300\sqrt{3}$，約為 3983.6。

老師： 單位可以寫為“lb-ft”，意思是「磅-英呎」。
如果你用”牛頓”及”公尺”表示力和距離的單位，功的單位就會是” NT-m”，意思是「牛頓-公尺」。在物理課本中，你會學到 1 牛頓-公尺等於 1 焦耳。

例題三

說明：練習柯西不等式解題。

（英文）Given $x^2 + y^2 = 3$, maximize $x + 2y$, and find the value of x and y, respectively.
（中文）已知 $x^2 + y^2 = 3$, 求 $x + 2y$ 最大值，及此時的 x 與 y 值。

Teacher: We will apply Cauchy’s inequality to solve this question. The Cauchy’s inequality is $(x_1x_2 + y_1y_2)^2 \leq (x_1{}^2 + y_1{}^2)(x_2{}^2 + y_2{}^2)$
The expression $x + 2y$ can be written as the inner (dot) product of (x, y) and $(1, 2)$.

$$x + 2y = (x, y) \cdot (1, 2) = x_1x_2 + y_1y_2$$

Compare the coefficients, and we find that $x_1 = x, x_2 = 1,\ y_1 = 1,\ y_2 = 2$.
Then, we can create the inequality $(x + 2y)^2 \leq (x^2 + y^2)(1^2 + 2^2)$. I think you know what to do next. Please follow up.

Student: $(x + 2y)^2 \leq 3 \cdot 5$, then $x + 2y \leq \sqrt{15}$. The maximum of $x + 2y$ is $\sqrt{15}$.

Teacher: The equality happens when $\frac{x_1}{y_1}=\frac{x_2}{y_2}$. What is the next step?

Student: The equality works when $\frac{x}{y}=\frac{1}{2}$. By cross multiplication, $y=2x$.

Teacher: Good, then we replace $y=2x$in the equation $x^2+y^2=3$. $5x^2=3$.

Solve for x, and we get $x=\pm\frac{\sqrt{15}}{5}$.

Don't forget the negative value. Now please find the answers.

Student: When $x=\frac{\sqrt{15}}{5}$, $y=\frac{2\sqrt{15}}{5}$. When $x=-\frac{\sqrt{15}}{5}$, $y=-\frac{2\sqrt{15}}{5}$

Teacher: When $x=\frac{\sqrt{15}}{5}$, $y=\frac{2\sqrt{15}}{5}$, or $x=-\frac{\sqrt{15}}{5}$, $y=-\frac{2\sqrt{15}}{5}$, $x+2y$ has the maximum $\sqrt{15}$.

老師： 我們要使用柯西不等式來解這題。

柯西不等式是$(x_1x_2+y_1y_2)^2\leq({x_1}^2+{y_1}^2)({x_2}^2+{y_2}^2)$

老師： $x+2y$ 可以寫成 (x,y) 和 $(1,2)$ 的內積。

$$x+2y=(x,y)\cdot(1,2)=x_1x_2+y_1y_2$$

比較係數後我們發現 $x_1=x, x_2=1, y_1=1, y_2=2$。

然後列出不等式 $(x+2y)^2\leq(x^2+y^2)(1^2+2^2)$。

我想你們應該知道接下來該做什麼，請接下去。

學生： $(x+2y)^2\leq 3\cdot 5$，然後 $x+2y\leq\sqrt{15}$ 的最大值是 $\sqrt{15}$。

老師： 等號成立時，$\frac{x_1}{y_1}=\frac{x_2}{y_2}$。下一步是什麼？

學生： 等號成立時，$\frac{x}{y}=\frac{1}{2}$，然後交叉相乘，$y=2x$。

老師： 很好，我們將 $y=2x$ 代入方程 $x^2+y^2=3$。$5x^2=3$，解得 $x=\pm\frac{\sqrt{15}}{5}$。

不要忘記負號。現在請算出答案。

學生： 當$x=\frac{\sqrt{15}}{5}$ 、$y=\frac{2\sqrt{15}}{5}$；當 $x=-\frac{\sqrt{15}}{5}$ 、$y=-\frac{2\sqrt{15}}{5}$

老師： 當$x=\frac{\sqrt{15}}{5}$，$y=\frac{2\sqrt{15}}{5}$，或 $x=-\frac{\sqrt{15}}{5}$，$y=-\frac{2\sqrt{15}}{5}$，$x+2y$ 的最大值為 $\sqrt{15}$。

單元八　平面向量的應用
Applications of Vectors

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■ 前言 Introduction

本單元以英文介紹平面向量的應用，先介紹二階行列式的公式及幾何意義，介紹兩向量張出的三角形面積公式並證明公式、二階行列式的運算規則及其與向量之間的關係，最後介紹克拉瑪公式-解二元一次方程式的新方法，並藉由克拉瑪公式討論解的種類。

■ 詞彙 Vocabulary

※粗黑體標示為此單元重點詞彙

單字	中譯	單字	中譯
determinant	行列式	second-order determinant	二階行列式
principal diagonal	主對角線	secondary diagonal	次對角線
Cramer’s Rule	克拉瑪公式		

■ 教學句型與實用句子 Sentence Frames and Useful Sentences

❶ ____ are interchanged.

例句：What would the determinant be if the two rows **are interchanged**?

如果兩列交換的話，行列式會變如何？

❷ ____ formed by ______.

例句：Sketch the parallelogram **formed by** the two vectors $(1, 2)$ and $(-2, 3)$.

畫出由兩向量 $(1, 2)$ 和 $(-2, 3)$ 所張出的平行四邊形。

❸ Use ______.

例句：**Use** Cramer’s rule to solve the system of linear equations.

利用克拉瑪公式解方程組。

■ 問題講解 Explanation of Problems

說明

[Second-order determinant]

In this section, we will learn the applications of vectors: the area of triangles and solving equations. Before that, I would like to introduce a new operation, second-order determinant.

The second-order determinant, denoted by $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, has four elements a_1, a_2, b_1, b_2. There are two rows and two columns in this determinant. We sometimes call the dimension 2×2 (read as two by two). The first row contains a_1 and b_1, and the second row contains a_2 and b_2. The first column contains a_1 and a_2, and the second column contains b_1 and b_2.

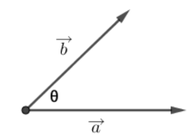
The diagonal from the top left corner to the bottom right corner is called the *principal diagonal*, or *main diagonal*. The elements along the principal diagonal are a_1 and b_2. The diagonal from the top right corner to the bottom left corner is called the *secondary diagonal*. The elements along the secondary diagonals are b_1 and a_2.

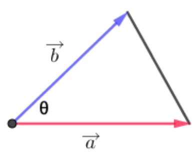
The value of a second-order determinant is defined as *the product of the elements on the principal diagonal minus the product of the elements on the secondary diagonal*, as follows:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - b_1a_2$$

[Area of triangle using vectors]

If a triangle is formed by two vectors $\vec{a}$ and $\vec{b}$, how do we find the area of the triangle using vectors?





In trigonometry, we learned that the area of the triangle is $\frac{1}{2}absin\theta$.

We replace the length with the magnitude of vectors, and we have $\frac{1}{2}absin\theta = \frac{1}{2}|\vec{a}||\vec{b}|sin\theta$. The area of the parallelogram formed by $\vec{a}$ and $\vec{b}$ is $|\vec{a}||\vec{b}|sin\theta$.

Let's start with the parallelogram.

$$S = |\vec{a}||\vec{b}|sin\theta = \sqrt{|\vec{a}|^2|\vec{b}|^2 sin^2\theta} = \sqrt{|\vec{a}|^2|\vec{b}|^2(1 - cos^2\theta)}$$

$$= \sqrt{|\vec{a}|^2|\vec{b}|^2 - |\vec{a}|^2|\vec{b}|^2 cos^2\theta}$$

Since $\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|cos\theta$, $S = \sqrt{|\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2}$.

Assume that $\vec{a} = (a_1, a_2)$, $\vec{b} = (b_1, b_2)$, then

$S^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2 = ({a_1}^2 + {a_2}^2)({b_1}^2 + {b_2}^2) - (a_1b_1 + a_2b_2)^2$

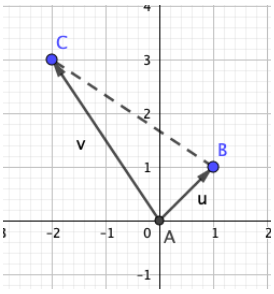
Simplify and we will have

$$S^2 = (a_1b_2 - a_2b_1)^2$$

The area of parallelogram is $S = |a_1 b_2 - a_2 b_1| = \left|\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\right|$, which can be expressed by the determinant. The absolute value sign guarantees the positive value for “area”.

As long as any two adjacent vectors of a triangle are given, the area of triangle is $\frac{1}{2}\left|\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\right|$.

Let’s apply it to an example. $(-2, 3)$ and $(1, 1)$ are vectors of two adjacent sides of a triangle.



Sketch these two vectors at the origin, and connect the point B and C. The area of $\triangle ABC$ is $\frac{1}{2}\left|\begin{vmatrix} -2 & 1 \\ 3 & 1 \end{vmatrix}\right| = \frac{1}{2}|-2 \cdot 1 - (3 \cdot 1)| = \frac{5}{2}$

[Properties of determinants]

Here, I would like to introduce some properties of determinants. We can examine the properties with real number examples.

(1) If both rows and columns are interchanged, then the value of the determinant remains the same.

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$\begin{vmatrix} -2 & 1 \\ 3 & 1 \end{vmatrix} = -2 - 3 = -5 \,;\, \begin{vmatrix} -2 & 3 \\ 1 & 1 \end{vmatrix} = -2 - 3 = -5$$

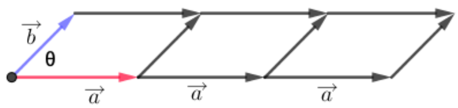
(2) If two rows (or two columns) of the determinant are interchanged, then the sign of the determinant changes.

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = -\begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix}, \quad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = -\begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix}$$

$$\begin{vmatrix} -2 & 1 \\ 3 & 1 \end{vmatrix} = -5 \,;\, \begin{vmatrix} 1 & -2 \\ 1 & 3 \end{vmatrix} = 3 - (-2) = 5$$

(3) If each element of a row or a column is multiplied by a constant m, then its value of determinant is multiplied by m.

$$\begin{vmatrix} ma_1 & b_1 \\ ma_2 & b_2 \end{vmatrix} = m\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$



The first parallelogram is formed by $\vec{a}$ and $\vec{b}$.

If we form a parallelogram by $3\vec{a}$ and $\vec{b}$, the parallelogram is enlarged. One side stays the same, but the other side is tripled. What will happen to the area of parallelogram? You can observe the figure and find that the area will be tripled, too. Assume that $\vec{a}=(a_1,a_2)$, $\vec{b}=(b_1,b_2)$. The area of new parallelogram can be expressed by $\begin{vmatrix}3a_1 & b_1\\3a_2 & b_2\end{vmatrix}=3\begin{vmatrix}a_1 & b_1\\a_2 & b_2\end{vmatrix}$, while $\begin{vmatrix}a_1 & b_1\\a_2 & b_2\end{vmatrix}$ represents the area of the first parallelogram.

(4) If some elements of a row or a column are expressed as the sum of two terms, then the determinant can be expressed as the sum of two determinants.

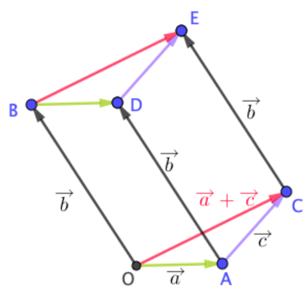
$$\begin{vmatrix}a_1+c_1 & b_1\\a_2+c_2 & b_2\end{vmatrix}=b_2(a_1+c_1)-b_1(a_2+c_2)=(a_1b_2-b_1a_2)+(c_1b_2-c_2b_1)$$

$$=\begin{vmatrix}a_1 & b_1\\a_2 & b_2\end{vmatrix}+\begin{vmatrix}c_1 & b_1\\c_2 & b_2\end{vmatrix}$$

Let's check the geometric meaning.

$OADB$ is a parallelogram formed by $\vec{a}$ and $\vec{b}$, and its area is $\begin{vmatrix}a_1 & b_1\\a_2 & b_2\end{vmatrix}$.

$ACED$ is a parallelogram formed by $\vec{c}$ and $\vec{b}$, and its area is $\begin{vmatrix}c_1 & b_1\\c_2 & b_2\end{vmatrix}$. The sum of the areas of these two parallelograms will be equivalent to the area of the parallelogram formed by $\vec{a}+\vec{c}$ and $\vec{b}$.



$OCEB=OADB+ACED$. This can be described with determinants, please see the formula above.

Can you tell that $\triangle OAC$ and $\triangle BDE$ are congruent? You can discuss it with your classmates.

(5) If the elements of a column are multiplied by a constant r and added to the corresponding elements of the other column, then the value of the determinant remains the same. Similarly, if the elements of one row are multiplied by a constant r and added to the corresponding elements of the other row, then the value of the determinant remains the same.

$$\begin{vmatrix} a_1 & ra_1+b_1 \\ a_2 & ra_2+b_2 \end{vmatrix} = \begin{vmatrix} a_1 & ra_1 \\ a_2 & ra_2 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = r\begin{vmatrix} a_1 & a_1 \\ a_2 & a_2 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$
$$= 0 + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & b_1 \\ ra_1+a_2 & rb_1+b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ ra_1 & rb_1 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = r\begin{vmatrix} a_1 & a_1 \\ a_2 & a_2 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$
$$= 0 + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

[Cramer's rule]

We have learned some ways to solve the system of linear equations: substitution and elimination. Here, I would like to introduce a new method "Cramer's rule" to solve the equations with determinants.

The system of linear equations $\begin{cases} a_1x+b_1y=c_1 \\ a_2x+b_2y=c_2 \end{cases}$ is given. If you apply either elimination or substitution, you can find the solution $x=\frac{c_1b_2-c_2b_1}{a_1b_2-a_2b_1}$, $y=\frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$. Each denominator and numerator in the solution can be expressed as a determinant.

$$x=\frac{c_1b_2-c_2b_1}{a_1b_2-a_2b_1}=\frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}},\ y=\frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1}=\frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

We usually denote the determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ as Δ(read as delta).

The numerator of x is denoted by Δ_x. $\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ is formed by using the constants c_1 and c_2 as replacements for the coefficients of x, which are a_1 and a_2.

The numerator of y is denoted by Δ_y. $\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$ is formed by using the constants c_1 and c_2 as replacements for the coefficients of y, which are b_1 and b_2.

There are three cases for the solution of the system of linear equations $\begin{cases}a_1x+b_1y=c_1\\a_2x+b_2y=c_2\end{cases}$.

(1) When $\Delta=\begin{vmatrix}a_1 & b_1\\a_2 & b_2\end{vmatrix}\neq 0$, the system has exactly one solution $x=\frac{\Delta_x}{\Delta}$, $y=\frac{\Delta_y}{\Delta}$.

(2) When $\Delta=0$, $\Delta_x\neq 0$, $\Delta_y\neq 0$, the system has no solution.

(3) When $\Delta=\Delta_x=\Delta_y=0$, the system has infinitely many solutions.

How do we explain Cramer's rule with "vectors"?

I would like to take case (1) as an example. The coefficient of x can be written as vector $\vec{a}=(a_1,a_2)$. The coefficient of y can be written as vector $\vec{b}=(b_1,b_2)$. The constants can be written as vector $\vec{c}=(c_1,c_2)$. If we rewrite the system of linear equations in terms of "vector", we would have $(c_1,\ c_2)=(a_1x+b_1y,\ a_2x+b_2y)=x(a_1,\ a_2)+y(b_1,b_2)$

The system of linear equations can be expressed as $\vec{c}=\vec{a}x+\vec{b}y$.

Does this look familiar to you? A linear combination of vectors is a result of combining the vector using scalar multiplications. When $\vec{a}$, $\vec{b}$ are non-zero vectors, unparalleled, coplanar vectors, there exists only one solution (x,y), such that $\vec{c}=\vec{a}x+\vec{b}y$.

You can try to explain the other two cases with vectors.

運算問題的講解

例題一

說明：練習用二階行列式的絕對值求出向量所張出的平行四邊形之面積

（英文）Find the area of the parallelogram formed by two vectors $\vec{a}=(-4,5)$ and $\vec{b}=(1,2)$.

（中文）求以向量 $\vec{a}=(-4,5)$ 與 $\vec{b}=(1,2)$ 為兩鄰邊所張出的平行四邊形面積。

Teacher: The parallelogram is formed by two vectors, and its area is $\left|\begin{vmatrix}a_1 & b_1\\a_2 & b_2\end{vmatrix}\right|$. The absolute value of the determinant. Please find the answer.

Student: $\left|\begin{vmatrix}-4 & 1\\5 & 2\end{vmatrix}\right|=|-4\cdot 2-5\cdot 1|=13.$

Teacher: Good.

Student: Do I have to arrange the vectors by columns? Can I arrange the vectors by rows?

Like this $\left|\begin{vmatrix} -4 & 5 \\ 1 & 2 \end{vmatrix}\right|$?

Teacher: I like your idea. Why don't we try it together?

Student: $\left|\begin{vmatrix} -4 & 5 \\ 1 & 2 \end{vmatrix}\right| = |-4 \cdot 2 - 1 \cdot 5| = 13$. They are the same.

Teacher: When you switch, you only change the order of numbers in the second part, from $5 \cdot 1$ to $1 \cdot 5$. According to the commutative property, the change in the order of two numbers in a multiplication doesn't change the product.

Student: Ok, I see.

老師： 這個平行四邊形是由兩個向量所形成的，它的面積是 $\left|\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\right|$。這是行列式的絕對值。請算出答案。

學生： $\left|\begin{vmatrix} -4 & 1 \\ 5 & 2 \end{vmatrix}\right| = |-4 \cdot 2 - 5 \cdot 1| = 13$.

老師： 很好

學生： 我一定要直的排列向量嗎？可以橫著排嗎？像這樣 $\left|\begin{vmatrix} -4 & 5 \\ 1 & 2 \end{vmatrix}\right|$？

老師： 我喜歡你的想法。我們一起試著算算看如何？

學生： $\left|\begin{vmatrix} -4 & 5 \\ 1 & 2 \end{vmatrix}\right| = |-4 \cdot 2 - 1 \cdot 5| = 13$。跟剛才答案一樣！

老師： 當你交換時，你只改變了第二部分數字的順序，從 $5 \cdot 1$ 到 $1 \cdot 5$。根據乘法的交換性，改變兩個數字在乘法中的順序不會改變乘積。

學生： 原來如此，我明白了。

例題二

說明：練習用二階行列式的運算

（英文）Assume that $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = 10$. Calculate $\begin{vmatrix} 2a-c & 4a+3c \\ 2b-d & 4b+3d \end{vmatrix}$.

（中文）已知 $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = 10$。求 $\begin{vmatrix} 2a-c & 4a+3c \\ 2b-d & 4b+3d \end{vmatrix}$ 的值。

Teacher: According to the property, if some elements of a row or a column are expressed as the sum of two terms, then the determinant can be expressed as the sum of two determinants. We are going to rewrite the determinant. Let's separate the first column and keep the second column the same.

$$\begin{vmatrix} 2a-c & 4a+3c \\ 2b-d & 4b+3d \end{vmatrix} = \begin{vmatrix} 2a & 4a+3c \\ 2b & 4b+3d \end{vmatrix} + \begin{vmatrix} -c & 4a+3c \\ -d & 4b+3d \end{vmatrix}$$

Student: Can we separate the second column and keep the first column the same?

Teacher: Yes, we can do that. You can try it later, and the answer would be the same.

Next, we are going to apply the property (5). If the elements of a column are multiplied by a constant r and added to the corresponding elements of the other column, then the value of the determinant remains the same.

In the determinant $\begin{vmatrix} 2a & 4a+3c \\ 2b & 4b+3d \end{vmatrix}$, I will multiply each element in the first column by -2, and add to the second column. $\begin{vmatrix} 2a & 2a\cdot(-2)+4a+3c \\ 2b & 2b\cdot(-2)+4b+3d \end{vmatrix} = \begin{vmatrix} 2a & 3c \\ 2b & 3d \end{vmatrix}$.

Why do I multiply each element by -2?

Student: It looks like you want to eliminate 4.

Teacher: Indeed. I want to simplify the determinant. Now, I am going to apply property (3). If each element of a column is multiplied by a constant m, then its value of the determinant is multiplied by m. In $\begin{vmatrix} 2a & 3c \\ 2b & 3d \end{vmatrix}$, the scalar of the first column is 2, and the scalar of the second column is 3. I would like to take out these constants.

Therefore, $\begin{vmatrix} 2a & 3c \\ 2b & 3d \end{vmatrix} = 2\cdot 3\begin{vmatrix} a & c \\ b & d \end{vmatrix} = 6\begin{vmatrix} a & c \\ b & d \end{vmatrix}$. Because $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = 10$ is given, $\begin{vmatrix} 2a & 3c \\ 2b & 3d \end{vmatrix} = 6\cdot 10 = 60$.

Can you work on $\begin{vmatrix} -c & 4a+3c \\ -d & 4b+3d \end{vmatrix}$?

Student: Can I eliminate 3?

Teacher: Which number is required to eliminate 3?

Student: -3.

Teacher: Multiply each element in the first column by 3, and add to the other column.

Student: $\begin{vmatrix} -c & -c \cdot 3 + 4a + 3c \\ -d & -d \cdot 3 + 4b + 3d \end{vmatrix} = \begin{vmatrix} -c & 4a \\ -d & 4b \end{vmatrix}$

Teacher: Good. Now take out the scalar from each column.

Student: $\begin{vmatrix} -c & 4a \\ -d & 4b \end{vmatrix} = -1 \cdot 4 \begin{vmatrix} c & a \\ d & b \end{vmatrix} = -4 \begin{vmatrix} c & a \\ d & b \end{vmatrix} = -40$

Teacher: Is $\begin{vmatrix} c & a \\ d & b \end{vmatrix}$ equivalent to $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$?

Student: The columns are interchanged.

Teacher: According to the property, if two columns of the determinant are exchanged, then the sign of the determinant changes. Do you want to change the answer?

Student: $\begin{vmatrix} c & a \\ d & b \end{vmatrix} = -\begin{vmatrix} a & c \\ b & d \end{vmatrix} = -10$. $\begin{vmatrix} -c & 4a \\ -d & 4b \end{vmatrix} = -4 \begin{vmatrix} c & a \\ d & b \end{vmatrix} = 40$.

Teacher: $\begin{vmatrix} 2a - c & 4a + 3c \\ 2b - d & 4b + 3d \end{vmatrix} = 6 \begin{vmatrix} a & c \\ b & d \end{vmatrix} + -4 \begin{vmatrix} c & a \\ d & b \end{vmatrix}$.

What is the value of $\begin{vmatrix} 2a - c & 4a + 3c \\ 2b - d & 4b + 3d \end{vmatrix}$?

Student: $60 + 40 = 100$.

Teacher: Correct. Please practice the properties of determinants more. Practice makes perfect!

老師：根據行列式性質，如果某一行或某一列可以表示為兩個項的和，那麼原行列式可以拆開成兩個行列式的和。我們將題目行列式改寫，分開第一行，第二行不變。

$$\begin{vmatrix} 2a - c & 4a + 3c \\ 2b - d & 4b + 3d \end{vmatrix} = \begin{vmatrix} 2a & 4a + 3c \\ 2b & 4b + 3d \end{vmatrix} + \begin{vmatrix} -c & 4a + 3c \\ -d & 4b + 3d \end{vmatrix}$$

學生：我們能夠分開第二行，並保持第一行不變嗎？

老師：可以，等一下你們再自己試試看，答案一樣不變。

接下來，我們要應用性質（5）。如果某一行乘以一個常數 r 並加到另一行，則行列式的值不變。

行列式 $\begin{vmatrix} 2a & 4a + 3c \\ 2b & 4b + 3d \end{vmatrix}$ 中，我將第一行的乘以-2，然後加到第二行。

$$\begin{vmatrix} 2a & 2a \cdot (-2) + 4a + 3c \\ 2b & 2b \cdot (-2) + 4b + 3d \end{vmatrix} = \begin{vmatrix} 2a & 3c \\ 2b & 3d \end{vmatrix}$$ 。

為什麼我要將第一行乘以-2？

學生：　看起來是想消去 4。

老師：　沒錯，我想簡化行列式。現在，我們要應用性質（3）。如果某一行都乘以一個常數 m，那麼行列式的值就會乘以 m。

$\begin{vmatrix} 2a & 3c \\ 2b & 3d \end{vmatrix}$ 中，第一行的純量為 2，第二行的純量為 3。我想拿出這些常數。

因此，$\begin{vmatrix} 2a & 3c \\ 2b & 3d \end{vmatrix} = 2\cdot 3\begin{vmatrix} a & c \\ b & d \end{vmatrix} = 6\begin{vmatrix} a & c \\ b & d \end{vmatrix}$。已知，$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = 10$，則 $\begin{vmatrix} 2a & 3c \\ 2b & 3d \end{vmatrix} = 6\cdot 10 = 60$。

那你們會做 $\begin{vmatrix} -c & 4a+3c \\ -d & 4b+3d \end{vmatrix}$ 嗎？

學生：　我可以消去 3 嗎？

老師：　需要哪個數字來消去 3？

學生：　-3。

老師：　將第一行乘以 3，加到另一行。

學生：　$\begin{vmatrix} -c & -c\cdot 3+4a+3c \\ -d & -d\cdot 3+4b+3d \end{vmatrix} = \begin{vmatrix} -c & 4a \\ -d & 4b \end{vmatrix}$

老師：　很好。現在從每一行取出純量。

學生：　$\begin{vmatrix} -c & 4a \\ -d & 4b \end{vmatrix} = -1\cdot 4\begin{vmatrix} c & a \\ d & b \end{vmatrix} = -4\begin{vmatrix} c & a \\ d & b \end{vmatrix} = -40$

老師：　$\begin{vmatrix} c & a \\ d & b \end{vmatrix}$ 等於 $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$ 嗎？

學生：　這兩行交換了。

老師：　根據性質，如果改變行列式的兩行，那麼行列式的符號也會改變。再確認一下答案

學生：　$\begin{vmatrix} c & a \\ d & b \end{vmatrix} = -\begin{vmatrix} a & c \\ b & d \end{vmatrix} = -10$，$\begin{vmatrix} -c & 4a \\ -d & 4b \end{vmatrix} = -4\begin{vmatrix} c & a \\ d & b \end{vmatrix} = 40.$

老師：　很好。因此 $\begin{vmatrix} 2a-c & 4a+3c \\ 2b-d & 4b+3d \end{vmatrix} = 6\begin{vmatrix} a & c \\ b & d \end{vmatrix} + -4\begin{vmatrix} c & a \\ d & b \end{vmatrix}$

$\begin{vmatrix} 2a-c & 4a+3c \\ 2b-d & 4b+3d \end{vmatrix}$ 的值是多少呢？

學生：　60+40=100。

老師：　正確。請多練習行列式的性質。熟能生巧！

應用問題 / 學測指考題

例題一

說明：這題利用克拉瑪公式判斷方程組的解。

（英文）Determine which system of linear equations has exactly one solution.

(1) $\begin{cases} 4x - y = -2 \\ 8x - 2y = -4 \end{cases}$　(2) $\begin{cases} 4x - 3y = -10 \\ 6x + 9y = 12 \end{cases}$　(3) $\begin{matrix} x + 2y = 5 \\ -2x - 4y = 1 \end{matrix}$

（中文）判斷以上方程組何者有唯一解。

Teacher: How do you determine which system of linear equations has only one solution?

Student: We can solve each system of linear equations, and then we would know.

Teacher: That's one way. We can also try Cramer's rule and the determinants. Find $\Delta,\ \Delta_x,\ \Delta_y$ in question (1).

Student: $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 8 & -2 \end{vmatrix} = -8 - (-8) = 0,$

$\Delta_x = \begin{vmatrix} -2 & -1 \\ -4 & -2 \end{vmatrix} = 4 - (-1)(-4) = 0,$

$\Delta_y = \begin{vmatrix} 4 & -2 \\ 8 & -4 \end{vmatrix} = -16 - (-16) = 0.$

They are all 0.

Teacher: When $\Delta = \Delta_x = \Delta_y = 0$, the system has infinitely many solutions. Now move on to question (2).

Student: $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} 4 & -3 \\ 6 & 9 \end{vmatrix} = 36 - (-18) = 54.$

$\Delta_x = \begin{vmatrix} -10 & -3 \\ 12 & 9 \end{vmatrix} = -90 + 36 = -54,$

$\Delta_y = \begin{vmatrix} 4 & -10 \\ 6 & 12 \end{vmatrix} = 48 - (-60) = 108.$

Student: $x = \frac{\Delta_x}{\Delta} = \frac{-54}{54} = -1,\ y = \frac{\Delta_y}{\Delta} = \frac{108}{54} = 2$. The solution is $(-1,2)$.

Teacher: Yes. Actually, when you found that $\Delta \neq 0$, you can tell that there exists exactly one solution. Now move on to question (3).

Student: $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ -2 & -4 \end{vmatrix} = -4 - (-4) = 0$

Student: $\Delta_x = \begin{vmatrix} 5 & 2 \\ 1 & -4 \end{vmatrix} = -20 - 2 = -22,\ \Delta_y = \begin{vmatrix} 1 & 5 \\ -2 & 1 \end{vmatrix} = 1 - (-10) = 11.$

Teacher: What is the solution?

Student: $x = \frac{\Delta_x}{\Delta} = \frac{-22}{0}$. The fraction has 0 as the denominator. It is undefined.

Teacher: $x = \frac{-22}{0}$. $y = \frac{\Delta_y}{\Delta} = \frac{11}{0}$. Both fractions are undefined, so the system of linear equations has no solution. Question (2) has exactly one solution.

老師： 如何確定哪個線性方程組只有一個解？

學生： 我們可以解每個方程組，然後就會知道。

老師： 這是一種方法。我們也可以嘗試克拉瑪公式和行列式。
現在找出題目(1)的Δ、Δx、Δy。

學生：
$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 8 & -2 \end{vmatrix} = -8 - (-8) = 0,$

$\Delta_x = \begin{vmatrix} -2 & -1 \\ -4 & -2 \end{vmatrix} = 4 - (-1)(-4) = 0,$

$\Delta_y = \begin{vmatrix} 4 & -2 \\ 8 & -4 \end{vmatrix} = -16 - (-16) = 0.$

它們都是 0。

老師： 當 $\Delta = \Delta_x = \Delta_y = 0$，該方程組有無限多個解。現在看題目(2)。

學生： $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} 4 & -3 \\ 6 & 9 \end{vmatrix} = 36 - (-18) = 54.$

$\Delta_x = \begin{vmatrix} -10 & -3 \\ 12 & 9 \end{vmatrix} = -90 + 36 = -54,$

$\Delta_y = \begin{vmatrix} 4 & -10 \\ 6 & 12 \end{vmatrix} = 48 - (-60) = 108.$

學生： $x = \frac{\Delta_x}{\Delta} = \frac{-54}{54} = -1$、$y = \frac{\Delta_y}{\Delta} = \frac{108}{54} = 2$. 解為 $(-1,2)$.

老師： 是的。實際上，當你發現 $\Delta \neq 0$ 時，就可以知道方程組有唯一解。
來算題目(3)。

學生： $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ -2 & -4 \end{vmatrix} = -4 - (-4) = 0$

學生： $\Delta_x = \begin{vmatrix} 5 & 2 \\ 1 & -4 \end{vmatrix} = -20 - 2 = -22,\ \Delta_y = \begin{vmatrix} 1 & 5 \\ -2 & 1 \end{vmatrix} = 1 - (-10) = 11.$

老師： 解是多少？

學生： $x=\frac{\Delta_x}{\Delta}=\frac{-22}{0}$，分母為 0，它是未定義的。

老師： $x=\frac{-22}{0}$ ，$y=\frac{\Delta_y}{\Delta}=\frac{11}{0}$。兩個分數都是未定義的，所以這個方程組沒有解。題目(2)有唯一解。

例題二

說明：利用克拉瑪公式解日常生活問題。

（英文）A chemist is mixing two alcohol solutions together. He has one solution containing 50% alcohol and a second one containing 20% alcohol. How much of each should be used to make 100 liters of 44 % alcohol.

（中文）一位化學家混合兩種酒精溶液，第一種濃度為 50%，第二種濃度為 20%。各需要多少的酒精溶液才能混合成 100 公升濃度為 44%的酒精溶液？

Teacher: A 50% alcohol solution means 50ml alcohol is dissolved in 50ml of water. Alcohol is the solute and water is the solvent. The alcohol solution is the mixture of alcohol and water. The amount of alcohol solution is the sum of the amount of “alcohol” and the amount of “water”.

Student: Aren’t we in math class now? Why do I see chemistry questions?

Teacher: This question shows you how math can be applied in other fields.

Teacher: Since we have no idea about the amount of each solution, let’s assume x is the number of liters of 50% alcohol, y is the number of liters of 20% alcohol. The total amount will be 100 liters. How do you write the equation to describe the amount of solutions?

Student: $x+y=100$.

Teacher: Next, we are going to see the amount of alcohol in each solution. The first solution contributes $0.5x$ liters of alcohol. How much alcohol does the second solution contribute?

Student: $0.2y$.

Teacher: In 100 liters of 44 % alcohol, what is the total amount of alcohol?

Student: $100\cdot 44\%=44$.

Teacher: Can you write the equation to describe the amount of “alcohol”?

Student: $0.5x + 0.2y = 44$ is it correct?

Teacher: Yes. Now we have a system of linear equations: $\begin{cases} x + y = 100 \\ 0.5x + 0.2y = 44 \end{cases}$.

Now please apply the Cramer’s rule to find the solution. Find the determinants first.

Student: $\Delta = \begin{vmatrix} 1 & 1 \\ 0.5 & 0.2 \end{vmatrix} = 0.2 - 0.5 = -0.3$. It is a negative number. Am I right?

Teacher: Be confident. Let’s find out Δ_x and Δ_y.

Student: $\Delta_x = \begin{vmatrix} 100 & 1 \\ 44 & 0.2 \end{vmatrix} = 20 - 44 = -24$. $\Delta_y = \begin{vmatrix} 1 & 100 \\ 0.5 & 44 \end{vmatrix} = 44 - 50 = -6$.

They are all negative numbers.

Teacher: $x = \frac{\Delta_x}{\Delta}$, and $y = \frac{\Delta_y}{\Delta}$. Negative number divided by a negative number is a positive number. Don’t worry. We are on the right track. Please find out the answer.

Student: $x = \frac{\Delta_x}{\Delta} = \frac{-24}{-0.3} = 80$, $y = \frac{\Delta_y}{\Delta} = \frac{-6}{-0.3} = 20$.

Teacher: Please double check the answer.

Student: $80 + 20 = 100$, and $0.5 \cdot 80 + 0.2 \cdot 20 = 44$. Both equations are satisfied.

Teacher: Great. The quantity of the first solution is 80 liters, and the second solution is 20 liters.

老師： 一個 50%酒精溶液表示 50 毫升的酒精溶解在 50 毫升的水中。酒精是溶質，水是溶劑。這個酒精溶液是酒精和水的混合物。酒精溶液的量等於“酒精”的量和“水”的量的總和。

學生： 現在不是在上數學課嗎？為什麼有化學題目？

老師： 這個問題告訴我們數學如何應用於其他領域。

老師： 既然我們都不知道這些溶液的量，讓我們先假設 x 是 50%酒精的公升數，y 是 20%酒精的公升數。總量為 100 公升。如何列方程式來描述溶液的量？

學生： $x + y = 100$。

老師： 接下來，我們要看每種溶液中的酒精量。第一個溶液含有 $0.5x$ 公升的酒精。第二個溶液有多少酒精？

學生： 0.2y。

老師： 很好。100 公升濃度為 44%的酒精，總共有多少酒精？

學生： $100 \cdot 44\% = 44$。

老師： 你能列出描述「酒精量」的方程式嗎？

學生： $0.5x + 0.2y = 44$，對嗎？

老師： 沒錯。現在我們有一個線性方程組：$\begin{cases} x + y = 100 \\ 0.5x + 0.2y = 44 \end{cases}$。現在請使用克拉瑪公式算出答案。首先找出行列式值。

學生： $\Delta = \begin{vmatrix} 1 & 1 \\ 0.5 & 0.2 \end{vmatrix} = 0.2 - 0.5 = -0.3$。算出來是一個負數，對嗎？

老師： 要有自信。讓我們找出 Δ_x和 Δ_y。

學生： $\Delta_x = \begin{vmatrix} 100 & 1 \\ 44 & 0.2 \end{vmatrix} = 20 - 44 = -24.\ \Delta_y = \begin{vmatrix} 1 & 100 \\ 0.5 & 44 \end{vmatrix} = 44 - 50 = -6$。

它們也都是負數。

老師： $x = \frac{\Delta_x}{\Delta}$ 、 $y = \frac{\Delta_y}{\Delta}$.。兩個負數相除是正數。別擔心，我們目前算的都是對的。

請算出答案。

學生： $x = \frac{\Delta_x}{\Delta} = \frac{-24}{-0.3} = 80,\ y = \frac{\Delta_y}{\Delta} = \frac{-6}{-0.3} = 20$。

老師： 請再次仔細檢查答案。

學生： $80 + 20 = 100$，且 $0.5 \cdot 80 + 0.2 \cdot 20 = 44$。兩個方程式都滿足。

老師： 太好了。因此第一種溶液需要 80 升，第二種溶液需要 20 升。

國內外參考資源 More to Explore

國家教育研究院樂詞網	
查詢學科詞彙 https://terms.naer.edu.tw/search/	
教育雲：教育媒體影音	
為教育部委辦計畫雙語教學影片 https://video.cloud.edu.tw/video/co_search.php?s=%E9%9B%99%E8%AA%9E	
Oak Teacher Hub	
國外教學及影音資源，除了數學領域還有其他科目 https://teachers.thenational.academy/	
CK-12	
國外教學及影音資源，除了數學領域還有自然領域 https://www.ck12.org/student/	
Twinkl	
國外教學及影音資源，除了數學領域還有其他科目，多為小學及學齡前內容 https://www.twinkl.com.tw/	

Khan Academy	
可汗學院，有分年級數學教學影片及問題的討論 https://www.khanacademy.org/	
Open Textbook (Math)	
國外數學開放式教學資源 http://content.nroc.org/DevelopmentalMath.HTML5/Common/toc/toc_en.html	
MATH is FUN	
國外教學資源，還有數學相關的小遊戲 https://www.mathsisfun.com/index.htm	
PhET: Interactive Simulations	
國外教學資源，互動式電腦模擬。除了數學領域，還有自然科 https://phet.colorado.edu/	
Eddie Woo YouTube Channel	
國外數學教學影音 https://www.youtube.com/c/misterwootube	

國立臺灣師範大學數學系陳界山教授網站	
國高中數學雙語教學相關教材 https://math.ntnu.edu.tw/~jschen/index.php?menu=Teaching_Worksheets	
2024 年第五屆科學與科普專業英文(ESP)能力大賽	
科學專業英文相關教材，除了數學領域，還有其他領域 https://sites.google.com/view/ntseccompetition/%E5%B0%88%E6%A5%AD%E8%8B%B1%E6%96%87%E5%AD%B8%E7%BF%92%E8%B3%87%E6%BA%90/%E7%9B%B8%E9%97%9C%E6%95%99%E6%9D%90?authuser=0	
Desmos Classroom	
國外教學資源，也有免費繪圖功能 https://teacher.desmos.com/?lang=en	

高中數學領域雙語教學資源手冊：英語授課用語

[十一年級上學期]

A Reference Handbook for Senior High School Bilingual Teachers in the Domain of Mathematics: Instructional Language in English

[11th grade 1st semester]

- 研編單位：國立臺灣師範大學雙語教學研究中心
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- 撰稿：周慧蓮、印娟娟、吳珮蓁
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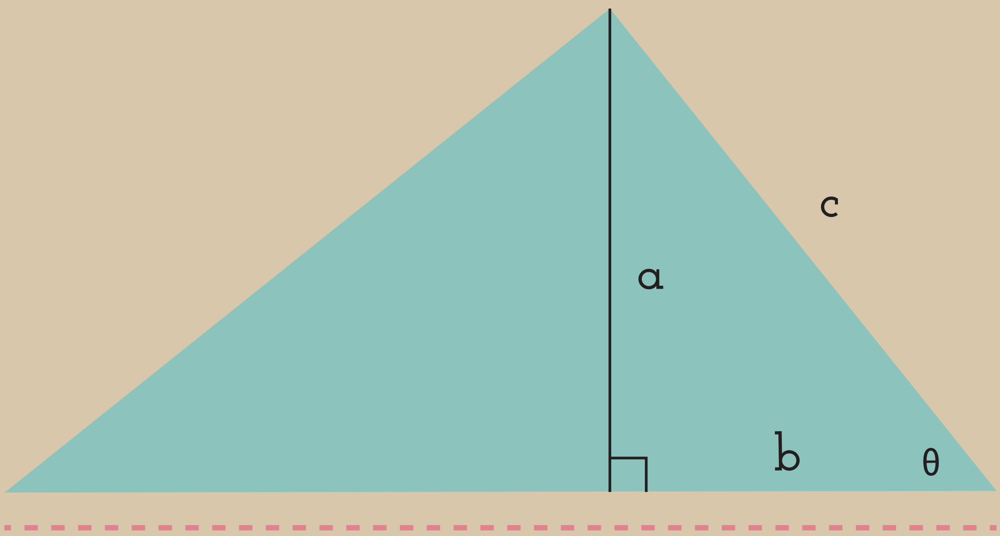
- 語言諮詢：Ramon Mislang
- 綜合規劃：王宏均
- 排版：吳依靜
- 封面封底：JUPE Design



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